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THE EXTREME POINTS OF A CLASS OF FUNCTIONS WITH POSITIVE REAL PART

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Let P_1 be the class of holomorphic functions on the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = 1$ and $\operatorname{Re} f > 0$. Let also P_n be the corresponding class on the unit disc U^n. The inequality $|a_k| \leqslant 2$ is known for the Taylor coefficients in the class P_1. In this paper, it is generalised for the class P_n. If $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$, with $\rho_1, \rho_2, \ldots, \rho_n$ nonegative integers whose greatest common divisor is equal to 1, we describe the form of the functions $f \in P_n$ under the restriction $|a_\rho| = 2$. Under the same restriction, we give conditions for a function to be an extreme point of the class P_n.

1. INTRODUCTION

Let U be the open unit disc in the complex plane C. If n is any natural number, P_n represents the class of all holomorphic functions in U^n, which have positive real part and assume the value 1 at the origin $\theta = (0, 0, 0, \ldots, 0)$. These functions can be expanded in Taylor series of the form

$$f(z) = \sum_{k \geqslant 0} a_k z^k$$

where: $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $z^k = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$, $a_\theta = 1$ and $\operatorname{Re} f > 0$.

In the case of $n = 1$, the Carathéodory-Toeplitz determinants describe the behaviour of the Taylor coefficients of class P_1. An immediate conclusion is the Carathéodory relation $|a_k| \leqslant 2$, $k = 1, 2, \ldots$ and that if $a_1 = 2e^{i\varphi}$, then $a_k = 2e^{ik\varphi}$ and $f(z) = \left(1 + e^{i\varphi}z\right)\left(1 - e^{i\varphi}z\right)^{-1}$. Moreover the functions of the above type constitute the extreme elements of the class P_1 (see [**3**]).

References [**1, 2**] deal with the problem of locating the extreme elements of the class P_2 and some of them were located.

In Section 2, by Theorem 2.3 of the present study, we achieve a generalisation of Carathéodory's conclusion for any class P_n in the case of $a_\rho = 2e^{i\varphi}$, where $\rho = (\rho_1, \rho_2, \ldots, \rho_n) \geqslant \theta$ and $\rho_1, \rho_2, \ldots, \rho_n$ are numbers prime to each other.

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The above are related to Theorem 3.

For the deduction of the results in Section 2 some simple relations from classical harmonic analysis will be used.

In Section 3, we will investigate the problem of locating the extreme points of any class P_n, by means of Theorem 2.3, and it will become possible to find some of them.

2. Some properties of class P_n

Lemma 2.1. *If $K(x) = K(x_1, x_2, \dots x_n)$ is a Lebesgue integrable function in $\mathbb{R}^n$, such that $\operatorname{Re} K \geqslant 0$, and $\widehat{K}(t) = \int_{\mathbb{R}^n} e^{-ixt} K(x)dx$ is the Fourier transform of $K(x)$, then the relation*

$$\left|\widehat{K}(t) + \overline{\widehat{K}}(-t)\right| \leqslant 2 \operatorname{Re} \widehat{K}(0)$$

holds for all $t \in \mathbb{R}^n$.

The proof is straight-forward by the definition of the Fourier transform.

Lemma 2.2. *If $K(x)$ is a function as in Lemma 2.1, such that the function $\widehat{K}$ has compact support and $K \geqslant 0$, then for every $f(z) = \sum_{\rho \geqslant \theta} a_\rho z^\rho \in P_n$ the inequaltiy*

$$\left|\sum_{\rho \geqslant \theta} a_\rho \widehat{K}(s - \rho) + \sum_{\rho \geqslant \theta} \overline{a}_\rho \overline{\widehat{K}}(-s - \rho)\right| \leqslant 2 \operatorname{Re} \sum_{\rho \geqslant \theta} a_\rho \widehat{K}(-\rho) \text{ holds for every } s \in \mathbb{R}^n.$$

Proof: If $z = re^{ix} = \left(r_1 e^{ix_1}, r_2 e^{ix_2}, \dots, r_n e^{ix_n}\right)$ and $F_r(x) = K(x) f\left(re^{ix}\right) = \left(\sum_{\rho \geqslant \theta} a_\rho r^\rho . e^{i\rho x}\right) K(x)$ then the function $F_r(x)$ satisfies the conditions of Lemma 2.1. We apply this Lemma, take the limit as $r \to (1, 1, \dots, 1)$ and the requested result is obtained.

Theorem 2.3. *If $f(z) = \sum_{k \geqslant \theta} a_k z^k \in P_n$, then:*

(a) $|a_k| \leqslant 2$, $k \in \mathbb{N}_0^n$

(b) *If a certain index $\rho = (\rho_1, \rho_2, \dots \rho_n)$ with $\rho_1, \rho_2, \dots, \rho_n$ numbers prime to each other, satisfies the relation $a_\rho = 2e^{i\varphi}$ then:*

  (i) $a_{\lambda\rho} = 2e^{i\lambda\varphi}$, for every natural number λ.

  (ii) If $\theta \leqslant k \leqslant \rho$, $k \neq \rho$, $k \neq \theta$ then $a_k = \overline{a}_{\rho-k} e^{i\varphi}$ and $a_{k+\lambda\rho} = a_k e^{i\lambda\varphi}$, for every natural number λ.

  (iii) If for some index k none of the relations $k \geqslant \rho$, $k \leqslant \rho$ is valid then $a_k = a_{k+\lambda\rho} = 0$, for every natural number λ.

Proof: (a) If we set $K(x) = \prod_{k=1}^{n} \left(\sin^2 \delta_k x_k\right)/x_k^2$, we obtain $\widehat{K}(t) = 1/\pi^n \prod \sup\left(0, 2\delta_k - |t_k|\right)$. By setting $\delta_1 = \delta_2 = \cdots = \delta_n = 1/2$ and applying Lemma 2.2 we have the requested result.

(b) Let A be a rotation transform of $\mathbb{R}^n$, such that $A\rho = (|\rho|, 0, 0\dots, 0)$. Since the inverse matrix of A is equal to its transpose, we have the relation $t.\ (A^{-1}x) = (At)x$. If we set $P(x) = K(Ax)$ and $x' = Ax$ we obtain:

$$\widehat{P}(t) = \int_{\mathbb{R}^n} K(Ax)e^{ixt}dx = \int_{\mathbb{R}^n} K(x')e^{it(A^{-1}x')}dx' = \widehat{K}(At) \text{ or}$$

$$\widehat{P}(A^{-1}t) = \widehat{K}(t) = \frac{1}{\pi^n}\prod_{k=1}^{n} \sup(0, 2\delta_k - |t_k|)$$

Since the matrix A^{-1} leaves lengths and angles invariant, the rectangular region $S = \prod_{k=1}^{n}(-2\delta_k, 2\delta_k)$ is transformed into $A^{-1}(S)$ which has the same dimensions as S.

If we set $\delta_1 = |\rho|$, then for suitably small $\delta_2, \delta_3, \dots \delta_n$ no integer indices are contained in $A^{-1}(S)$ other than $\theta, \rho, -\rho$. This is further supported by the fact that the numbers $\rho_1, \rho_2, \dots, \rho_n$ are prime to each other, so that the line segment $(-2\rho, 2\rho)$ contains no integer indices other than $\theta, \rho, -\rho$. In the above mentioned case we have:

$$\widehat{P}(\theta) = (2/\pi)^n\delta_1\delta_2\dots\delta_n,\ \widehat{P}(\rho) = \widehat{K}(|\rho|, 00\dots0)$$
$$= (2^{n-1}/\pi^n)\delta_1\delta_2\dots\delta_n = \widehat{P}(-\rho)$$

and $\widehat{P}(k) = 0$ for all the rest of the indices. Now applying Lemma 2.2 we obtain the relation:

$$|a_{k-\rho} + a_{k+\rho} + 2a_k + \overline{a}_{\rho-k}| \leqslant 2\,\mathrm{Re}\,(2 + a_\rho) \text{ for every } k \in \mathbb{N}_0^n$$

(it is understood that $a_s = 0$ when $s \not\geqslant \theta$).

If part (b) holds for the case $a_\rho = -2$ it holds generally. Indeed, if we consider the function $g(z) = f(\eta_1.z_1, z_2, \dots, z_n)$ where $a_\rho = 2\eta$ and $\eta = -\eta_1^{-\rho_1}(|\eta| = 1)$, we observe that the Taylor coefficient of order ρ of the function g is equal to -2.

Applying now (b) in this case, we obtain the required result for the function f.

The hypothesis $a_\rho = -2$ yields the relations:

$$\begin{array}{lllllllll}
a_{k-\rho} & + & a_{k+\rho} & + & 2a_k & + & \overline{a}_{-k-\rho} & = 0 & (0)\\
a_k & + & a_{k+2\rho} & + & a_{k+\rho} & & & = 0 & (1)\\
\cdot & & \cdot & & \cdot & & & & \\
\cdot & & \cdot & & \cdot & & & & \\
\cdot & & \cdot & & \cdot & & & & \\
a_{k+\lambda\rho} & + & a_{k+(\lambda+2)\rho} & + & a_{k+(\lambda+1)\rho} & & & = 0 & (\lambda+1)
\end{array}$$

Subtracting successively the above relations we have:

$$\omega = (a_{k+2\rho} - a_k) = -(a_{k+3\rho} - a_{k+\rho}) = +(a_{k+4\rho} - a_{k+2\rho}) = \dots$$

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and consequently:

$$a_{k+2\lambda\rho} = a_k + \lambda\omega, \qquad a_{k+\rho} + \lambda\omega = a_{k+(2\lambda+1)\rho}.$$

Since $|a_s| \leqslant 2$ for every index s we infer that $\omega = 0$, so that returning to equations (0) and (1) in the case of $k \leqslant \rho$, $k \neq \theta$, $k \neq \rho$ we obtain that $a_k = -\overline{a}_{\rho-k}$; furthermore, if $k \not\geqslant \rho$, $k \not\leqslant \rho$, $a_k = 0$ and for any $k \neq \theta$ it is true that $a_{k+\lambda\rho} = (-1)^\lambda a_k$. □

3. Extreme elements of class P_n

Let $S \subset N_0^n$ such that $\theta \in S$. HS represents the set of the holomorphic functions on U^n which assume the form:

$$f(z) = \sum_{n \in S} a_n z^n, \quad \text{with } a_\theta = 1.$$

By PS we denote the set $P_n \cap HS$. Let $S_\rho \subset N_0^n$ for which the following conditions hold:

- (α) $\rho = (\rho_1, \rho_2, \ldots, \rho_n) \in N_0^n$ with $\rho_1, \rho_2, \ldots, \rho_n$ numbers prime to each other;
- (β) $n \leqslant \rho$ and $n \neq \rho$ for every $n \in S_\rho$;
- (γ) $\theta \in S_\rho$;
- (δ) when $n \leqslant \rho$ and $n \neq \theta$ then exactly one of the indices n, $\rho - n$ belongs to S_ρ.

If $f \in HN_0^n$, which satisfies the propositions (i), (ii), (iii) of Theorem 2.3 for $\varphi = 0$, then, obviously,

$$f(z) = \left[p(z) + z^\rho \overline{p}\left(\frac{1}{\overline{z}}\right)\right](1 - z^\rho)^{-1} \text{ and } p \in HS_\rho.$$

The inverse is also obvious.

If $\Omega_\rho(\varphi) = \{\sum_{n \geqslant \theta} a_n z^n \in P_n : a_\rho = 2e^{i\varphi}\}$ then on the above basis we are able to define the class QS_ρ by means of the relation

$$\Omega_\rho(0) = \left\{\left[p(z) + z^\rho \overline{p}\left(\frac{1}{\overline{z}}\right)\right](1 - z^\rho)^{-1} : p \in QS_\rho\right\}.$$

If EA represents the set of the extreme points of the convex set A, it is evident that

(i) $\Omega_\rho(0) \cap EP_n = \{[p(z) + z^\rho \overline{p}(1/\overline{z})](1 - z^\rho)^{-1} : p \in EQS_\rho\}$;

(ii) $\Omega_\rho(\varphi) \cap EP_n = \{f(e^{i\varphi} z_1, z_2, \ldots, z_n) : f \in \Omega_\rho(0) \cap EP_n\}$.

The following theorem provides a useful necessary and sufficient condition for a function to belong to QS_ρ.

Theorem 3.1. *If $p \in HS_\rho$, $\rho^* = (\rho_2, \rho_3, \dots, \rho_n)$ and $\widetilde{\rho} = \rho_1 + \rho_2 + \cdots + \rho_n$ the following are equivalent:*

(i) $p \in QS_\rho$;

(ii) $\operatorname{Re} p\left(z^{-\rho^*}, z^{-\rho^*} z_2^{\widetilde{\rho}}, \dots, z^{-\rho^*} z_n^{\widetilde{\rho}}\right) \geqslant 0$ *for every* $z = (z_2, z_3, \dots, z_n) \in (\partial U)^{n-1}$.

Proof: (i) → (ii).

If $f(z) = [p(z) + z^\rho \overline{p}(1/\overline{z})](1 - z^\rho)^{-1}$, $p \in HS_\rho$ and $s = (s_2, s_3, \dots, s_n) \in (\partial U)^{n-1}$ then

$$f(z_1, s_2 z_1, \dots, s_n z_1) = \sum_{k=0}^{\widetilde{\rho}-1} \lambda_k(s)(\varepsilon_k + z_1)(\varepsilon_k - z_1)^{-1}$$

where $\{\varepsilon_k\}_{k=0}^{\widetilde{\rho}-1}$ are the solutions of the equation

$$z_1^{\rho_1}.s_2^{\rho_2} \dots s_n^{\rho_n} = 1 \text{ and } \lambda_k(s) = \operatorname{Re} p(\varepsilon_k, s_2\varepsilon_k, \dots, s_n\varepsilon_k)\widetilde{\rho}^{-1}.$$

Indeed, it is obvious that

$$f(z_1, s_1 z_1, \dots, s_n z_n) = \sum_{k=0}^{\widetilde{\rho}-1} \lambda_k(s)(\varepsilon_k + z_1)(\varepsilon_k - z_1)^{-1} + C(s)$$

where
$$\lambda_k(s) = \lim_{z_1 \to \varepsilon_k} f(z_1, s_2 z_1, \dots, s_n z_1).(\varepsilon_k - z_1)(\varepsilon_k + z_1)^{-1}$$
$$= \widetilde{\rho}^{-1} \operatorname{Re} p(\varepsilon_k, s_2\varepsilon_k, \dots, s_n\varepsilon_k).$$

Because $\sum_{k=0}^{\widetilde{\rho}-1} \widetilde{\rho}^{-1} \operatorname{Re} p(\varepsilon_k, s_2\varepsilon_k, \dots, s_n\varepsilon_k) = 1$ we have that $C(s) = 0$. If $f \in \Omega_\rho(0)$ and $f(z_1, s_2 z_1, \dots, s_n z_1) = 1 + \sum_{n=1}^{\infty} \beta_n(s) z_1^n$,

$$\overline{\beta}_{\widetilde{\rho}-n}(s) = 2\sum_{k=0}^{\widetilde{\rho}-1} \lambda_k(s)\overline{\varepsilon}_k^{\widetilde{\rho}-n} = s_2^{\rho_2} s_3^{\rho_3} \dots s_n^{\rho_n} \beta_n(s)$$

when $n = 1, 2, \dots, \widetilde{\rho} - 1$ and $\overline{\beta}_{\widetilde{\rho}}(s) = 2 s_2^{\rho_2} s_3^{\rho_3} \dots s_n^{\rho_n}$ so that $\Delta_{\widetilde{\rho}} = 0$, where

$$\Delta_{\widetilde{\rho}} = \begin{vmatrix} 2 & \beta_1(s) & \dots & \beta_\rho(s) \\ \overline{\beta}_1(s) & 2 & \dots & \overline{\beta}_{\rho-1}(s) \\ \vdots & \vdots & & \vdots \\ \overline{\beta}_{\widetilde{\rho}}(s) & \overline{\beta}_{\rho-1}(s) & \dots & 2 \end{vmatrix} = 0.$$

Since $\operatorname{Re} f > 0$ and the above Carathéodory-Toeplitz determinant $\Delta_{\widetilde{\rho}}$ is zero we have that

$$x_k(s) = \widetilde{\rho}^{-1} \operatorname{Re} p(\varepsilon_k, s_2\varepsilon_k, \ldots, s_n\varepsilon_k) \geqslant 0, \quad k = 0, 1, \ldots, \widetilde{\rho} - 1.$$

If we set $s_2 = z_2^{\widetilde{\rho}}$, $s_3 = z_3^{\widetilde{\rho}}, \ldots, s_n = z_n^{\widetilde{\rho}}$, $\varepsilon_0 = z_2^{-\rho_2} z_3^{-\rho_3} \ldots z_n^{-\rho_n} = z^{-\rho^*}$ where $z \in (\partial U)^{n-1}$ we obtain the result.

(ii) $\to$ (i).

Let $f(z) = [p(z) + z^{\rho}\overline{p}(1/\overline{z})](1 - z^{\rho})^{-1}$ and $p \in HS_{\rho}$.

If we prove that $x_k(s) \geqslant 0$, $k = 0, 1, \ldots, \widetilde{\rho} - 1$, we, essentially, have the desired conclusion, since we can apply the maximum principle for harmonic functions, for each variable separately, thus obtaining that $f \in \Omega_{\rho}(0)$.

We consider the system

$$-(\rho_2\theta_2 + \cdots + \rho_n\theta_n) = 2k\pi\widetilde{\rho}^{-1} - 2\pi\omega_1,$$
$$\widetilde{\rho}\theta_{\lambda} - (\rho_2\theta_2 + \rho_3\theta_3 + \cdots + \rho_n\theta_n) = 2k\pi\widetilde{\rho}^{-1} - 2\pi\omega_{\lambda}, \quad \lambda = 2, 3 \ldots n$$

and $k, \omega_1, \omega_2, \ldots, \omega_n$ integers.

The above system has a unique solution if $k = \sum_{\lambda=1}^{n} \rho_{\lambda}\omega_{\lambda}$. Since $\rho_1, \rho_2, \ldots \rho_n$ are numbers prime to each other, the above can be obtained by choosing suitable ω_{λ}.

Now, if we set $s_2 = z_2^{\widetilde{\rho}}$, $s_3 = z_3^{\widetilde{\rho}}, \ldots, s_n = z_n^{\widetilde{\rho}}$, $\varepsilon_0 = z_2^{-\rho_1} z_3^{-\rho_3} \ldots \overline{z}_n^{\rho_n} = z^{-\rho^*}$, then $x_0(s) \geqslant 0$ since $p \in QS_{\rho}$. If we replace each variable z_{λ} by $z_{\lambda}e^{i\theta_{\lambda}}$, where θ_{λ} are the solutions of the previous system, we have $\operatorname{Re} p(u_1, u_2, \ldots, u_n) \geqslant 0$ where

$$u_1 = \varepsilon_0 e^{-i(\rho_2\theta_2 + \rho_3\theta_3 + \cdots + \rho_n\theta_n)} = \varepsilon_0 e^{2\pi i k\widetilde{\rho}^{-1}} = \varepsilon_k \text{ and}$$
$$u_m = z_m^{\widetilde{\rho}} z^{-\rho^*} e^{i\widetilde{\rho}\theta_m - (\rho_2\theta_2 + \rho_3\theta_3 + \ldots \rho_n\theta_n)} = s_m\varepsilon_k$$

with $k = 1, 2, \ldots \widetilde{\rho} - 1$ and $m = 2, 3, \ldots, n$. □

REMARKS 3.2.

1. If $a = (\rho_1, \rho_2, \ldots, \rho_n, 0, 0, \ldots, 0) \in \mathbb{N}_0^m$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$, then $\Omega_{\rho}(0) = \Omega_a(0)$.
2. If $\rho \in \mathbb{N}_0^n$ with $\rho_1 = 1$ and $S_{\rho} = \{n \in \mathbb{N}_0^n, \theta \leqslant n \leqslant \rho, n \neq \rho, n_1 = 0\}$, $S(\rho^*) = \{k \in \mathbb{N}_0^{n-1}, k \leqslant \rho^*\}$ then it is obvious that HS_{ρ} is identical to $HS(\rho^*)$. Moreover, if $(\omega_2, \omega_3, \ldots, \omega_n) \in (\partial U)^{n-1}$ it is easy to find a $z = (z_2, z_3, \ldots, z_n) \in (\partial U)^{n-1}$ such that $\omega_{\lambda} = z^{-\rho^*} z_{\lambda}^{\widetilde{\rho}}$, $\lambda = 2, 3, \ldots, n$. Hence, for every $p \in QS_{\rho}$ the relation $\operatorname{Re} p(\varepsilon_0, \omega_2, \omega_3, \ldots, \omega_n) \geqslant 0$ holds. Since $n_1 = 0$, it is $\operatorname{Re} p(\omega_1, \omega_2, \ldots, \omega_n) \geqslant 0$ for every $(\omega_1, \omega_2, \ldots, \omega_n) \in (\partial U)^n$. Applying the maximum principle theorem for harmonic functions we obtain that $PS(\rho^*) = PS_{\rho} = QS_{\rho}$.

By means of this equality we conclude that the determination of the extreme elements of class $\Omega_\rho(0) \cap EP_n$ reduces to locating those of the class of polynomials $EPS(\rho^*)$.

Especially in the case $\rho = (n, 1)$ the problem reduces to locating the extreme elements of the class of polynomials of degree at most n, which belong to the class P_1.

3. In the case of two variables the problem of locating the extreme elements of the class $\Omega_\rho(0) \cap EP_2$ is always reduced to finding the extreme elements of a class of polynomials with one variable. Indeed, if

$$S_\rho = \{n \in N_0^2 : n_1\rho_2 - n_2\rho_1 < 0,\, n \leqslant \rho\} \cup \{\theta\} \quad \text{and}$$
$$S_1 = \{n_2\rho_1 - n_1\rho_2 : (n_1 n_2) \in S_\rho\}$$

we observe that the map $x(n_1 n_2) = n_2\rho_1 - n_1\rho_2$ is one-to-one from the set S_ρ onto S_1, since the numbers ρ_1, ρ_2 are prime to each other.

If $p(z) = \sum_{n \in S_\rho} a_n z^n$ and define $\mathcal{L}(p) = \sum_{n \in S_\rho} a_n z^{n_2\rho_1 - n_1\rho_2}$ then the transformation $\mathcal{L}$ is an isomorphism of the space HS_ρ onto the space HS_1.

By means of Theorem 3.1 we obtain $\mathcal{L}[QS_\rho] = PS_1$; hence

$$\Omega_\rho(0) \cap EP_2 = \{[\mathcal{L}^{-1}(p_1)(z) + z^\rho \mathcal{L}^{-1}(p_1)(1/\bar{z})](1 - z^\rho)^{-1} : p_1 \in EPS_1\}.$$

4. Applications

(a) If $s = (s_1, s_2, \dots, s_n) \in (\partial U)^n$ and $\underset{\sim}{s} = (x_1, x_2, \dots, x_{2n}) = (\rho_1, \rho_1, \rho_2, \rho_2, \dots, \rho_n, \rho_n) \in (\partial U)^{2n}$, we denote by $A_k(s)$ the sum of products which are formed by considering all the permutations of the components of the vector $\underset{\sim}{s}$ taken k at a time. Moreover we define $A_0(s) = 1$.

If $S_n = \{0, 1, 2, \dots, n\}$, then it is known that the class EPS_n is formed by the elements of the class PS_n which obey the relationship

$$\operatorname{Re} p(z) = K(-1)^n (s_1 s_2 \dots s_n)^{-1} z^{-n} (z - s_1)^2 \dots (z - s_n)^2$$

for all $z \in (\partial U)^n$. The number $K > 0$ is exactly determined through the relationship $p(0) = 1$ connected with $s \in (\partial U)^n$. On the basis of the above considerations we conclude the following relationship:

$$EPS_n = \{1 + 2(-1)^n A_n^{-1}(s) \sum_{k=1}^{n} A_{n-k}(s) z^k : s \in U^n\} \quad \text{(see [4])}.$$

If we take into consideration Theorem 3.1, the above relationship solves the problem of locating the elements of class $\Omega_\rho(0) \cap EP_2$, where $\rho = (n, 1)$ or $\rho = (1, n)$. Moreover, it solves the problem of finding the elements of class $\Omega_\rho(\varphi)\cap EP_2$. These elements have the form $p(e^{i\varphi}z_1, z_2)$, where $p(z_1, z_2) \in \Omega_\rho(0)\cap EP_2$. The elements of the class $\Omega(\varphi)\cap EP_2$ coincide with the elements of the class $\Omega_k(\varphi)\cap EP_\lambda$, where $k = (n, 1, 0, 0\dots 0) \in N_0^\lambda$.

For $\rho = (1, 1)$ we have

$$\Omega_\rho(0)\cap EP_2 = \{(1 + sz_1 + \bar{s}z_2 + z_1z_2)(1 - z_1z_2)^{-1} : |s| = 1\}.$$

The above result is a generalisation of the result [**2**, 3.3, p.280] (that is, the function of the form $(1 + sz_1 + \bar{s}z_2 + z_1z_2)(1 - z_1z_2)^{-1}$, $|s| = 1$, belongs to the class EP_2).

(b) In the case $\rho = (1, 1, 1)$, by means of Theorem 3.1, the problem is reduced to locating the elements of class EPS_ρ where

$$S_\rho = \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

If a polynomial belongs to the class EPS_ρ then there is a $(\omega_0\varphi_0) \in [0, 2\pi]^2$ such that

$$\operatorname{Re}\{1 + \alpha e^{i\omega_0} + \beta e^{i\varphi_0} + \gamma e^{i(\omega_0+\varphi_0)}\} = 0.$$

Without loss of generality we assume that $(\omega_0, \varphi_0) = (0, 0)$. Now, since there is a minimum at this position, we obtain the following relationships:

$$1 + \alpha + \beta + \gamma = iA,\ \alpha + \gamma = B - 1,\ \beta + \gamma = \Gamma,\ (A, B, \Gamma) \in \mathbb{R}^3.$$

The relationship $\operatorname{Re}\{1 + \alpha z_1 + \beta z_2 + \gamma z_1 z_2\} > 0$ for all $(z_1, z_2) \in U^2$ by means of the maximum principle for harmonic functions, is equivalent to the relationship $|a + \gamma e^{i\varphi}| \leqslant 1 + \operatorname{Re}\beta e^{i\varphi}$, $\varphi \in [0, 2\pi)$, so $\beta^2 = B^2 + A^2 \leqslant 1$. After some algebraic manipulations we obtain

$$\Delta_1\cos\varphi + \Delta_2\sin\varphi + \Delta_3\sin^2\varphi - \Delta_2\cos\varphi\sin\varphi - \Delta_1 \geqslant 0$$

where $\Delta_1 = 2(\Gamma^2 + A^2 + \Gamma + B)$, $\Delta_2 = -2AB$, $\Delta_3 = A^2 - B^2$.

Now, by setting $x = \tan(\varphi/2)$ $(-\infty < x < \infty)$ we take $\Delta_1^2 - 2\Delta_1\Delta_3 - 4\Delta_2^2 \geqslant 0$, $\Delta_1 \leqslant 0$ which are equivalent to $\Delta_1 \leqslant 2B^2$. Next, if $H = \{(A, B, \Gamma) \in \mathbb{R}^3 : \Gamma^2 + A^2 + \Gamma + \Gamma B \leqslant 0,\ A^2 + B^2 \leqslant 1\}$ we prove that

$$\operatorname{Re}\{1 + (iA - 1 - \Gamma)z_1 + (iA - B)z_2 + (\Gamma + B - iA)z_1z_2\} > 0$$

for all $(A, B, \Gamma) \in H$ and $(z_1, z_2) \in U^2$.

Now using a direct reversal process we can prove the converse of the previous result.

Obviously for the extreme elements which have the above form, the following holds: $f(A, B, \Gamma) \equiv \Gamma^2 + A^2 + \Gamma + \Gamma B = 0$. If $(A, B, \Gamma) \in H$ with $f(A, B, \Gamma) = 0$ and

$s = (s_1, s_2, s_3) \neq (0, 0, 0)$ the polynomial $\tau(\lambda) = f(A + \lambda s_1, B + \lambda s_2, \Gamma + \lambda s_3)$ has the form $\lambda[\lambda(s_1^2 + s_2^2 + s_1 s_2) + k]$, $k \in \mathbb{R}$.

By the fact that $\tau(\lambda)$ assumes nonpositive values which are all to the right (or all to the left) of zero we obtain the converse result. Finally, the set EPS_ρ is composed of polynomials of the form

$$p(z) = 1 + (iA - 1 - \Gamma)e^{i\omega} z_1 + (iA - B)e^{i\varphi} z_2 + (\Gamma + B - iA)e^{i(\omega+\varphi)} z_1 z_2$$

where $(\omega, \varphi) \in [0, 2\pi)^2$, $(A, B, \Gamma) \in \mathbb{R}^3$, $\Gamma^2 + A^2 + \Gamma + \Gamma B = 0$, $A^2 + B^2 \leqslant 1$.

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