AN INCLUSION THEOREM FOR GENERALIZED CESÀRO AND RIESZ MEANS

A. MEIR

For a positive integer, p, a strictly increasing unbounded sequence of positive numbers $\{\lambda_n: n \geqslant 1\}$ and an arbitrary sequence of complex numbers $\{a_n\}$ let

(1) $$A^p(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^p a_\nu,$$

(2) $$C_n^p = \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \ldots (\lambda_{n+p} - \lambda_\nu) a_\nu.$$

The series $\sum a_\nu$ is said to be (R, λ, p) summable to s if

(3) $$R^p(\omega) \equiv \omega^{-p} A^p(\omega) \to s \qquad \text{as } \omega \to \infty,$$

and (C, λ, p) summable to s if

(4) $$t_n^p \equiv (\lambda_{n+1} \lambda_{n+2} \ldots \lambda_{n+p})^{-1} C_n^p \to s \qquad \text{as } n \to \infty.$$

D. C. Russell **(2)** proved that $(C, \lambda, p) \subseteq (R, \lambda, p)$ for any $\{\lambda_n\}$ and any $p \geqslant 0$. In the opposite direction he showed that $(R, \lambda, p) \subseteq (C, \lambda, p)$ for $p \geqslant 3$ if the sequence $\{\lambda_n\}$ satisfies the condition

(5) $$\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} = O\left(\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}\right),$$

and for all $\{\lambda_n\}$ if $p = 0, 1, 2$. In a recent note, D. Borwein **(1)** established the same inclusion relation under another (independent) condition:

(6) $$\lambda_{n+1} = O(\lambda_n).$$

We shall prove here that the inclusion relation holds for all $p \geqslant 0$ without any restriction on the sequence $\{\lambda_n\}$.

THEOREM. $(R, \lambda, p) \subseteq (C, \lambda, p)$ *for all sequences* λ *and* $p \geqslant 0$.

The proof of the theorem is an immediate consequence of the following lemma.

LEMMA. *For every* $n \geqslant 1$ *there exist real numbers* $C_j^{(n)}, \omega_j^{(n)}$ $(j = 0, 1, \ldots, p)$ *satisfying*

(7) $$\sum_{j=0}^{p} C_j^{(n)} = 1,$$

(8) $$|C_j^{(n)}| \leqslant H, \qquad j = 0, 1, \ldots, p;$$

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where H depends on p but not on n,

(9) $$\lambda_n \leqslant \omega_j^{(n)} \leqslant \lambda_{n+p}, \qquad j = 0, 1, \ldots, p,$$

and

(10) $$t_n^p = \sum_{j=0}^{p} C_j^{(n)} R^p(\omega_j^{(n)}).$$

Proof of the lemma. We may take $p \geqslant 1$ in the proof. Let n be any fixed integer. We distinguish between two cases.

Case (i). Suppose that

(11) $$\lambda_{n+p}/\lambda_n \leqslant (p+1)^p.$$

From equations (3), (4), and (5) of **(1)** it follows that there exist $y_j^{(n)}$ and $\omega_j^{(n)}$ satisfying

(12) $$\lambda_n \leqslant \omega_j^{(n)} \leqslant \lambda_{n+p},$$

(13) $$|y_j^{(n)}| \leqslant (p+1)!\,(p+1)^{2p},$$

(14) $$C_n^p = \sum_{j=0}^{p} y_j^{(n)} A^p(\omega_j^{(n)}).$$

Also it follows from the construction of the y_j's that

(15) $$\sum_{j=0}^{p} y_j^{(n)} (\omega_j^{(n)})^p = \lambda_{n+1} \ldots \lambda_{n+p}.$$

Dividing both sides of (14) by $\lambda_{n+1} \ldots \lambda_{n+p}$ we have that

(16) $$t_n^p = \sum_{j=0}^{p} C_j^{(n)} R^p(\omega_j^{(n)}),$$

where

$$C_j^{(n)} = y_j^{(n)} \frac{(\omega_j^{(n)})^p}{\lambda_{n+1} \ldots \lambda_{n+p}}.$$

From (15) it follows that

(17) $$\sum_{j=0}^{p} C_j^{(n)} = 1$$

and from (11), (12), and (13) that

(18) $$|C_j^{(n)}| \leqslant (p+1)!\,(p+1)^{2p+p^2}.$$

(12), (16), (17), and (18) prove the lemma in case (i).

Case (ii). Suppose that

$$\lambda_{n+p}/\lambda_n > (p+1)^p.$$

Then there exists an integer r, $0 \leqslant r \leqslant p-1$, such that

(19) $$\lambda_{n+r+1}/\lambda_{n+r} > p+1$$

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and

$$\lambda_{n+j+1}/\lambda_{n+j} \leqslant p+1, \qquad j = 0, 1, \ldots, r-1. \tag{20}$$

We define the numbers $\omega_j^{(n)}$ for $j = 0, 1, \ldots, p$ by

$$\omega_j^{(n)} = (j+1)\lambda_{n+r}, \tag{21}$$

and the numbers $C_j^{(n)}$ by

$$\prod_{k=1}^{p}\left(1 - \frac{x}{\lambda_{n+k}}\right) \equiv \sum_{j=0}^{p} C_j^{(n)}\left(1 - \frac{x}{\omega_j^{(n)}}\right)^p. \tag{22}$$

It is easily seen that the identity (22) is equivalent to the system of equations:

$$\sum_{j=0}^{p} C_j^{(n)}(j+1)^{-k} = \beta_k^{(n)}, \qquad k = 0, 1, \ldots, p, \tag{23}$$

where

$$\beta_k^{(n)} = \binom{p}{k}^{-1} \sum (\lambda_{\nu_1} \cdot \lambda_{\nu_2} \cdot \ldots \cdot \lambda_{\nu_k})^{-1} \lambda_{n+r}^k$$

and the summation extends to all $\binom{p}{k}$ combinations of k integers $\nu_1, \nu_2, \ldots, \nu_k$ from $n+1, n+2, \ldots, n+p$. By (20) we have for all ν obeying

$$n+1 \leqslant \nu \leqslant n+p$$

that

$$0 < \lambda_{n+r}/\lambda_\nu \leqslant (p+1)^p,$$

from which it follows easily that for $0 \leqslant k \leqslant p$

$$0 < \beta_k^{(n)} \leqslant (p+1)^{kp} \leqslant (p+1)^{p^2}.$$

Using Cramer's formula to solve (23) for $C_j^{(n)}$ we conclude by an elementary argument that there exists a constant $A = A_p$ independent of n such that

$$|C_j^{(n)}| \leqslant A. \tag{24}$$

It also follows from (23), with $k = 0$, that

$$\sum_{j=0}^{p} C_j^{(n)} = 1, \tag{25}$$

and from (18) and (20) that

$$\lambda_{n+r} \leqslant \omega_j^{(n)} < \lambda_{n+r+1}, \qquad j = 0, 1, \ldots, p. \tag{26}$$

Now on putting $x = \lambda_\nu$ in (22), multiplying by a_ν, and summing over $1 \leqslant \nu \leqslant n+r$,

$$t_n^p = \sum_{j=0}^{p} C_j^{(n)} R^p(\omega_j^{(n)}),$$

which, together with (24), (25), and (26), concludes the proof of the lemma in case (ii).

References

1. D. Borwein, *On a generalized Cesàro summability method of integral order*, Tôhoku Math. J., *18* (1966), 71–73.

2. D. C. Russell, *On generalized Cesàro means of integral order*, Tôhoku Math. J., *17* (1965), 410–442.

The University of Alberta
Edmonton, Alberta