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A GENERAL INCLUSION THEOREM FOR l-l NÖRLUND SUMMABILITY

BY

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ABSTRACT. Nörlund methods of summability are studied as mappings from l_1 into l_1. Conditions are given for an arbitrary l-l method to include a Nörlund method. In particular necessary and sufficient conditions are given for a row finite l-l method to include a Nörlund mean.

As in [2], let p be a complex sequence, $p_0 \neq 0$, and let $P_n = \sum_{k=0}^{n} p_k$, $n = 0, 1, 2, \ldots$ Suppose P_n is eventually non-zero. If K is the least positive integer so that $P_n \neq 0$ for all $n \geq K$, define the Nörlund method of summability N_p by $N_p[n, k] = p_{n-k}/P_0$ if $0 \leq n < K$, $k \leq n$, p_{n-k}/P_n if $n \geq K$, $k \leq n$, and 0 otherwise. Let $\mathcal{N}$ denote the collection of all such Nörlund means. If we let $\hat{P}_n = P_0$ for $0 \leq n < K$ and $\hat{P}_n = P_n$ for $n \geq K$, then the N_p transform of the sequence x is given by $N_p x$, where

$$(N_p x)_n = (1/\hat{P}_n) \sum_{k=0}^{n} p_{n-k} x_k$$

for all $n \geq 0$. Throughout we write the sequence $\{\hat{P}_n\}$ as $\{P_n\}$.

Let $l \equiv l_1 = \{x \mid \sum_k \mid x_k \mid < \infty\}$. A matrix mapping A is called l-l if and only if $l \subseteq A^{-1}[l] \equiv l(A)$. In [6], Knopp and Lorentz proved that the matrix method A is l-l if and only if there exists some $M > 0$ such that

$$\sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} < M.$$

In [2] it is shown that for any $N_p \in \mathcal{N}$, N_p is l-l if and only if (i) $p \in l$, and (ii) $P_n \not\to 0$ as $n \to \infty$. Let $\mathcal{N}_l$ denote the collection of all l-l Nörlund methods.

In [2] it is shown that given $N_p, N_q \in \mathcal{N}_l$, $l(N_p) \subseteq l(N_q)$ if and only if $b \in l$, where $b(z) = p(z)/q(z) = \sum_n b_n z^n$. The main theorem of this paper gives conditions to ensure that $l(N_p) \subseteq l(A)$, A an arbitrary l-l matrix.

2. DEFINITION. The matrix A is absolutely translative for the sequence $\{x_k\}$ provided for $j = 0, 1, 2, \ldots$

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(i) there exists some $M_1>0$ such that

$$\sup_j \left\{ \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk}\, x_{k+j} \right| \right\} < M_1, \text{ and}$$

(ii) there exists some $M_2>0$ such that

$$\sup_j \left\{ \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk}\, x_{k-j} \right| \right\} < M_2,$$

where $x_i=0$ for $i<0$.
The matrix A is absolutely left translative (a.l.t) for the sequence $\{x_k\}$ provided (ii) holds and absolutely right translative for $\{x_k\}$ provided (i) holds. The matrix A is absolutely translative provided (i) and (ii) hold for each sequence in $l(A)$, with similar definitions for absolutely left and right translative.

We remark here that if A is an l-l matrix, then A is absolutely translative for each $x \in l$. The first theorem gives a class of l-l Nörlund methods that are absolutely right translative for bounded sequences in their summability fields.

THEOREM 1. *Suppose N_p has as its generating function the polynomial $p(z)=p_0+p_1z+\cdots+p_\nu z^\nu$ with $P_n \neq 0$ for all $n \geq 0$. Then N_p is absolutely right translative for each bounded sequence in $l(N_p)$.*

Proof. It suffices to show that if $x \in l(N_p)\cap m$, then the sequence $x^{(j)}=(x_j, x_{j+1}, \ldots) \in l(N_p)$ for each $j \geq 1$ and there exists some $M>0$ such that

$$\sup_j \left\{ \sum_{n=0}^{\infty} |(N_p x^{(j)})_n| \right\} < M.$$

Now for any $j \geq 1$ we have for all $n \geq \nu$,

$$\sum_{n=\nu}^{\infty} |(N_p x^{(j)})_n| \leq \sum_{n=\nu}^{\infty} |(N_p x)_n|.$$

If $0 \leq n < \nu$, then

$$(N_p x^{(j)})_n = [(N_p x)_{n+j}][P_{n+j}/P_n] - (p_{n+j}x_0+\cdots+p_{n+1}x_{j-1})/P_n.$$

Moreover, $p(z)$ being a polynomial implies $p \in l$. Hence there exists numbers $\varepsilon>0$ and $T>0$ such that $|P_n|>\varepsilon>0$ for all $n \geq 0$ and $\sum_n |p_n|<T$. Consequently, if $x \in l(N_p)\cap m$, we have independent of j,

$$\begin{aligned}\sum_{n=0}^{\nu-1} |(N_p x^{(j)})_n| &\leq \sum_{n=0}^{\nu-1} |(N_p x)_{n+j}|\, |P_{n+j}/P_n| \\ &\quad + \sum_{n=0}^{\nu-1} [(|p_{n+j}|\,|x_0|+\cdots+|p_{n+1}|\,|x_{j-1}|)/|P_n|] \\ &< (T/\varepsilon)\sum_{n=0}^{\nu-1} |(N_p x)_{n+j}| + \left[\left(\sup_k |x_k|\right)/\varepsilon\right]\left[\nu \sum_{k=0}^{\infty} |p_k|\right] < \infty\end{aligned}$$

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We remark here that x being bounded in the preceding theorem is necessary. To demonstrate this define the sequence p as follows: $p_0=1$, $p_1=-2$, $p_n=0$ for all $n\geq 2$. So that $P_0=1$, $P_n=-1$ for all $n\geq 1$. Define the sequence x by $x_n=2^n$, $n\geq 0$. Hence $x^{(j)}=(2^j, 2^{j+1},\ldots)$. It then follows that $(N_px^{(j)})_0=2^j$ and $(N_px^{(j)})_n=0$, $n\geq 1$. Thus $\sum_{n=0}^{\infty}|(N_px^{(j)})_n|=2^j$ and hence N_p is not absolutely right translative for the sequence x.

Moreover, we assert there exist l-l Nörlund means whose generating functions are not polynomials and which are absolutely right translative for bounded sequences in their summability field. In particular we have the following:

THEOREM 2. *Suppose N_s and N_q are l-l Nörlund means with $s(z)$ a polynomial and $q(z)$ not a polynomial. If $p(z)=s(z)q(z)$, then $l(N_s)\subseteq l(N_p)$. Moreover, if $l(N_q)=l$, then $l(N_p)=l(N_s)$.*

Proof. Let

$$\sum_n b_nz^n\equiv p(z)/s(z)=\sum_n q_nz^n.$$

Since $b\in l$, by [2, Theorem 2] we have $l(N_s)\subseteq l(N_p)$.

Now let $h(z)=1/q(z)\equiv\sum_n h_nz^n$. If $l(N_q)=l$, then by [2, Corollary 3], $h\in l$. Thus if $a(z)=s(z)/p(z)\equiv\sum_n a_nz^n$, by [2, Theorem 2], $l(N_p)\subseteq l(N_s)$. Thus $l(N_p)=l(N_s)$.

Consider the following example. Let $s(z)=1+z$. Then N_s is the l-l Nörlund mean defined by $s_0=s_1=1$, $s_n=0$ for $n\geq 2$. Moreover, by [2, Corollary 3], $l\subsetneqq l(N_s)$. It will also follow from Theorem 6 that $l(N_s)\subseteq m$. Now let $q(z)=\sum_n(1/2^n)z^n$. By [2, Corollary 3] it follows that $l(N_q)=l$. If we now let $p(z)=(1+z)q(z)$, by Theorem 2, $l(N_p)=l(N_s)$.

Let $x\in m\cap l(N_p)$. Then $x\in m\cap l(N_s)$, since $l(N_p)=l(N_s)$. By Theorem 1, $x^{(j)}\in l(N_s)=l(N_p)$, and N_p is absolutely right translative for bounded sequences.

THEOREM 3. *If $N_p\in\mathcal{N}_l$, then N_p is absolutely left translative for each $x\in l(N_p)$.*

Proof. Since $N_p\in\mathcal{N}_l$ there exists some $T>0$ such that $|P_{n-j}/P_n|<T$ for all $n\geq j$, independent of j. If $(x_{(j)})_n=0$, if $0\leq n\leq j-1$ and x_{n-j} if $n\geq j$, then

$$(N_px_{(j)})_n=[(N_px)_{n-j}][P_{n-j}/P_n]$$

where $(N_px_{(j)})_n=0$ for $0\leq n<j$. It now follows that N_p is a.l.t for each $x\in l(N_p)$.

The next theorem gives sufficient conditions for an arbitrary l-l method A to include an l-l Nörlund method N_p.

THEOREM 4. *Suppose N_p is an l-l Nörlund method and A is an arbitrary l-l*

summability method. Let $p(z)\equiv\sum_n p_n z^n$ *and* $1/p(z)\equiv\sum_n \beta_n z^n$. *If*

(i) *there exists some* $M>0$ *such that*

$$\sup_j \left\{\sum_{n=0}^{\infty}\left|\sum_{k=j}^{\infty} a_{nk}\beta_{k-j}\right|\right\}<M,$$

and

(ii) $$\lim_{\lambda\to\infty}\sum_{j=0}^{\infty}\left|\sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j}\right|=0,$$

then $l(N_p)\subseteq l(A)$.

Proof. Suppose $s\in l(N_p)$ and let

$$t_n=(1/P_n)\sum_{k=0}^{n} p_{n-k}s_k.$$

Then for small $|z|$,

$$\sum_{n=0}^{\infty} t_nP_nz^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}s_k\right)z^n=p(z)s(z),$$

where $s(z)\equiv\sum_{n=0}^{\infty} s_nz^n$. Then

$$s(z)=(1/p(z))\sum_{n=0}^{\infty} t_nP_nz^n$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} P_kt_k\beta_{n-k}\right)z^n,$$

and it follows that $s_n=\sum_{k=0}^{n} P_kt_k\beta_{n-k}$ for $n\geq 0$. Let

$$\sigma_n\equiv(As)_n\equiv\sum_{k=0}^{\infty} a_{nk}s_k=\sum_{k=0}^{\infty}\left[a_{nk}\left(\sum_{j=0}^{k} P_jt_j\beta_{k-j}\right)\right].$$

We now assert that

$$\sigma_n=\sum_{j=0}^{\infty}\left[t_jP_j\sum_{k=j}^{\infty} a_{nk}\beta_{k-j}\right].$$

Consider the following array:

$$\begin{array}{ll} & t_0P_0a_{n0}\beta_0+t_0P_0a_{n1}\beta_1+\cdots \\ & \quad\vdots \qquad\qquad \vdots \\ + & \quad\vdots \qquad\qquad \vdots \\ + & t_kP_ka_{nk}\beta_0+t_kP_ka_{n,k+1}\beta_1+\cdots \\ & \quad\vdots \qquad\qquad \vdots \qquad\qquad . \end{array}$$

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Then by [5, Theorem 10] it suffices to show that for any $n \geq 0$,

(a) $\sum_{j=0}^{\infty} |t_j P_j \sum_{k=j}^{\infty} a_{nk}\beta_{k-j}| < \infty$,

(b) $|\sum_{j=0}^{\infty} t_j P_j a_{nj}\beta_k| < \infty$ for each fixed k, and

(c) $\lim_{\lambda\to\infty}[\sum_{j=0}^{\infty} t_j P_j \sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j}] = 0$.

Now, since N_p is l-l, $p \in l$ which implies $\sup_j |P_j| < \infty$. The sequence s being in $l(N_p)$ implies $\{t_j\} \in l$. By (i), $|\sum_{k=j}^{\infty} a_{nk}\beta_{k-j}| < M$ for all $j \geq 0$, and some M. Then

$$\begin{aligned}\sum_{j=0}^{\infty}\left|t_j P_j \sum_{k=j}^{\infty} a_{nk}\beta_{k-j}\right| &< M \sum_{j=0}^{\infty} |t_j|\,|P_j| \\ &< M\left(\sup_j |P_j|\right)\sum_{j=0}^{\infty}|t_j| \\ &< \infty.\end{aligned}$$

Now for each fixed k,

$$\left|\sum_{j=0}^{\infty} t_j P_j a_{nj}\beta_k\right| \leq |\beta_k| \left(\sup_j \sum_{n=0}^{\infty} |a_{nj}|\right) \sum_{j=0}^{\infty} |t_j|\,|P_j|.$$

Since A is an l-l method it follows that (b) holds. Finally

$$\begin{aligned}\left|\sum_{j=0}^{\infty} t_j P_j \sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j}\right| &\leq \sum_{j=0}^{\infty} |t_j|\,|P_j| \left|\sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j}\right| \\ &< \left(\sup_j |t_j|\right)\left(\sup_j |P_j|\right) \sum_{j=0}^{\infty}\left|\sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j}\right|.\end{aligned}$$

We see that the right hand member tends to zero by appealing to (ii). This completes the proof of the assertion.

We can now write

$$\begin{aligned}\sigma_n &= \sum_{j=0}^{\infty} t_j\left(P_j \sum_{k=j}^{\infty} a_{nk}\beta_{k-j}\right) \\ &= \sum_{j=0}^{\infty} t_j e_{nj},\end{aligned}$$

where $e_{nj} = P_j \sum_{k=j}^{\infty} a_{nk}\beta_{k-j}$. Then in order to show $l(N_p) \subseteq l(A)$ it suffices to show that the matrix (e_{nj}) defines an l-l method.

Combining (i) with the fact that $\{P_n\}$ is bounded, we have there exists some $M' > 0$ such that

$$\sup_j \left\{\sum_{n=0}^{\infty}\left|P_j \sum_{k=j}^{\infty} a_{nk}\beta_{k-j}\right|\right\} < M',$$

which is

$$\sup_j \left\{\sum_{n=0}^{\infty} |e_{nj}|\right\} < M'.$$

Thus by the Knopp-Lorentz Theorem, (e_{nj}) is an l-l matrix. This completes the proof.

It is an open question as to whether (i) and/or (ii) are necessary conditions in Theorem 4. However if we now assume that A is row-finite and l-l, we have that $l(N_p) \subseteq l(A)$ if and only if A is absolutely left translative on the sequence β. That is,

THEOREM 5. *Suppose N_p is an l-l Nörlund method and A is an arbitrary row-finite l-l matrix. Let $1/p(z) = \sum_n \beta_n z^n$. Then $l(N_p) \subseteq l(A)$ if and only if there exists some $M > 0$ such that*

$$\sup_j \left\{ \sum_{n=0}^{\infty} \left| \sum_{k=j}^{m_n} a_{nk}\beta_{k-j} \right| \right\} < M,$$

where m_n is the column index of the last non-zero term in the nth row of A.

Proof. Since the summability methods N_p and A are both row-finite, following the proof of Theorem 4, we can write

$$\begin{aligned} \sigma_n &= \sum_{k=0}^{\infty} a_{nk}s_k = \sum_{k=0}^{m_n} a_{nk}s_k \\ &= \sum_{k=0}^{m_n} a_{nk}\left(\sum_{j=0}^{k} t_j P_j \beta_{k-j} \right) \\ &= \sum_{j=0}^{m_n} t_j P_j \left(\sum_{k=j}^{m_n} a_{nk}\beta_{k-j} \right). \end{aligned}$$

Hence, if $\sigma_n = \sum_{j=0}^{m_n} t_j e_{nj}$, where $e_{nj} = P_j \sum_{k=j}^{m_n} a_{nk}\beta_{k-j}$ and $l(N_p) \subseteq l(A)$, then the matrix (e_{nj}) defines and l-l summability method, and therefore there exists some $M > 0$ such that

$$\sup_j \left\{ \sum_{n=0}^{\infty} |P_j| \left| \sum_{k=j}^{m_n} a_{nk}\beta_{k-j} \right| \right\} < M.$$

Since N_p is l-l the result follows.

Conversely, if there exists such an M, then the matrix (e_{nj}) defines an l-l summability method and hence, $l(N_p) \subseteq l(A)$.

COROLLARY 1. *Suppose $N_p, N_q \in \mathcal{N}_l$, and let $1/p(z) = \sum_n \beta_n z^n$. Then $l(N_p) \subseteq l(N_q)$ if and only if $\beta \in l(N_q)$.*

Proof. This follows immediately from Theorem 3 and Theorem 5.

We remark here that if $N_p, N_q \in \mathcal{N}_l$, $l(N_q) \subseteq l(N_p)$ if and only if $h \in l(N_p)$, where $h(z) = 1/q(z) = \sum_n h_n z^n$.

The next theorem follows from the proof of Theorem 5. In it we show that under certain rather broad conditions, N_p maps only bounded sequences into l.

THEOREM 6. *Suppose N_p is an l-l method. Let $1/p(z)=\beta(z)=\sum_n \beta_n z^n$. If β is a bounded sequence, then $l(N_p)$ is contained in the space of bounded sequences.*

Proof. Let $s \in l(N_p)$. From the proof of Theorem 5, we have that for each $n \geq 0$,

$$s_n = \sum_{k=0}^{n} P_k t_k \beta_{n-k}.$$

Let $\sup_k |P_k| < T$ and $\sup_k |\beta_k| < B$. Then

$$\begin{aligned} |s_n| &\leq \sum_{k=0}^{n} |P_k t_k \beta_{n-k}| \\ &< TB \sum_{k=0}^{\infty} |t_k| \\ &< TBM \end{aligned}$$

say, where $\sum_k |t_k| < M$.

EXAMPLE. Consider the Binary method of summability: that is the l-l Nörlund method generated by the sequence p given by $p_0 = p_1 = 1$, $p_n = 0$ for all $n \geq 2$. Therefore $p(z) = 1 + z$ and

$$\beta(z) = \sum_n (-1)^n z^n \quad \text{for} \quad |z| < 1.$$

Thus β is bounded and hence by Theorem 6, $l(N_p)$ is contained in the space of bounded sequences.

3. In [2, Theorem2], it was shown that for $N_p, N_q \in \mathcal{N}_l$, $l(N_p) \subseteq l(N_q)$ if and only if the sequence $b \in l$, $b(z) = q(z)/p(z) = \sum_n b_n z^n$. The next lemma says that Theorem 2 of [2] and Corollary 1 are equivalent.

LEMMA 1. *Suppose $N_p, N_q \in \mathcal{N}_l$. Then $\beta \in l(N_q)$ if and only if $b \in l$.*

Proof. The proof is straight forward.

We remark here that if $N_p, N_q \in \mathcal{N}_l$ and $r = p * q$ (i.e., $r_n = p_0 q_n + \cdots + p_n q_0$ for $n \geq 0$), then $N_r \in \mathcal{N}_l$ (see Lemma 3 of [2]). Moreover

$$\begin{aligned} (N_r \beta)_n &= (1/\hat{R}_n) \sum_{k=0}^{n} r_{n-k} \beta_k \\ &= q_n / \hat{R}_n \end{aligned}$$

since $p(z)\beta(z) \equiv 1$. Thus $\beta \in l(N_r)$ and $l(N_p) \subseteq l(N_r)$ by Corollary 1. Similarly $l(N_q) \subseteq l(N_r)$.

Now suppose that $N_p, N_q, N_s \in \mathcal{N}_l$. Let $\nu = q * s$ and $\mu = p * s$. We need the following notation.

(i) $p(z) = \sum_n p_n z^n$, $q(z) = \sum_n q_n z^n$, $s(z) = \sum_n s_n z^n$,

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(ii) $1/p(z)=\beta(z)$, $1/s(z)=\gamma(z)$, $1/\mu(z)=\sum_n c_n z^n$, and
(iii) $\hat{V}_n=\sum_{k=0}^n \nu_n$ if $V_n\neq 0$ and $\hat{V}_n=V_0$ if $V_n=0$.

If $l(N_p)\subseteq l(N_q)$, then by Corollary 1, $\beta\in l(N_q)$. We assert that $l(N_\mu)\subseteq l(N_\nu)$. It suffices to show $c\in l(N_\nu)$. Since $1/\mu(z)=\{1/p(z)\}\{1/s(z)\}$ for small $|z|$, it implies $c=\beta*\gamma$. Also

$$\hat{V}_n(N_\nu c)_n=\sum_{k=0}^n \nu_{n-k}c_k.$$

Therefore the sequence $\{\hat{V}_n(N_\nu c)_n\}$ is given by,

$$(\beta*\gamma)*(q*s)=(\beta*q)*(\gamma*s)=\beta*q.$$

But $\beta\in l(N_q)$ and hence $c\in l(N_\nu)$. Thus $l(N_\mu)\subseteq l(N_\nu)$. By the remark immediately after Corollary 1 and a similar argument as above it follows that if $l(N_p)\subsetneqq l(N_q)$ then $l(N_\mu)\subsetneqq (N_\nu)$. We now have,

THEOREM 7 [2, Theorem 6]. *With "strictly l-weaker than" as order relation and "$*$" as the binary operation, $\mathcal{N}_l$ is an ordered abelian semigroup.*

4. This section was suggested by J. Fridy. We give a class of matrix summability methods that include certain Nörlund methods. In [3] J. Fridy introduced the following class of methods.

Let t be a sequence such that $0<t_n<1$ for all $n\geq 0$. Define $A_t=(a_{nk})$ by $a_{nk}=t_n(1-t_n)^k$. It is easy to see that A_t is an l-l method if and only if $t\in l$.

We now have,

THEOREM 8. *Suppose p is a non-negative sequence in l, $p_0>0$, and let $1/p(z)=\sum_n \beta_n z^n$. If $\limsup_k|\beta|^{1/k}\leq 1$ and $t\in l$, then $l(N_p)\subseteq l(A_t)$.*

Proof. We need to verify that the two conditions of Theorem 4 hold. First consider

$$\begin{aligned}\sum_{k=j}^{\infty} a_{nk}\beta_{k-j}&=\sum_{k=j}^{\infty} t_n(1-t_n)^k\beta_{k-j}\\&=t_n(1-t_n)^j\sum_{k=j}^{\infty}(1-t_n)^{k-j}\beta_{k-j}\\&=t_n(1-t_n)^j\sum_{i=0}^{\infty}(1-t_n)^i\beta_i\\&=\{t_n(1-t_n)^j\}\{1/p(1-t_n)\}\end{aligned}$$

since $\beta(z)=1/p(z)$ and $\limsup_k|\beta_k|^{1/k}\leq 1$. Now

$$\begin{aligned}\sum_{n=0}^{\infty}[t_n(1-t_n)]/p(1-t_n)&\leq\sum_{n=0}^{\infty}[t_n/p(1-t_n)]\\&<\infty\end{aligned}$$

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provided $p(1-t_n)$ is bounded away from zero. Since $p_n \geq 0, p_0 > 0$ we have $p(z) = \sum_{n=0}^{\infty} p_n z^n \geq p_0$ for every sequence z such that $z_n \geq 0$. Then $1/p(1-t_n) \leq 1/p_0$ for all n. Moreover

$$\sum_{n=0}^{\infty} |t_n(1-t_n)^j/p(1-t_n)| \leq \sum_{n=0}^{\infty} |t_n(1-t_n)/p(1-t_n)|$$

and thus condition (i) of Theorem 4 holds.

To verify the second condition consider

$$\begin{aligned}\sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j} &= \sum_{k=j+\lambda}^{\infty} t_n(1-t_n)^k \beta_{k-j} \\ &= t_n(1-t_n)^j \sum_{i=\lambda}^{\infty} (1-t_n)^i \beta_i.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=0}^{\infty} \left| \sum_{k=j+\lambda}^{\infty} a_{nk}\beta_{k-j} \right| &= t_n \left| \sum_{i=\lambda}^{\infty} (1-t_n)^i \beta_i \right| \sum_{j=0}^{\infty} (1-t_n)^j \\ &= \left| \sum_{i=\lambda}^{\infty} (1-t_n)^i \beta_i \right|.\end{aligned}$$

But the series $\sum_i (1-t_n)^i \beta_i$ converges and hence $\sum_{i=\lambda}^{\infty} (1-t_n)^i \beta_i \to 0$ as $\lambda \to \infty$. Thus by Theorem 4 we have $l(N_p) \subseteq l(A_t)$.

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