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PRINCIPAL RINGS WITH THE DUAL OF THE ISOMORPHISM THEOREM

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Abstract. A ring R satisfies the dual of the isomorphism theorem if $R/Ra \cong \mathtt{l}(a)$ for every element $a \in R$. We call these rings left morphic, and say that R is left P-morphic if, in addition, every left ideal is principal. In this paper we characterize the left and right P-morphic rings and show that they form a Morita invariant class. We also characterize the semiperfect left P-morphic rings as the finite direct products of matrix rings over left uniserial rings of finite composition length. J. Clark has an example of a commutative, uniserial ring with exactly one non-principal ideal. We show that Clark's example is (left) morphic and obtain a non-commutative analogue.

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1. Introduction. If R is a ring, the isomorphism theorem asserts that $R/\mathtt{l}(a) \cong Ra$ for every element $a \in R$ (where $\mathtt{l}(a)$ denotes the left annihilator). Dually, R is called left *morphic* if it satisfies the following equivalent conditions [**6**, Lemma 1].

- *$R/Ra \cong \mathtt{l}(a)$ for every $a \in R$;*
- *For every $a \in R$ there exists $b \in R$ such that $Ra = \mathtt{l}(b)$ and $\mathtt{l}(a) = Rb$.*

Every unit regular ring is left and right morphic and $\mathbb{Z}_4$ is an example of a non-regular morphic ring. We call a ring R left *special* if it satisfies the following equivalent conditions [**6**, Theorem 9].

- *R is left uniserial of finite length*;
- *R is left morphic, local and the Jacobson radical J is nilpotent*;
- *The left ideal lattice is $R \supset J \supset J^2 \supset \cdots \supset J^m = 0$*;
- *R is local and $J = Rc$ where $c \in R$ is nilpotent.*

In this case $J^n = Rc^n$ for each $n \geq 0$. The ring $\mathbb{Z}_{p^n}$ of integers modulo p^n is a commutative special ring for every prime p, and other examples appear below.

The left morphic rings were studied in [**6**] where, among other things, the left artinian, left and right morphic rings were characterized as the finite products of matrix rings over left and right special rings. Call a ring R left *principally morphic* (left *P-morphic*) if it is left morphic and every left ideal is principal. The local, left P-morphic rings are just the left special rings, but matrix rings over left P-morphic rings need not be left morphic. Nonetheless, we prove the following results.

THEOREM. (*Theorem* 13) *Let R be a left P-morphic ring. Then R is semiperfect if and only if it is a finite product of matrix rings over left special rings (and so is left artinian).*

THEOREM. (*Theorem* 15) *A ring is left and right P-morphic if and only if it is a finite product of matrix rings over left and right special rings.*

Thus matrix rings over left and right special rings are left and right principal ideal rings.

In the second half of the paper we turn our attention to the local, left morphic rings in which J is not assumed to be nilpotent, and prove the following theorem.

THEOREM. (*Theorem* 18) *A ring R with left ideal lattice*

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset V \subset \cdots \subset U_2 \subset U_1 \subset U_0 = R$$

is left morphic if and only if the left and right socles of R are equal.

Such a ring is 'almost' left P-morphic in the sense that V is the only non-principal left ideal. The example of Clark [**2**] is a commutative ring of this type.

Throughout this paper every ring R is associative with unity and all modules are unitary. If M is an R-module we write $J(M)$, $soc(M)$ and $Z(M)$ for the Jacobson radical, the socle, and the singular submodule of M, respectively. The length of a module means the composition length, and we write $N \subseteq^{ess} M$ if N is an essential submodule of M. We abbreviate $J(R) = J$, $soc({}_RR) = S_l$ and $soc(R_R) = S_r$, and we write $U = U(R)$ for the group of units of R. The left and right annihilators of a subset $X \subseteq R$ are denoted by $\mathtt{l}(X)$ and $\mathtt{r}(X)$ respectively. We write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n. See Lam [**5**] for other ring-theoretic notions.

2. Left principally morphic rings. Recall that a ring R is called left morphic if $R/Ra \cong \mathtt{l}(a)$ for every $a \in R$. We call a ring R left *P-morphic* if it is left morphic and every left ideal is principal. The ring $\mathbb{Z}_{p^n}$ of integers modulo p^n is left P-morphic for each prime p and each $n \geq 0$. More generally [**6**, Theorem 9] gives

EXAMPLE 1. Every left special ring R is left P-morphic.

EXAMPLE 2. Every semisimple artinian ring is left and right P-morphic.

Proof. This is because matrix rings over division rings are left and right morphic [**6**, Theorem 17], products of left (or right) morphic rings are left (or right) morphic [**6**, Example 2], and direct summand left (or right) ideals in a semisimple ring are principal. □

EXAMPLE 3. If $R_1, \ldots, R_n$ are rings then $R_1 \times R_2 \times \cdots \times R_n$ is left P-morphic if and only if each R_i is left P-morphic.

Proof. This follows because products of left (or right) morphic rings are left (or right) morphic [**6**, Example 2]; and if R is left (or right) morphic, the same is true for eRe for every idempotent $e^2 = e \in R$ [**6**, Theorem 15]. □

We are going to give some characterizations of left P-morphic rings; the following new description of the left morphic rings will be needed and has independent interest.

LEMMA 4. *The following are equivalent for a ring R.*
(1) *R is left morphic.*
(2) *If $L \subseteq R$ is a left ideal and $R/L \cong Ra$, $a \in R$, then $R/Ra \cong L$.*

Proof. (1)$\Rightarrow$(2). Let $\sigma : R/L \to Ra$ be an isomorphism. If $b = (1 + L)\sigma$ then $Rb = Ra$ and $\mathtt{l}(b) = L$. By (1), there exists $c \in R$ such that $Rc = \mathtt{l}(b)$ and $\mathtt{l}(c) = Rb$. Then $R/Ra = R/\mathtt{l}(c) \cong Rc = L$, proving (2).

(2)$\Rightarrow$(1). If $a \in R$ we have $R/\mathtt{l}(a) \cong Ra$, so $R/Ra \cong \mathtt{l}(a)$ by (2). $\square$

LEMMA 5. *The following are equivalent for a ring R.*
(1) *R is left P-morphic.*
(2) *R is left morphic and every principal left R-module embeds in ${}_RR$.*
(3) *R is left morphic and, if $L \subseteq R$ is a left ideal, then $L = ker(\alpha)$ for some $\alpha : {}_RR \to {}_RR$.*

Proof. (1)$\Rightarrow$(2). If ${}_RM$ is principal, then $M \cong R/Ra$ for $a \in R$ by (1), and $R/Ra \cong \mathtt{l}(a)$ by Lemma 4. But there exists $b \in R$ such that $\mathtt{l}(a) = Rb$, again by (1), so ${}_RM$ embeds in ${}_RR$.

(2)$\Rightarrow$(3). If $\sigma : R/L \to {}_RR$ is monic, take $\alpha = \theta\sigma$ where $\theta : R \to R/L$ is the coset map.

(3)$\Rightarrow$(1). If $L \subseteq R$ is a left ideal let $L = ker(\alpha)$ for some $\alpha : {}_RR \to {}_RR$. If $a = 1\alpha$, then $L = \mathtt{l}(a)$ so $L = Rb$ for some $b \in R$ because R is left morphic. $\square$

Since simple left modules are principal, we have

COROLLARY 6. *If a ring R is left P-morphic then every simple left R-module embeds in ${}_RR$. In particular, the left socle S_l is nonzero.*

Every left special ring is left P-morphic but the following example [**6**, Example 8] displays a left special ring that is not right P-morphic. It will be referred to several times.

EXAMPLE 7. Let F be a field with an isomorphism $x \mapsto \bar{x}$ from F to a subfield $\bar{F} \neq F$. Let R denote the left F-space on basis $\{1, c\}$ where $c^2 = 0$ and $cx = \bar{x}c$ for all $x \in F$. Then R is a left special ring that is not right morphic, and $M_2(R)$ is not left morphic.

Proof. The only left ideals of R are 0, $J = Rc = Fc$ and R. If $y \in F - \bar{F}$ and $a = yc$, there is no $b \in R$ such that $aR = \mathtt{r}(b)$ and $\mathtt{r}(a) = bR$. The last observation is [**6**, Example 16]. $\square$

EXAMPLE 8. If p is a prime, then $M_n(\mathbb{Z}_{p^k})$ is left and right P-morphic for all $n \geq 1$ and $k \geq 1$.

Proof. Write $R = \mathbb{Z}_{p^k}$. Then R is left and right special, so $M_n(R)$ is left and right morphic by [**6**, Theorem 17]. But every submodule of $M = R^n$ can be generated by n or fewer elements (by the fundamental theorem of finite abelian groups) so every left or right ideal of $M_n(R)$ is principal (for example, see [**4**, Chapter 2, Corollary 1.4]). $\square$

In Theorem 15 below, we will show more generally that $M_n(R)$ is a left and right P-morphic ring if R is *any* left and right special ring.

If R is left morphic and $e^2 = e \in R$, the corner ring eRe is again left morphic by [**6**, Theorem 15]. However, if R is left P-morphic we do not know whether eRe must be left P-morphic, even if we assume that $ReR = R$.

THEOREM 9. *Let L and L' be left ideals of the left P-morphic ring R. Then $R/L \cong R/L'$ if and only if $L \cong L'$.*

Proof. If $R/L \cong R/L'$ then, by Lemma 5, there exists $a \in R$ such that $R/L \cong Ra$. Hence $R/L' \cong Ra$ too, so $L \cong R/Ra \cong L'$, by Lemma 4.

Conversely, if $L \cong L'$ then Lemma 5 provides a and a' in R such that $R/L \cong Ra$ and $R/L' \cong Ra'$. It follows by Lemma 4 that $R/Ra \cong L \cong L' \cong R/Ra'$. Hence $Ra \cong Ra'$ by the first half of this proof. But then $R/L \cong R/L'$, as required. □

We do not know whether a left principal ideal ring R is left morphic if it satisfies the condition in Theorem 9.

Before proceeding we prove two technical lemmas about local rings that will be needed in the next theorem and again in Section 3.

LEMMA 10. *Let R denote a local ring in which $J = Rc$, $c \in R$, and J is not nilpotent. Write $V = \cap_{n\geq 0} J^n$.*

(1) *$J^m = Rc^m$ for every $m \geq 0$.*

(2) *$Rc^m - Rc^{m+1} = Uc^m$ for every $m \geq 0$.*

(3) *If $L \nsubseteq V$ is a left ideal then $L = J^m$ for some $m \geq 0$.*

(4) *$\mathtt{l}(c^t) \subseteq V$ for every $t \geq 0$.*

Proof. (1). If $J^m = Rc^m$ then $J^{m+1} = J(Rc^m) = Jc^m = (Rc)c^m = Rc^{m+1}$.

(2). We have $Rc^{m+1} \subset Rc^m$ by (1) because $c \in J$ and J is not nilpotent. Let x be an element of $Rc^m - Rc^{m+1}$, say $x = uc^m$, $u \in R$. Then $u \notin Rc = J$ because $x \notin Rc^{m+1}$, so u is a unit because R is local. Hence $x \in Uc^m$. Conversely, if $x = uc^m$, $u \in U$, then $x \notin Rc^{m+1}$ because otherwise we would have $c^m = u^{-1}x \in Rc^{m+1}$.

(3). Assume $L \nsubseteq V$. Since $L \subseteq Rc^0 = R$ and $L \nsubseteq Rc^n$ for some $n \geq 0$, there exists $m \geq 0$ such that $L \subseteq Rc^m$ and $L \nsubseteq Rc^{m+1}$. If $x \in L - Rc^{m+1}$ then (2) gives $x = uc^m$, $u \in U$, so $c^m = u^{-1}x \in L$. Hence $L = Rc^m$.

(4). If $\mathtt{l}(c^t) \nsubseteq V$ then $\mathtt{l}(c^t) = Rc^m$ for some m by (3). This means $Rc^{t+m} = 0$, that is $J^{t+m} = 0$, contrary to hypothesis. □

LEMMA 11. *Let R be a local left morphic ring with a simple left ideal, in which J is not nilpotent. If $Rk \subseteq R$ is simple choose $c \in R$ such that $Rc = \mathtt{l}(k)$ and $\mathtt{l}(c) = Rk$. Then $\mathtt{l}(c^t) \subset \mathtt{l}(c^{t+1})$ for every $t \geq 0$.*

Proof. Such an element c exists because R is left morphic. We have $R/Rc = R/\mathtt{l}(k) \cong Rk$ so Rc is maximal. As R is local, $J = Rc$. Now $\mathtt{l}(c^t) \subset \mathtt{l}(c^{t+1})$ is clear if $t = 0$ because $\mathtt{l}(c) = Rk \neq 0$. So assume that $t \geq 1$ and choose $v \in R$ such that $Rv = \mathtt{l}(c^t)$ and $Rc^t = \mathtt{l}(v)$. Note that $v \in V = \cap_{n\geq 0} J^n$ by (4) of Lemma 10. We have $vc^t = 0$ so, since $vc^0 = v \neq 0$, let $vc^m \neq 0$ but $vc^{m+1} = 0$ where $0 \leq m < t$. But $v \in V \subseteq J^{t-m} = Rc^{t-m}$, say $v = rc^{t-m}$. Then $rc^t = vc^m \neq 0$, but $rc^{t+1} = vc^{m+1} = 0$. Hence $r \in \mathtt{l}(c^{t+1}) - \mathtt{l}(c^t)$, as required. □

With this we can characterize the local, left P-morphic rings.

THEOREM 12. *The following are equivalent for a ring R.*

(1) *R is local and left P-morphic.*

(2) *R is local, left morphic, with a simple left ideal and ACC on left annihilators.*

(3) *R is left special.*

Proof. (1) ⇒ (2). The ACC holds because left morphic rings are left noetherian. As R is a left principal ideal ring, let $J = Rc$, $c \in R$. Since R is left morphic there

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exists $k \in R$ such that $Rc = \mathtt{l}(k)$ and $Rk = \mathtt{l}(c)$. Since $R/\mathtt{l}(c) \cong Rc$, Lemma 4 gives $Rk = \mathtt{l}(c) \cong R/Rc = R/J$. Since R is local, this shows that Rk is simple.

(2) ⇒ (3). Let Rk be simple, $k \in R$, and choose $c \in R$ such that $Rc = \mathtt{l}(k)$ and $Rk = \mathtt{l}(c)$. Then $R/Rc = R/\mathtt{l}(k) \cong Rk$, so Rc is maximal. As R is local we have $Rc = J$ so, by Lemma 11, if J is not nilpotent then $\mathtt{l}(c) \subset \mathtt{l}(c^2) \subset \mathtt{l}(c^3) \subset \cdots$, contrary to (2). Hence J is nilpotent and (3) follows by [**6**, Theorem 9].

(3) ⇒ (1). This is clear by the definition of left special rings. □

Recall that a ring R is called an *exchange ring* if ${}_RR$ (equivalently R_R) has the finite exchange property (see [**8**]). This is a large class of rings, containing every *semiregular* ring R (that is, R/J is regular and idempotents can be lifted modulo J). However, we have

THEOREM 13. *The following conditions are equivalent for a left P-morphic ring R.*
(1) *R is an exchange ring.*
(2) *R is a semiperfect ring.*
(3) *$R \cong \Pi_{i=1}^{k} M_{n_i}(S_i)$ where each S_i is left special.*
(4) *R is left artinian.*

Proof. (3) ⇒ (4) because left special rings are left artinian; (4) ⇒ (1) is well known, and (1) ⇒ (2) by [**1**, Corollary 2] because R is left noetherian. So we prove (2) ⇒ (3).

Given (2), [**6**, Theorem 29] shows that $R \cong \Pi_{i=1}^{k} M_{n_i}(S_i)$ where, for each i, $M_{n_i}(S_i)$ is left morphic and $S_i \cong e_iRe_i$ for a local idempotent $e_i \in R$. Hence each $M_{n_i}(S_i)$ is left P-morphic, so (3) follows from the

Claim. If $R = M_n(S)$ is left P-morphic and S is local then S is left special.

Proof. We may assume that $S = eRe$ where $e^2 = e \in R$ satisfies $ReR = R$. Then S is local, left morphic (by [**6**, Theorem 15]), and left noetherian so, by Theorem 12, it suffices to show that $soc({}_SS) \neq 0$. We have $JS_l = 0$, so $eS_le \subseteq \mathtt{r}_S(eJe) = soc({}_SS)$ (since S is semilocal). But $eS_le \neq 0$ because $R(eS_le)R = Re(RS_lR)eR = RS_lR = S_l \neq 0$ by Corollary 6. □

Note that the ring R in Example 7 is left artinian and left P-morphic but $M_2(R)$ is not left morphic by [**6**, Example 16]. Hence the rings identified in Theorem 13 do not form a Morita invariant class. However, being left and right P-morphic does turn out to be a Morita invariant property, and we now determine the structure of these rings. The following result will be needed and is of interest in itself.

THEOREM 14. *Let R be a left and right special ring. If $0 \neq {}_RM \subseteq R^n$ then M is a direct sum of at most n principal submodules.*

Proof. Suppose first that $M = C_1 \oplus \cdots \oplus C_r$ where each ${}_RC_i$ is principal. Since ${}_RR$ is uniserial, it follows that $soc({}_RC_i)$ is simple for each i, so $soc(M) = soc(C_1) \oplus \cdots \oplus soc(C_r)$ has composition length r. Since $M \subseteq R^n$ and $soc(R^n)$ has length n, we have $r \leq n$, as asserted.

Hence it suffices to show that M is a finite direct sum of principal submodules.[1] The proof proceeds by induction on the composition length of M.

Claim. We may assume that M is faithful.

[1]A more general result is proved in Facchini's book [**3**, Theorem 5.6]: If R is artinian and serial then every module is a direct sum of cyclic uniserial modules, and any two such direct sum decompositions are isomorphic. However we give a short, direct proof in this special case.

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Proof. Suppose $\mathtt{l}(M) = J^t$ and write $\bar{R} = R/J^t$. Then $\bar{R}$ is left and right special and M is a faithful left $\bar{R}$-module via $(r + J^t)m = rm$ for all $r \in R$ and $m \in M$. By hypothesis ${}_{\bar{R}}M = {}_{\bar{R}}C_1 \oplus \cdots \oplus {}_{\bar{R}}C_r$ where each ${}_{\bar{R}}C_i$ is principal, so ${}_RM = {}_RC_1 \oplus \cdots \oplus {}_RC_r$ where each ${}_RC_i$ is principal. This proves the Claim.

If ${}_RM$ is faithful let $S_lx \neq 0$ where $x \in M$. Then the map $r \mapsto rx$ is an epimorphism $R \to Rx$ with kernel $\mathtt{l}(x)$, and $\mathtt{l}(x) = 0$ because $S_l \nsubseteq \mathtt{l}(x)$. Hence $Rx \cong {}_RR$ is injective, and so $M = Rx \oplus N$ for some ${}_RN \subseteq M$. By induction N is a finite direct sum of principal submodules (or $N = 0$), so the same is true of M, and the proof is complete. □

It is interesting to note that the proof of Theorem 14 goes through for rings for which every image is a left selfinjective ring with finite left composition length, and every principal left module has a simple socle. Note further that the ring in Example 7 is left special but not left selfinjective (not even left P-injective). Here R is called left *P-injective* if every map $Ra \to {}_RR$, $a \in R$, is right multiplication by an element of R. Moreover, by [**6**, Proposition 27], a left morphic ring R is left selfinjective if and only if it is left P-injective, if and only if it is right morphic.

We can now give our main structure theorem for left and right P-morphic rings.

THEOREM 15. *A ring R is left and right P-morphic if and only if $R \cong M_{n_1}(R_1) \times \cdots \times M_{n_k}(R_k)$ where each R_i is left and right special.*

Proof. If R is left and right P-morphic, then it is left and right noetherian and so is left and right artinian by [**6**, Theorem 31]. It follows by [**6**, Theorem 29] that $R \cong M_{n_1}(R_1) \times \cdots \times M_{n_k}(R_k)$ where each R_i is local, left and right morphic, and left and right artinian, and hence right and left special by [**6**, Theorem 9].

Conversely, if R is left and right special then, for each $n \geq 1$, every submodule of R^n is n-generated by Theorem 14, and hence every left ideal of $M_n(R)$ is principal by [**4**, Chapter 2, Corollary 1.4]. Since $M_n(R)$ is left and right morphic by [**6**, Proposition 17], it follows that it is left and right P-morphic. Now the result follows by Corollary 6. □

Remarkably, Theorem 15 isolates the same class of rings as in Theorem 35 of [**6**], so we obtain several other characterizations of the left and right P-morphic rings.

COROLLARY 16. *The following conditions are equivalent for a ring R.*

(1) *R is left and right P-morphic.*

(2) *R is left artinian and left and right morphic.*

(3) *R is semiprimary and left and right morphic.*

(4) *R is left perfect and left and right morphic.*

(5) *R is a semiperfect, left and right morphic ring in which J is nil and $S_r \subseteq^{ess} R_R$.*

(6) *R is a semiperfect, left and right morphic ring with ACC on principal left ideals in which $S_r \subseteq^{ess} R_R$.*

(7) *R is a finite direct product of matrix rings over left and right special rings.*

Thus, for example, a left (or right) perfect, left and right morphic ring is a left and right principal ideal ring. Another surprising fact is now given.

COROLLARY 17. *Being left and right P-morphic is a Morita invariant. In addition, if R is left and right P-morphic the same is true of eRe for any idempotent $e \in R$.*

Proof. Let R be left and right P-morphic. If $e^2 = e \in R$ then eRe, is left and right morphic by [**6**, Theorem 15], so eRe is left and right P-morphic by [**6**, Theorem 15] and (2) of Corollary 16. Next, let $R \cong \Pi_{i=1}^{k} M_{n_i}(R_i)$ where each R_i is left and right special. Then $M_n(R) \cong \Pi_{i=1}^{k} M_{nn_i}(R_i)$, so $M_n(R)$ is left and right P-morphic by (7) of Corollary 16. □

3. On an example of Clark. Clark [**2**] gives an example of a commutative local ring R with ideal lattice

$$0 \subset Rv_1 \subset Rv_2 \subset \cdots \subset V \subset \cdots \subset Rc^2 \subset Rc \subset R$$

which answers a question of Faith (in the negative). We are going to show that R is a morphic ring, so R is 'almost' P-morphic because V is the only non-principal ideal. In fact, we give a noncommutative version of this example. Recall that a ring is called left *duo* if every left ideal is two-sided.

THEOREM 18. *Suppose that R is a ring with left ideal lattice*

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset V \subset \cdots \subset U_2 \subset U_1 \subset U_0 = R.$$

(1) *Then* $U_m = J^m$ *and* $L_m = \mathtt{r}(J^m)$ *for each m, and V is the only non-principal left ideal. Moreover, R is a left duo ring.*
(2) *R is left morphic if and only if* $\mathtt{r}(J) = \mathtt{l}(J)$, *equivalently* $S_l = S_r$.

Proof. (1). Note first that U_m and L_m are principal left ideals for each $m \geq 0$ (for example $U_m = Rx$ for any $x \in U_m - U_{m+1}$). In particular, $U_1 = J = Rc$ for some $c \in R$. Moreover, J is not nilpotent (otherwise R would be left special by [**6**, Theorem 9], and so left artinian). Hence the conclusions of Lemma 10 all hold for R.

Now J^m/J^{m+1} is semisimple (R is semilocal) and so is simple or zero (being uniserial). But $J^m = J^{m+1}$ implies $J^m = 0$ because (Lemma 10) $J^m = Rc^m$ for each $m \geq 1$. Hence J^{m+1} is maximal in J^m for each $m \geq 0$, and it follows by induction that $U_m = J^m$ for each m.

Next we show that $L_m = \mathtt{r}(J^m)$ for each $m \geq 0$. This is clear if $m = 0$; if $m = 1$ we have $L_1 = S_l = \mathtt{r}(J)$ because R is semilocal. In general, if $L_m = \mathtt{r}(J^m)$ then L_m is a two-sided ideal. If we write $\bar{R} = R/L_m$, then the ring $\bar{R}$ satisfies the hypotheses of the theorem. Since $soc({}_RM) = \{m \mid Jm = 0\}$ for any module ${}_RM$ over a semilocal ring R, we have

$$\begin{aligned} L_{m+1}/L_m &= soc[{}_R(R/L_m)] = soc({}_{\bar{R}}\bar{R}) = \mathtt{r}_{\bar{R}}(J/L_m) = \{r + L_m \mid Jr \subseteq \mathtt{r}(J^m)\} \\ &= \mathtt{r}(J^{m+1})/L_m. \end{aligned}$$

It follows that $L_{m+1} = \mathtt{r}(J^{m+1})$, as required.

Finally, we have $\cup_m L_m \subseteq V$, and $\cup_m L_m \neq L_n$ for every n. Hence $V = \cup_m L_m$ and it follows that V is not principal (indeed, not finitely generated). Moreover this shows that R is left duo, and so completes the proof of (1).

(2). As $J = Rc$, $c \in R$, we have $J^m = Rc^m$ for all $m \geq 0$ by Lemma 10, and so $\mathtt{r}(J^m) = \mathtt{r}(c^m)$. Observe that $\mathtt{r}(c^m)$ is a principal left ideal for each $m \geq 0$ by (1), say $\mathtt{r}(c^m) = Rv_m$ where $v_m \in V$ by Lemma 10.

Claim 1. $Rc^m = \mathtt{l}(v_m)$ *for each* $m \geq 0$.

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Proof. We have $Rc^m \subseteq \mathtt{l}(v_m)$ because $v_m \in \mathtt{r}(c^m)$. Hence, by hypothesis, $\mathtt{l}(v_m) = Rc^t = J^t$ where $0 \leq t \leq m$. Hence $J^t\mathtt{r}(c^m) = J^tRv_m \subseteq J^tv_m = 0$, so $\mathtt{r}(c^m) \subseteq \mathtt{r}(J^t) = \mathtt{r}(Rc^t) = \mathtt{r}(c^t)$. It follows that $\mathtt{r}(c^m) = \mathtt{r}(c^t)$, and hence that $t = m$ by (1). This proves Claim 1.

Claim 2. $\mathtt{l}(d) = \mathtt{l}(Rd)$ *for any* $d \in R$.

Proof. If $a \in \mathtt{l}(d)$ then $aR \subseteq \mathtt{l}(d)$ because $\mathtt{l}(d)$ is a right ideal by (1). Hence $aRd = 0$, whence $a \in \mathtt{l}(Rd)$. Thus $\mathtt{l}(d) \subseteq \mathtt{l}(Rd)$; as the other inclusion is clear, this proves Claim 2.

To prove (2), assume first that R is left morphic. Since $R/Rc \cong Rv_1 = \mathtt{r}(c)$ by Claim 1, Lemma 4 gives $R/\mathtt{r}(c) \cong Rc$. Hence there exists an epimorphism $\gamma : R \to Rc$ with $ker(\gamma) = \mathtt{r}(c)$. In particular 1γ is a generator of Rc so $1\gamma = uc$, $u \in U$ (R is local). Hence $\mathtt{r}(c) = ker(\gamma) = \mathtt{l}(uc) = \mathtt{l}(c)u^{-1} = \mathtt{l}(c)$ because $\mathtt{l}(c)$ is a right ideal by (1). Using Claim 2, this means $\mathtt{r}(J) = \mathtt{l}(J)$, that is $S_l = S_r$ (because R is semilocal).

Conversely, assume that $S_l = S_r$, that is $\mathtt{r}(c) = \mathtt{l}(c)$ using Claim 2; we must prove that R is left morphic. It follows by induction that $\mathtt{r}(c^m) = \mathtt{l}(c^m)$ for each $m \geq 0$.

Claim 3. *The left ideals of* R *are pairwise non-isomorphic.*

Proof. V is the only non-principal left ideal. The fact that Rv_m has length m for each m has two consequences: (a) $Rv_n \cong Rv_m$ implies that $n = m$; and (b) Rv_m cannot be isomorphic to Rc^n for any n. Finally observe that $Rv_m = \mathtt{r}(c^m) = \mathtt{l}(c^m) = \mathtt{l}(J^m)$ for each m (using Claim 2). Hence if $Rc^m \cong Rc^n$, that is $J^m \cong J^n$, then $Rv_m = \mathtt{l}(J^m) = \mathtt{l}(J^n) = Rv_n$, and again $m = n$. This proves Claim 3.

To prove that R is left morphic it suffices by Lemma 4 to show that, if $R/L \cong Ra$ where $a \in R$ and L is a left ideal of R, then $R/Ra \cong L$. We have $R/V \ncong Ra$ for any $a \in R$ because the left submodule lattices are not isomorphic. If $L \neq V$ then either $L = \mathtt{r}(c^m)$ or $L = Rc^m$. Observe that

$$R/\mathtt{r}(c^m) = R/\mathtt{l}(c^m) \cong Rc^m \quad \text{and} \quad R/Rc^m = R/\mathtt{l}(v_m) \cong Rv_m = \mathtt{r}(c^m).$$

This, with Claim 3, shows that R is left morphic, and so proves (2). □

Since the Clark example is commutative, Theorem 18 gives

COROLLARY 19. *The Clark example is a commutative, local, morphic ring with exactly one non-principal ideal.*

As mentioned above, the left special rings R are local, left morphic rings in which $J = Rc$, $c \in R$, and J is nilpotent. These rings have left ideal lattice $R \supset J \supset J^2 \supset \cdots \supset J^m = 0$. We now investigate the situation where J is not nilpotent, and prove in Theorem 23 a type of converse of Theorem 18. Lemmas 10 and 11 require that $S_l \neq 0$. However, if R is left morphic then $S_r \subseteq S_l$ by [**6**, Theorem 24] and the proof of our theorem requires the stronger condition that $S_r \neq 0$. The following technical result gives several equivalent conditions.

LEMMA 20. *Let* R *be a local, left morphic ring with* $S_l \neq 0$. *If* Rk *is simple,* $k \in R$, *choose* $c \in R$ *such that*

$$Rk = \mathtt{l}(c) \quad \textit{and} \quad Rc = \mathtt{l}(k).$$

Then $J = Rc = Z(R_R)$, $S_l = \mathtt{r}(c)$, *and* $R/Rk \cong J$ *as left* R-*modules. Moreover, the following conditions are equivalent.*

(1) $S_r = S_l$.
(2) $S_r \neq 0$.
(3) *S_l is simple as a left R-module.*
(4) $\mathtt{r}(c) = \mathtt{l}(c)$.
(5) *Rk is a right ideal.*
(6) $J = \mathtt{r}(k)$.
(7) *kR is simple.*
(8) $J = Z({}_RR)$.

When these conditions hold, S_l is simple and essential in ${}_RR$ and $S_l = Rk = \mathtt{l}(c)$.

Proof. Since $R/Rc = R/\mathtt{l}(k) \cong Rk$ is simple, Rc is a maximal left ideal. Hence $J \subseteq Rc$, and this is equality because R is local. We have $J = Z(R_R)$ in any left morphic ring by [**6**, Theorem 24]. As R is semilocal, $S_l = \mathtt{r}(J) = \mathtt{r}(Rc) = \mathtt{r}(c)$. Finally, $R/Rk = R/\mathtt{l}(c) \cong Rc = J$.

(1) ⇒ (2). Clear as $S_l \neq 0$ by hypothesis.

(2) ⇒ (3). If $0 \neq a \in R$, choose $b \in R$ such that $Ra = \mathtt{l}(b)$ and $Rb = \mathtt{l}(a)$. Then b is not a unit, so $b \in J$ (because R is local) and it follows that $S_r = \mathtt{l}(J) \subseteq \mathtt{l}(b) = Ra$. Hence $S_r \subseteq^{ess} {}_RR$ by (2). But S_r (viewed as a left ideal) is a direct summand of S_l, so it follows that $S_r = S_l$. Thus S_l is simple and essential in ${}_RR$, and this certainly proves (3).

(3) ⇒ (4). Given (3), we have $Rk = S_l$, that is $\mathtt{l}(c) = \mathtt{r}(c)$.

(4) ⇒ (5). Given (4), $Rk = \mathtt{l}(c) = \mathtt{r}(c)$ is a right ideal.

(5) ⇒ (6). Given (5), $kJ = kRc \subseteq Rkc = 0$, so $J \subseteq \mathtt{r}(k)$. Hence $J = \mathtt{r}(k)$ because R is local.

(6) ⇒ (7). Given (6), $kR \cong R/\mathtt{r}(k) = R/J$ is simple because R is local.

(7) ⇒ (8). Clearly (7) ⇒ (2), so $S_r \subseteq^{ess} {}_RR$ by the proof of (2) ⇒ (3). But if $a \in J$ then $S_r \subseteq \mathtt{l}(J) \subseteq \mathtt{l}(a)$, whence $a \in Z({}_RR)$. Hence $J \subseteq Z({}_RR)$, and (8) follows because R is local.

(8) ⇒ (1). We have $S_r \subseteq S_l$ by [**6**, Theorem 24]. If $a \in J$ then $\mathtt{l}(a) \subseteq^{ess} {}_RR$ by (8), so $S_l \subseteq \mathtt{l}(a)$. It follows that $S_l \subseteq \mathtt{l}(J) = S_r$, proving (1).

Finally, if these conditions hold then S_l is simple and essential in ${}_RR$ by the proof of (2) ⇒ (3), so $S_l = Rk$ because Rk is simple. □

Since $S_r \subseteq S_l$ in every left morphic ring by [**6**, Theorem 24], Lemma 20 gives:

Corollary 21. *Let R be a local, left morphic ring. Then $S_r \neq 0$ if and only if $S_l = S_r \neq 0$, and in this case S_l is simple and essential in ${}_RR$.*

We do not know an example of a local, left morphic ring in which $S_r \neq S_l$.

If R is local and left morphic with $S_l \neq 0$, conditions (1)–(8) in Lemma 20 hold in the following situations.

(1) R is left duo (then Rk is a right ideal).

(2) R is left perfect (then $S_r \subseteq^{ess} {}_RR$).

(3) R has ACC on right annihilators (then $J = Z(R_R)$ is nilpotent [**5**, Theorem 7.15], and hence R is left perfect).

(4) R is left mininjective (then $S_l \subseteq S_r$ by [**7**, Theorem 1.14]). Here R is called left *mininjective* if every R-linear map $\gamma : Ra \to {}_RR$, where Ra is simple, is right multiplication by some element of R. Examples include all semiprime and all left P-injective rings.

Lemma 22. *Let R be a local, left morphic ring with $S_r \neq 0$. If Rk is simple, $k \in R$, choose $c \in R$ such that $Rk = \mathtt{l}(c)$ and $Rc = \mathtt{l}(k)$. With an eye on Lemma 10 and Clark's* *example, define*

$$V = \cap_{m\geq 0} Rc^m = \cap_{m\geq 0} J^m \quad \textit{and} \quad W = \cup_{m\geq 0} \mathtt{r}(c^m).$$

Then all the properties in Lemmas 10 and 11 hold, and in addition:

(1) $\mathtt{l}(J^m) = \mathtt{l}(c^m) = \mathtt{r}(c^m) = \mathtt{r}(J^m)$ *for all* $m \geq 0$.

(2) $W \subseteq \mathtt{l}(V)$.

(3) *For* $m \geq 0$, *choose* v_m *such that* $\mathtt{l}(c^m) = Rv_m$ *and* $\mathtt{l}(v_m) = Rc^m$. *Then* $Rv_{m+1} - Rv_m = Uv_{m+1}$.

Proof. We have $R/Rc \cong Rk$, so $J = Rc$ because R is local. Hence the hypotheses of Lemmas 10 and 11 are satisfied.

(1). We have $\mathtt{r}(c^m) = \mathtt{r}(J^m)$ by Lemma 10, and $\mathtt{l}(c) = \mathtt{r}(c)$ by Lemma 20. Now $\mathtt{l}(c^m) = \mathtt{r}(c^m)$ follows by induction on m. Clearly $\mathtt{l}(J^m) \subseteq \mathtt{l}(c^m)$. If $ac^m = 0$ then $c^m a = 0$ so $c^m ar = 0$ for all $r \in R$. But then $arc^m = 0$, and we have $aJ^m = aRc^m = 0$. Hence $\mathtt{l}(c^m) \subseteq \mathtt{l}(J^m)$.

(2). We must prove that $WV = 0$, equivalently that $\mathtt{l}(c^m)V = 0$ for every $m \geq 0$. But $V \subseteq Rc^m$ so $\mathtt{l}(c^m)V \subseteq \mathtt{l}(c^m)Rc^m \subseteq R\mathtt{l}(c^m)c^m = 0$ because $\mathtt{l}(c^m)$ is a right ideal by (1).

(3). Note that each $v_m \in V$ by Lemma 10. Let $b \in Rv_{m+1} - Rv_m$. Then $b = uv_{m+1}$, $u \in R$, and we claim that $u \in U$. For otherwise, $u \in Rc$ so, since Rc^m is a right ideal, $c^m b \in c^m Rcv_{m+1} \subseteq Rc^m cv_{m+1} \subseteq Rc^{m+1}\mathtt{l}(c^{m+1}) = 0$. Hence $b \in Rv_m$, a contradiction. □

We can now prove a 'converse' to Theorem 18.

THEOREM 23. *Let R be a local, left morphic ring in which $S_r \neq 0$ and J is not nilpotent. If R contains a unique non-principal left ideal then there exists $c \in R$ such that the lattice of left ideals of R is*

$$0 \subset \mathtt{l}(c) \subset \mathtt{l}(c^2) \subset \cdots \subset V \subset \cdots \subset Rc^2 \subset Rc \subset R.$$

In particular R is a left duo ring.

Proof. We have $J = Rc$ where $\mathtt{l}(c) = S_l = S_r$ by Lemma 20. As in Lemma 22, write $V = \cap_{m\geq 0} J^m = \cap_m Rc^m$ and $W = \cup_{m\geq 0}\mathtt{l}(c^m)$. We have $W \subseteq V$ since, otherwise, $W = Rc^k$ for some k by Lemma 10 whence $c^k \in \mathtt{l}(c^m)$ for some m, a contradiction.

Claim. V is not principal.

Proof. If $V = Ra$ then $R/V \cong \mathtt{l}(a)$ by Lemma 4. But $\mathtt{l}(a) \supseteq \mathtt{l}(V) \supseteq W$ by Lemma 22, and ${}_RW$ is not noetherian by Lemma 11. Hence $R/V \cong \mathtt{l}(a)$ is also not noetherian, a contradiction. This proves the Claim.

Now W is not left principal by Lemma 11, so $V = W$ by hypothesis. If L is a left ideal of R and $L \nsubseteq V$ then $L = Rc^m$ for some m by Lemma 10. Suppose now that $L \subset V$. By hypothesis $L = Rb$, $b \in W$, so $b \in \mathtt{l}(c^{m+1}) - \mathtt{l}(c^m)$ for some $m \geq 0$. Using the notation of (3) of Lemma 22, this means $b = uv_{m+1}$, $u \in U$, and we obtain $Rb = Rv_{m+1} = \mathtt{l}(c^{m+1})$, as required. □

REMARK. Clark points out that $V = Vc^n$ for each $n \geq 1$. In fact this holds if R is any local ring in which $J = Rc$ and J is not nilpotent. Clearly, $Vc \subseteq V$. If $v \in V$ then $v = yc$, $y \in R$. Then $y \in V$ because otherwise $y = uc^m$, $u \in U$, by Lemma 10, so $c^{m+1} = u^{-1}v \in V \subseteq Rc^{m+2}$. It follows that $c^{m+1} = 0$, so $J^{m+1} = 0$ by Lemma 10, a contradiction.

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The left module V/W is mysterious if it is not zero (using the notation of Lemma 22). Here are some facts about it.

PROPOSITION 24. *Let R be a local, left morphic ring in which J is not nilpotent and $S_r \neq 0$. Suppose that $V \neq W$ as in Lemma 22.*

(1) *If $v \in V - W$ then $W \subset \mathtt{l}(v) \subset V$ and $W \subset Rv \subset V$.*

(2) *V/W is not left artinian.*

(3) *If R is left duo then V/W is not left noetherian.*

Proof. (1). Since neither W nor V is principal and R is left morphic, it suffices to show that $W \subseteq \mathtt{l}(v) \subseteq V$ and $W \subseteq Rv \subseteq V$. Write $\mathtt{l}(v) = Rb$ where $\mathtt{l}(b) = Rv$. We have $W \subseteq \mathtt{l}(V) \subseteq \mathtt{l}(v)$ by Lemma 22. If $\mathtt{l}(v) \nsubseteq V$, we have $\mathtt{l}(v) = Rc^m$ for some $m \geq 0$ by Lemma 10 so $b = uc^m$, $u \in U$. But then, since $\mathtt{l}(c^m)$ is a right ideal by Lemma 22, $Rv = \mathtt{l}(b) = \mathtt{l}(c^m)u^{-1} = \mathtt{l}(c^m) \subseteq W$, a contradiction. Hence $W \subseteq \mathtt{l}(v) \subseteq V$.

Next, $Rv \subseteq V$ is clear, so it remains to show that $W \subseteq Rv = \mathtt{l}(b)$, that is $Wb = 0$. Since $WV = 0$ by Lemma 22, it suffices to show that $b \in V$. But $b \notin V$ means $b = uc^m$ with $u \in U$ and $m \geq 0$, so $Rv = \mathtt{l}(b) = \mathtt{l}(uc^m) = \mathtt{l}(c^m)u^{-1} = \mathtt{l}(c^m) = \mathtt{r}(c^m) \subseteq W$, contradicting the choice of v.

(2). We show that V/W is not artinian by showing that $(Rc^m v + W)/W \supset (Rc^{m+1}v + W)/W$ for each $m \geq 1$. If these are equal then $c^m v - rc^{m+1}v \in W$ for some r, whence $c^m v \in W$ (because $c \in J$). But then $c^m v \in \mathtt{r}(c^n)$ for some n, whence $v \in \mathtt{r}(c^{m+n}) \subseteq W$, a contradiction.

(3). Choose $v \in V - W$. Then $\mathtt{l}(cv) \subseteq \mathtt{l}(c^2 v) \subseteq \mathtt{l}(c^3 v) \subseteq \cdots$ so it suffices to show that $\mathtt{l}(c^m v) \subset \mathtt{l}(c^{m+1}v)$ for each $m \geq 1$. Write $\mathtt{l}(c^m v) = Ra$ where $\mathtt{l}(a) = Rc^m v$, and $\mathtt{l}(c^{m+1}v) = Rd$ where $\mathtt{l}(d) = Rc^{m+1}v$. If $\mathtt{l}(c^m v) = \mathtt{l}(c^{m+1}v)$ then $Ra = Rd$ so, since R is local, $a = ud$ where $u \in U$. But then $\mathtt{l}(a) = \mathtt{l}(ud) = \mathtt{l}(d)u^{-1} = \mathtt{l}(d)$ because R is left duo, that is $Rc^m v = Rc^{m+1}v$. Since $c \in J$, this implies that $c^m v = 0$, so $v \in \mathtt{r}(c^m) \subseteq W$, a contradiction. □

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