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ON AN EIGENVALUE PROBLEM FOR AN ANISOTROPIC ELLIPTIC EQUATION INVOLVING VARIABLE EXPONENTS

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Abstract. We study the eigenvalue problem $-\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = \lambda |u|^{q(x)-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, λ is a positive real number, and $p_1, \ldots, p_N$, q are continuous functions satisfying the following conditions: $2 \le p_i(x) < N$, $1 < q(x)$ for all $x \in \Omega$, $i \in \{1, \ldots, N\}$; there exist $j,\ k \in \{1, \ldots, N\}$, $j \neq k$, such that $p_j \equiv q$ in $\overline{\Omega}$, q is independent of x_j and $\max_{\overline{\Omega}} q < \min_{\overline{\Omega}} p_k$. The main result of this paper establishes the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue, while no $\lambda \in (0, \lambda_0)$ can be an eigenvalue of the above problem.

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1. Introduction. The goal of this paper is to study the existence of solutions of the following anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(\left|\partial_{x_i} u\right|^{p_i(x)-2}\partial_{x_i} u\right) = \lambda |u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive number, and p_i, q are continuous functions on $\overline{\Omega}$ such that $2 \le p_i(x) < N$ and $q(x) > 1$ for all $x \in \overline{\Omega}$ and $i \in \{1, \ldots, N\}$.

Our study is motivated by some recent advances on the eigenvalue problems for anisotropic operators involving variable exponent growth conditions obtained in [**19**]. Considering different cases regarding the variable exponents $p_i(x)$ and $q(x)$ involved in equation (1), the authors of [**19**] found certain interesting results that will be briefly presented in what follows:

- In the case when $\max\{\max_{\overline{\Omega}} p_1, \ldots, \max_{\overline{\Omega}} p_N\} < \min_{\overline{\Omega}} q$ and q has a subcritical growth, a mountain pass argument can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (1) (see [**19**, Theorem 2]).
- In the case when $\min_{\overline{\Omega}} q < \min\{\min_{\overline{\Omega}} p_1, \ldots, \min_{\overline{\Omega}} p_N\}$ and q has a subcritical growth, using Ekeland's variational principle, one can prove the existence of a

continuous family of eigenvalues lying in a neighbourhood of the origin (see [**19**, Theorem 4]).

- In the case when $\max_{\overline{\Omega}} q < \min\{\min_{\overline{\Omega}} p_1, \dots, \min_{\overline{\Omega}} p_N\}$ it can be proved that the energy functional associated with problem (1) has a non-trivial (global) minimum point for any positive λ large enough and, consequently, any positive λ large enough is an eigenvalue of problem (1) (see [**19**, Theorem 3]). Obviously, in this case the above result can also be applied and, thus, in this situation there exist two positive constants $\lambda^\star$ and $\lambda^{\star\star}$ such that every $\lambda \in (0, \lambda^\star) \cup (\lambda^{\star\star}, \infty)$ is an eigenvalue of problem (1) (see [**19**, Corollary 1]).

Our paper supplements the above results on problem (1) by considering a new case, when there exist $j,\ k \in \{1, \dots, N\}$ with $j \neq k$ such that p_j is independent of x_j,

$$p_j(x) = q(x), \quad \forall\, x \in \overline{\Omega} \quad \text{and} \quad \max_{\overline{\Omega}} q < \min_{\overline{\Omega}} p_k\,.$$

In this situation it will be proved that small values of λ cannot be eigenvalues of problem (1) while every λ large enough is an eigenvalue of problem (1).

On the other hand, we point out that our study extends to the case of anisotropic equations the results obtained in [**22**] and generalizes some other existing results on eigenvalue problems involving variable exponent growth conditions [**11, 12, 20, 21, 23**]. Finally, we note that equations of type (1) are models for various phenomena which arise from the study of electrorheological fluids (see [**7, 14, 20, 29, 30**]), image processing (see [**6**]), or the theory of elasticity (see [**35**]).

2. Abstract framework. In this section we recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We will also introduce an adequate functional space where problems of type (1) can be studied. Such a space will be called an anisotropic variable exponent Sobolev space and it can be characterized as a functional space of Sobolev's type in which different space directions have different roles.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \{u : u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

which is a separable and reflexive Banach space. If $|\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \leq p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$. For every $u \in L^{p(\cdot)}(\Omega)$ and $v \in$

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$L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2}$$

is valid.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u_n, u \in L^{p(\cdot)}(\Omega)$ then the following implications hold

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{p(\cdot)}, \tag{3}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p^-}_{p(\cdot)}, \tag{4}$$

$$|u_n - u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0, \tag{5}$$

since $p^+ < \infty$.

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^1(\Omega)$ under the norm

$$\|u\|_{1,p(\cdot)} = |\nabla u|_{p(\cdot)}\,.$$

We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(\cdot)} = \sum_{i=1}^N \left|\partial_{x_i} u\right|_{p(\cdot)},$$

provided that $p(x) \ge 2$ for all $x \in \overline{\Omega}$ (see **[18]**). Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable, reflexive Banach space. Note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, where $p^\star(x) = Np(x)/[N - p(x)]$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \ge N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact. For proofs, details and further results on variable exponent Lebesgue and Sobolev spaces we refer to Musielak's book **[24]** and the papers of Kováčik and Rákosník **[17]**, Edmunds et al. **[8–10]**, Samko and Vakulov **[31]**, while for applications of such kind of spaces to the study of partial differential equations we refer to **[1–7, 15, 19–23, 26, 29, 30, 35]**.

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study problem (1) with sufficient accuracy. For this purpose, let us denote by $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\vec{p} = (p_1, \ldots, p_N)$. We define $W_0^{1,\vec{p}(\cdot)}(\Omega)$, the *anisotropic variable exponents Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}(\cdot)} = \sum_{i=1}^N \left|\partial_{x_i} u\right|_{p_i(\cdot)}.$$

As it was pointed out in **[19]**, $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

We also note that in the case when p_i are all constant functions, the resulting anisotropic Sobolev space is denoted by $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ is the constant vector $(p_1, \dots, p_N)$. The theory of such spaces was developed in [**13, 25, 27, 33, 34**].

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ we introduce $\vec{P}_+, \vec{P}_- \in \mathbb{R}^N$ as

$$\vec{P}_+ = (p_1^+, \dots, p_N^+), \quad \vec{P}_- = (p_1^-, \dots, p_N^-)$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\{p_1^+, \dots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, \dots, p_N^-\}, \quad P_-^- = \min\{p_1^-, \dots, p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1, \tag{6}$$

and define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

We recall that if $s \in C_+(\overline{\Omega})$ satisfies $1 < s(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact (see [**19**, Theorem 1]).

3. The main result. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1) if there exists $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega \left\{ \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} \varphi - \lambda |u|^{q(x)-2} u \varphi \right\} dx = 0$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. For $\lambda \in \mathbb{R}$ an eigenvalue of problem (1) the function u from the above definition will be called a *weak solution* of problem (1) corresponding to the eigenvalue λ.

In this paper our basic assumptions on the functions p_i, q involved in equation (1) will be the following:

(A1) Assume that there exists $j \in \{1, \dots, N\}$ such that $q(x) = q(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$ (i.e. q is independent of x_j) and $p_j(x) = q(x)$ for all $x \in \overline{\Omega}$.

(A2) Assume that there exists $k \in \{1, \dots, N\}$ ($k \neq j$ with j given in (A1)) such that

$$\max_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p_k(x).$$

Define the Rayleigh type quotients λ_0 and λ_1 associated with problem (1) by

$$\lambda_0 = \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N |\partial_i u|^{p_i(x)}\, dx}{\int_\Omega |u|^{q(x)}\, dx}, \quad \lambda_1 = \inf_{u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} |\partial_i u|^{p_i(x)}\, dx}{\int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx}.$$

The main result of this paper is given by the following theorem:

THEOREM 1. *Assume that conditions (A1) and (A2) are fulfilled. Then* $0 < \lambda_0 \le \lambda_1$ *and every* $\lambda \in (\lambda_1, \infty)$ *is an eigenvalue of problem (1), while no* $\lambda \in (0, \lambda_0)$ *can be an eigenvalue of problem (1).*

REMARK 1. At this stage, we are not able to say whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case, an interesting question concerns the existence of eigenvalues of problem (1) in the interval $[\lambda_0, \lambda_1]$. We propose to the reader the study of these open problems.

REMARK 2. The result of Theorem 1 also supplements some earlier *classical* results on eigenvalue problems. For instance, in the case when in equation (1) we consider $p_i(x) = q(x) = 2$ for all $x \in \overline{\Omega}$, $i \in \{1, \dots, N\}$, a basic result in the elementary theory of partial differential equations asserts that the spectrum of the negative Laplace operator (in $H_0^1(\Omega)$) is *discrete* (if Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary). This celebrated result goes back to the Riesz–Fredholm theory of self-adjoint and compact operators on Banach spaces. Furthermore, in the case when in equation (1) we have $p_i(x) = q(x) = p$ for all $x \in \overline{\Omega}$, $i \in \{1, \dots, N\}$, with $p > 1$ a given constant, then the operator involved in the equation is similar with the p-Laplace operator, i.e. $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. In this case the Lusternik–Schnirelman theory asserts that the spectrum of the negative p-Laplace operator contains at least an unbounded sequence of positive eigenvalues, say $0 < \mu_1 < \mu_2 \le \cdots \le \mu_n \le \cdots$. Unfortunately, to our best knowledge, nothing is known in general about the possible existence of other eigenvalues in (μ_1, ∞). However, it is known (see [**4**]) that μ_1 is an isolated point of the spectrum (actually, μ_1 is given by the infimum of the Rayleigh quotient which defines λ_1 above).

We point out that in the two cases presented above the two Rayleigh quotients, which define λ_1 and λ_0, are equal and consequently, in these two cases, we have $\lambda_1 = \lambda_0$. Clearly, that fact is a consequence of the homogenity of the equations in these two particular cases. The loss of homogenity in the case emphasized in Theorem 1 will lead to a *continuous* spectrum for problem (1).

4. An auxiliary result. A key result in proving Theorem 1 is given by the following proposition which extends the result of relation (11) in [**13**]. The proof of this result is inspired by the proof of relation (11) in [**13**].

PROPOSITION 1. *Assume that condition (A1) is fulfilled. Then there exists a positive constant* $C = C(a_j, q^+)$ *such that*

$$\int_\Omega |u|^{q(x)}\, dx \le C \int_\Omega |\partial_{x_j} u|^{q(x)}\, dx, \quad \forall\, u \in C_0^1(\Omega).$$

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Proof. First, we recall the definition of the *width* of the domain Ω in a direction. Consider that $\{e_1, \dots, e_N\}$ is the canonical basis in $\mathbb{R}^N$. We say that Ω has *width* $a_i > 0$ in the e_i direction if

$$\sup_{x,y\in\Omega}(x-y, e_i) = a_i\,.$$

Without loss of generality, we assume that

$$\Omega \subset \{x \in \mathbb{R}^N; \quad 0 < x_j \le a_j\}.$$

For each $u \in C_0^1(\Omega)$ we put

$$v(x) = u(x)\partial_{x_j}u(x).$$

Next, we extend u and v on the whole $\mathbb{R}^N$ by setting 0 outside $\text{supp}(u)$ and $\text{supp}(v)$. For each $x = (x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_N) \in \mathbb{R}^N$ let us denote $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N) \in \mathbb{R}^{N-1}$. In order to emphasize the jth component of x we will write $x = (x_j, x')$.

With the above notation we have $q(x) = q(x')$ for all $x \in \mathbb{R}^N$. Note that

$$\begin{aligned} 0 = \frac{|u(a_j, x')|^{q(x')} - |u(0, x')|^{q(x')}}{q(x')} &= \int_0^{a_j} |u(t, x')|^{q(x')-2} v(t, x')\, dt \\ &= \int_0^{a_j} |u(t, x')|^{q(x')-2} v^+(t, x')\, dt - \int_0^{a_j} |u(t, x')|^{q(x')-2} v^-(t, x')\, dt\,, \end{aligned}$$

where $v^{\pm}(t, x') = \max\{0, \pm v(t, x')\}$.

On the other hand, the following equality holds true

$$\begin{aligned} \int_0^{a_j} |u(t, x')|^{q(x')-2} |v(t, x')|\, dt = &\int_0^{a_j} |u(t, x')|^{q(x')-2} v^+(t, x')\, dt \\ &+ \int_0^{a_j} |u(t, x')|^{q(x')-2} v^-(t, x')\, dt. \end{aligned}$$

The above equalities imply

$$\int_0^{a_j} |u(t, x')|^{q(x')-2} v^+(t, x')\, dt = \frac{1}{2}\int_0^{a_j} |u(t, x')|^{q(x')-2} |v(t, x')|\, dt.$$

Using the last relation and some elementary estimates we deduce

$$\begin{aligned} |u(x_j, x')|^{q(x')} &= q(x')\int_0^{x_j} |u(t, x')|^{q(x')-2} v(t, x')\, dt \\ &\le q(x')\int_0^{x_j} |u(t, x')|^{q(x')-2} v^+(t, x')\, dt \\ &\le q(x')\int_0^{a_j} |u(t, x')|^{q(x')-2} v^+(t, x')\, dt \\ &= \frac{q(x')}{2}\int_0^{a_j} |u(t, x')|^{q(x')-1} |\partial_{x_j} u(t, x')|\, dt\,, \end{aligned}$$

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for all $x_j \in (0, a_j)$. Now, using Young's inequality, we deduce that

$$|u(t,x')|^{q(x')-1}|\partial_{x_j}u(t,x')| \leq \frac{q(x')-1}{q(x')}\varepsilon^{\frac{q(x')}{q(x')-1}}|u(t,x')|^{q(x')} + \frac{1}{q(x')\varepsilon^{q(x')}}|\partial_{x_j}u(t,x')|^{q(x')},$$

for all $(t, x') \in \mathbb{R}^N$ and all $\varepsilon > 0$.

The last two relations yield

$$|u(x_j,x')|^{q(x')} \leq \frac{q(x')-1}{2}\varepsilon^{\frac{q(x')}{q(x')-1}}\int_0^{a_j}|u(t,x')|^{q(x')}\,dt + \frac{1}{2\varepsilon^{q(x')}}\int_0^{a_j}|\partial_{x_j}u(t,x')|^{q(x')}\,dt,$$

for all $x_j \in (0, a_j)$, all $x' \in \mathbb{R}^N$ and all $\varepsilon > 0$. Integrating the above inequality with respect to $x_j \in (0, a_j)$ we get

$$\int_0^{a_j}|u(t,x')|^{q(x')}\,dt \leq a_j\frac{q(x')-1}{2}\varepsilon^{\frac{q(x')}{q(x')-1}}\int_0^{a_j}|u(t,x')|^{q(x')}\,dt + \frac{a_j}{2\varepsilon^{q(x')}}\int_0^{a_j}|\partial_{x_j}u(t,x')|^{q(x')}\,dt,$$

for all $x' \in \mathbb{R}^N$ and all $\varepsilon > 0$. Next, for all $\varepsilon \in (0, 1)$ we find

$$\left[1 - a_j\frac{q^+-1}{2}\varepsilon^{\frac{q^+}{q^+-1}}\right]\int_0^{a_j}|u(t,x')|^{q(x')}\,dt \leq \frac{a_j}{2\varepsilon^{q^+}}\int_0^{a_j}|\partial_{x_j}u(t,x')|^{q(x')}\,dt,$$

for all $x' \in \mathbb{R}^N$. Obviously, there exists $\varepsilon_0 \in (0, 1)$, small enough, such that

$$\alpha := 1 - a_j\frac{q^+-1}{2}\varepsilon_0^{\frac{q^+}{q^+-1}} > 0.$$

Thus, we find

$$\int_0^{a_j}|u(t,x')|^{q(x')}\,dt \leq \frac{a_j}{2\alpha\varepsilon_0^{q^+}}\int_0^{a_j}|\partial_{x_j}u(t,x')|^{q(x')}\,dt.$$

Finally, letting $C = \frac{a_j}{2\alpha\varepsilon_0^{q^+}}$ and integrating the last inequality with respect to $x' \in \mathbb{R}^N$ we conclude

$$\int_\Omega |u|^{q(x)}\,dx \leq C\int_\Omega |\partial_{x_j}u|^{q(x)}\,dx,$$

for every $u \in C_0^1(\Omega)$.

The proof of Proposition 1 is complete. □

5. Proof of the main result. From now on E denotes the anisotropic variable exponent Orlicz–Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Define the functionals $J, I, J_1, I_1 : E \to \mathbb{R}$

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by

$$J(u)=\int_\Omega\sum_{i=1}^N\frac{1}{p_i(x)}\left|\partial_{x_i}u\right|^{p_i(x)}\,dx,\quad I(u)=\int_\Omega\frac{1}{q(x)}|u|^{q(x)}\,dx,$$
$$J_1(u)=\int_\Omega\sum_{i=1}^N\left|\partial_{x_i}u\right|^{p_i(x)}\,dx,\quad I_1(u)=\int_\Omega|u|^{q(x)}\,dx.$$

Standard arguments imply that $J, I\in C^1(E,\mathbb{R})$ and their Fréchet derivatives are given by

$$\langle J_\lambda'(u),v\rangle=\int_\Omega\sum_{i=1}^N\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v\,dx,\quad \langle I_\lambda'(u),v\rangle=\int_\Omega|u|^{q(x)-2}uv\,dx\,,$$

for all $u,\ v\in E$.

- First, we note that by Proposition 1 we can easily infer that

$$\lambda_0=\inf_{u\in E\setminus\{0\}}\frac{J_1(u)}{I_1(u)}>0\quad\text{and}\quad\lambda_1=\inf_{u\in E\setminus\{0\}}\frac{J(u)}{I(u)}>0.$$

- Second, we point out that no $\lambda\in(0,\lambda_0)$ can be an eigenvalue of problem (1). Indeed, assuming by contradiction that there exists $\lambda\in(0,\lambda_0)$ an eigenvalue of problem (1) it follows that there exists a $w_\lambda\in E\setminus\{0\}$ such that

$$\langle J'(w_\lambda),v\rangle=\lambda\langle I'(w_\lambda),v\rangle,\quad\forall\ v\in E.$$

Thus, for $v=w_\lambda$ we find

$$\langle J'(w_\lambda),w_\lambda\rangle=\lambda\langle I'(w_\lambda),w_\lambda\rangle,$$

that is,

$$J_1(w_\lambda)=\lambda I_1(w_\lambda).$$

The fact that $w_\lambda\in E\setminus\{0\}$ assures that $I_1(w_\lambda)>0$. Since $\lambda<\lambda_0$, the above information yields

$$J_1(w_\lambda)\ge\lambda_0I_1(w_\lambda)>\lambda I_1(w_\lambda)=J_1(w_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Consequently, no $\lambda\in(0,\lambda_0)$ can be an eigenvalue of problem (1).

- Third, we will prove that every $\lambda\in(\lambda_1,\infty)$ is an eigenvalue of problem (1).

  In order to do that, we need the following auxiliary result.

  LEMMA 1.

$$\lim_{\|u\|_{\vec{p}(\cdot)}\to\infty}\frac{J(u)}{I(u)}=\infty.$$

Proof. Assume by contradiction that the conclusion of Lemma 1 does not hold true. Then there exists an $M>0$ such that for each $n\in\mathbb{N}^*$ there exists a $u_n\in E$ with

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$\|u_n\|_{\vec{p}(\cdot)} > n$ and

$$\frac{J(u_n)}{I(u_n)} \leq M. \tag{7}$$

While $\|u_n\|_{\vec{p}(\cdot)} = \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(\cdot)} \to \infty$ as $n \to \infty$, the sequence $\{|\partial_{x_k} u_n|_{p_k(\cdot)}\}_n$ (with k given by condition (A2)) is either bounded or unbounded.

On the other hand, it is not difficult to see that

$$\int_\Omega |u|^{q(x)} \leq \int_\Omega |u|^{q^-}\, dx + \int_\Omega |u|^{q^+}\, dx, \quad \forall\, u \in E.$$

Next, using relation (11) in **[13]** we find that there exists a positive constant c_1 such that

$$\int_\Omega |u|^{q^-}\, dx + \int_\Omega |u|^{q^+}\, dx \leq c_1 \left(\int_\Omega |\partial_{x_k} u|^{q^-}\, dx + \int_\Omega |\partial_{x_k} u|^{q^+}\, dx \right), \quad \forall\, u \in E.$$

Since by condition (A2) we have $q^+ < p_k^- \leq P_-^+ \leq P_{-,\infty}$ we deduce that $L^{p_k(\cdot)}$ is continuously embedded in $L^{q^\pm}(\Omega)$. The above pieces of information lead to the existence of a positive constant c_2 such that

$$\int_\Omega |u|^{q(x)} \leq c_2 [|\partial_{x_k} u|_{p_k(\cdot)}^{q^+} + |\partial_{x_k} u|_{p_k(\cdot)}^{q^-}], \quad \forall\, u \in E. \tag{8}$$

If $\{|\partial_{x_k} u_n|_{p_k(\cdot)}\}_n$ is bounded then by inequality (8) we have that $\{I(u_n)\}_n$ is also bounded while by relation (19) in **[19]** we have that

$$J(u_n) \geq c_3 \|u_n\|_{\vec{p}(\cdot)}^{P_-^-} - c_4, \quad \forall\, n \in \mathbb{N}^*,$$

where c_3 and c_4 are two positive constants. Consequently, in this case we obtain that $\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty$ which contradicts (7).

Now, we assume that $|\partial_{x_k} u_n|_{p_k(\cdot)} \to \infty$, as $n \to \infty$, on a subsequence of u_n denoted again u_n. We can assume that $|\partial_{x_k} u_n|_{p_k(\cdot)} > 1$ for all n. Using relations (3) and (8) we find

$$\frac{J(u_n)}{I(u_n)} \geq \frac{c_5 \int_\Omega |\partial_{x_k} u_n|^{p_k(x)}\, dx}{c_2 \big[|\partial_{x_k} u_n|_{p_k(\cdot)}^{q^+} + |\partial_{x_k} u_n|_{p_k(\cdot)}^{q^-} \big]} \geq \frac{c_5 |\partial_{x_k} u_n|_{p_k(\cdot)}^{p_k^-}}{c_2 \big[|\partial_{x_k} u_n|_{p_k(\cdot)}^{q^+} + |\partial_{x_k} u_n|_{p_k(\cdot)}^{q^-} \big]} \quad \forall\, u \in E, \quad n \in \mathbb{N}^*,$$

where c_5 is a positive constant. Since by condition (A2) we have $p_k^- > q^+$ the above inequalities show that $J(u_n)/I(u_n) \to \infty$, as $n \to \infty$, which contradicts again (7).

Therefore, the conclusion of Lemma 1 is valid. □

Now, we are prepared to show that every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (1).

Let $\lambda \in (\lambda_1, \infty)$ be arbitrary but fixed. Define $T_\lambda : E \to \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly, $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T_\lambda'(u), v \rangle = \langle J'(u), v \rangle - \lambda \langle I'(u), v \rangle, \quad \forall\, u \in E.$$

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Thus, λ is an eigenvalue of problem (1) if and only if there exists $u_\lambda \in E \setminus \{0\}$ a critical point of T_λ.

By Lemma 1 we get that T_λ is coercive, i.e. $\lim_{\|u\|_{\vec{p}(\cdot)} \to \infty} T_\lambda(u) = \infty$. On the other hand, similar arguments as those used in the proof of [**20**, Lemma 3.4] show that the functional T_λ is weakly lower semi-continuous. These two facts enable us to apply [**32**, Theorem 1.2] in order to prove that there exists $u_\lambda \in E$ a global minimum point of T_λ and thus, a critical point of T_λ. In order to conclude that λ is an eigenvalue of problem (1) it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in E\setminus\{0\}} \frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a non-trivial critical point of T_λ, that is λ is an eigenvalue of problem (1).

- Finally, we note that by the above arguments we can infer that $\lambda_0 \leq \lambda_1$.

The proof of Theorem 1 is complete.

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REFERENCES

1. E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth, *Arch. Ration. Mech. Anal.* **156** (2001), 121–140.

2. E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$–Laplacean system, *J. Reine Angew. Math.* **584** (2005), 117–148.

3. C. O. Alves and M. A. Souto, Existence of solutions for a class of problems in $\mathbb{R}^N$ involving the $p(x)$–Laplacian, in *Contributions to nonlinear analysis, a tribute to D.G. de Figueiredo on the occasion of his 70th birthday* (Cazenave T., Costa D., Lopes O., Manàsevich R., Rabinowitz R., Ruf R. and Tomei C., Editors), Progress in Nonlinear Differential Equations and Their Applications, Vol. **66** (Birkhäuser, Basel, Switzerland, 2006), 17–32.

4. A. Anane, Simplicité et isolation de la première valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris Sér.* I **305** (1987), 725–728.

5. J. Chabrowski and Y. Fu, Existence of solutions for $p(x)$–Laplacian problems on a bounded domain, *J. Math. Anal. Appl.* **306** (2005), 604–618.

6. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006), 1383–1406.

7. L. Diening, Theoretical and numerical results for electrorheological fluids, PhD thesis (University of Frieburg, Germany, 2002).

8. D. E. Edmunds, J. Lang and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. R. Soc. Lond. Ser.* A **455** (1999), 219–225.

9. D. E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. R. Soc. Lond. Ser.* A **437** (1992), 229–236.

10. D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, *Studia Math.* **143** (2000), 267–293.

11. X. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **352** (2009), 85–98.

12. X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005), 306–317.

13. I. Fragalà, F. Gazzola and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **21** (2004), 715–734.

14. T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761–766.

15. P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen, The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, *Potential Anal.* **25** (2006), 79–94.

16. P. Hästö, On the density of continuous functions in variable exponent Sobolev spaces, *Rev. Mat. Iberoamericana* **23** (2007), 74–82.

17. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czech. Math. J.* **41** (1991), 592–618.

18. M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Appl. Anal.* **89** (2010), 257–271.

19. M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687–698.

20. M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. Lond. Ser.* A **462** (2006), 2625–2641.

21. M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007), 2929–2937.

22. M. Mihăilescu and V. Rădulescu, Continuous spectrum for a class of nonhomogeneous differential operators, *Manuscr. Math.* **125** (2008), 157–167.

23. M. Mihăilescu and V. Rădulescu, Spectrum in an unbounded interval for a class of nonhomogeneous differential operators, *Bull. Lond. Math. Soc.* **40** (6) (2008), 972–984.

24. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. **1034** (Springer, Berlin, 1983).

25. S. M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, *Russ. Math. Surv.* **16** (1961), 55–104.

26. C. Pfeiffer, C. Mavroidis, Y. Bar–Cohen and B. Dolgin, Electrorheological fluid based force feedback device, in *Proc. 1999 SPIE Telemanipulator and Telepresence Technologies VI Conf.*, vol. 3840 (Boston, MA, 1999), pp. 88–99.

27. J. Rákosník, Some remarks to anisotropic Sobolev spaces I, *Beitr. Anal.* **13** (1979), 55–68.

28. J. Rákosník, Some remarks to anisotropic Sobolev spaces II, *Beitr. Anal.* **15** (1981), 127–140.

29. K. R. Rajagopal and M. Ruzicka, Mathematical modelling of electrorheological fluids, *Contin. Mech. Thermodyn.* **13** (2001), 59–78.

30. M. Ruzicka, *Electrorheological fluids: Modeling and mathematical theory* (Springer–Verlag, Berlin, 2002).

31. S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, *J. Math. Anal. Appl.* **310** (2005), 229–246.

32. M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems* (Springer, Heidelberg, 1996).

33. M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.* **18** (1969), 3–24.

34. L. Ven'–tuan, On embedding theorems for spaces of functions with partial derivatives of various degree of summability, *Vestn. Leningr. Univ.* **16** (1961), 23–37.

35. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **29** (1987), 33–66.