Introduction

In the moduli theory of curves, the main objects – *stable curves* – are projective curves C that satisfy two conditions:

- (local) the singularities are nodes, and
- (global) K_C is ample.

Generalizing this, Kollár and Shepherd-Barron (1988) posited that in higher dimensions the objects of the moduli theory – *stable varieties* – are projective varieties X such that

- (local) the singularities are semi-log-canonical, and
- (global) K_X is ample.

The theory of semi-log-canonical singularities is treated in Kollár (2013b). Once the objects of a moduli theory are established, we need to describe the families that we aim to understand. For curves, the answer is clear: flat, projective morphisms whose fibers are stable curves.

By contrast, there are too many flat, projective morphisms whose fibers are stable surfaces; basic numerical invariants are not always constant in such families. The correct notion of *(locally) stable families* of surfaces was defined in Kollár and Shepherd-Barron (1988). We describe these in all dimensions, first for one-parameter families in Chapter 2, and then over an arbitrary base in Chapter 3, where seven equivalent definitions of local stability are given in Definition–Theorem 3.1.

Stable curves with weighted points also appeared in many contexts, and, correspondingly, the general objects in higher dimensions are pairs (X, Δ), where X is a variety and $\Delta = \sum a_i D_i$ is a formal linear combination of divisors with rational or real coefficients. Such a pair (X, Δ) is *stable* iff

- (local) the singularities are semi-log-canonical, and
- (global) $K_X + \Delta$ is ample.

The main aim of this book is to complete the moduli theory of stable pairs in characteristic 0.

Defining the right notion of *(locally) stable families of pairs* turned out to be very challenging. The reason is that the divisorial part Δ is not necessarily flat over the base. Flatness was built into the foundations of algebraic geometry by Grothendieck, and many new results had to be developed.

Our solution goes back to the works of Cayley (1860, 1862), who associated a divisor in $\mathrm{Gr}(1,3)$ – the Grassmannian of lines in $\mathbb{P}^3$ – to any space curve. More generally, given any subvariety $X^d \subset \mathbb{P}^n$ and a divisor D on X, there is a Cayley hypersurface $\mathrm{Ca}(D) \subset \mathrm{Gr}(n-d, n)$. We declare a family of divisors $\{D_s : s \in S\}$ *C-flat* if the corresponding Cayley hypersurfaces $\{\mathrm{Ca}(D_s) : s \in S\}$ form a flat family. This turns out to work very well over reduced base schemes, leading to a complete moduli theory of stable families of pairs over such bases. This is done in Chapter 4. For the rest of the book, the key result is Theorem 4.76, which constructs the universal family of C-flat Mumford divisors over an arbitrary base. While C-flatness is defined using a projective embedding, it is independent of it over reduced bases, but most likely not in general.

Chapter 5 contains numerical criteria for various fiber-wise constructions to fit together into a flat family. For moduli theory the most important result is Theorem 5.1: a flat, projective morphism $f : X \to S$ is stable iff the fibers are stable and the volume of the fibers $(K^n_{X_s})$ is locally constant on S.

Chapter 6 discusses several special cases where flatness is the right notion for the divisor part of a family of stable pairs. This includes all the pairs $(X, \Delta := \sum a_i D_i)$ with $a_i > \frac{1}{2}$ for every i; see Theorem 6.29.

The technical core of the book is Chapter 7. We develop the notion of K-flatness, which is a version of C-flatness that is independent of the projective embedding; see Definition 7.1. It has surprisingly many good properties, listed in Theorems 7.3–7.5. We believe that this is the “correct” concept for moduli purposes. However, the proofs are rather nuts-and-bolts; a more conceptual approach would be very desirable.

All of these methods and results are put together in Chapter 8 to arrive at Theorem 8.1, which is the main result of the book: The notion of Kollár–Shepherd-Barron–Alexeev stability for families of stable pairs yields a good moduli theory, with projective coarse moduli spaces.

Section 8.8 discusses problems that complicate the moduli theory of pairs in positive characteristic; some of these appear quite challenging.

The remaining chapters are devoted to auxiliary results. Chapter 9 discusses hulls and husks, a generalization of quot schemes, that was developed to suit the needs of higher dimensional moduli theory. Chapter 10 collects sundry results for which we could not find good references, while Chapter 11 summarizes the key concepts and theorems of Kollár (2013b), as well as the main results of the minimal model program that we need.