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The category of uniform convergence spaces is cartesian closed

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This paper first assigns specific uniform convergence structures to the products and function spaces of pairs of uniform convergence spaces, and then establishes a bijection between corresponding sets of morphisms which yields cartesian closedness.

1. Introduction

Cartesian closed categories, being set-like, are useful in topology. Extensive bibliographies exist in Herrlich [3], Nel [5], and Wyler [6]. For a category that is not cartesian closed, embedding into one may be considered. For example: the category of uniform spaces is known to be not cartesian closed, but it can be embedded into the category U of uniform convergence spaces. This paper shows that U is cartesian closed Other properties of U will be discussed in a future article.

The definition of uniform convergence structure, given below and used throughout this paper, is due to Wyler [7], since the uniform convergence structure introduced by Cook and Fischer [1] contains the diagonal filter; thus, cannot be used if a function space with the continuous convergence structure is to be in U , as shown by Example 2.2 in Gazik, Kent, and Richardson [2].

2. Uniform convergence spaces

NOTATION. $[B]$ denotes the filter generated by the filter base B .

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Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on $X \times X$ including null filters [7]. Then $\mathcal{F}^{-1} = [\{F^{-1} \mid F \in \mathcal{F}\}]$ where $F^{-1} = \{(a, b) \mid (b, a) \in F\}$; and $\mathcal{F} \circ \mathcal{G} = [\{F \circ G \mid F \in \mathcal{F}, G \in \mathcal{G}\}]$ where

$F \circ G = \{(a, c) \mid$ there exist $b \in X$ with $(a, b) \in F$ and $(b, c) \in G\}$

represents the usual composition of relations in the reverse order. $\dot{x}$ denotes the ultrafilter generated by $\{x\}$. For every filter $\mathcal{F}$ of $U(Y, Z) \times U(Y, Z)$, and $\mathcal{G}$ of $Y \times Y$, together with the evaluation map $e : U(Y, Z) \times Y \to Z$, we define $(e \times e)(\mathcal{F} \times \mathcal{G})$ to be the filter $\mathcal{F}(\mathcal{G}) = [\{F(G) \mid F \in \mathcal{F}, G \in \mathcal{G}\}]$ where

$$F(G) = \{(fa, gb) \mid (f, g) \in F, (a, b) \in G\} .$$

Using the order notation of Kowalsky and Wyler, a filter $\mathcal{F}$ is less than or equal to a filter $\mathcal{G}$ if $\mathcal{F}$ is finer than $\mathcal{G}$. Also, $\mathcal{F} \cup \mathcal{G}$ denotes the filter $\{F \cup G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, which is the set-intersection of $\mathcal{F}$ and $\mathcal{G}$; and $\mathcal{F} \cap \mathcal{G}$ denotes the filter $\{F \cap G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$ which is generated by the set-union of $\mathcal{F}$ and $\mathcal{G}$. These symbols of filter operations carry over formal laws from set algebra to filter algebra in a natural way; see Wyler [7].

A *uniform convergence structure* $\mathcal{J}$ on a set X is a set of filters on $X \times X$ such that

(i) $\dot{x} \times \dot{x} \in \mathcal{J}$ for every $x \in X$;

(ii) if $\mathcal{F} \in \mathcal{J}$ and $\mathcal{G} \leq \mathcal{F}$, then $\mathcal{G} \in \mathcal{J}$;

(iii) if $\mathcal{F}, \mathcal{G} \in \mathcal{J}$, then $\mathcal{F} \cup \mathcal{G} \in \mathcal{J}$;

(iv) if $\mathcal{F} \in \mathcal{J}$, then $\mathcal{F}^{-1} \in \mathcal{J}$;

(v) if $\mathcal{F}, \mathcal{G} \in \mathcal{J}$, then $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}$.

A *uniform convergence space* is a pair $(X, \mathcal{J})$ of a set X and a uniform convergence structure $\mathcal{J}$ on X . A map f from a uniform convergence space $(X, \mathcal{J})$ to a uniform convergence space $(Y, \mathcal{K})$ is *uniformly continuous* iff $(f \times f)\mathcal{J} \subset \mathcal{K}$.

The collection U of all uniform convergence spaces and uniformly continuous maps form a category, the identity map for each uniform convergence space and the composition of uniformly continuous maps being uniformly continuous [1].

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Let V be the initial structure on $X \times Y$ with respect to uniformly continuous projections p and q from the product to X and Y, respectively. Then V consists of all filters on $(X \times Y) \times (X \times Y)$ generated by $(p \times p)^{-1}F \cap (q \times q)^{-1}G$ for filters $F \in J$, $G \in K$, and is known as the product uniform convergence structure [1].

LEMMA 1. *$(X \times Y, V)$ is a product of (X, J) and (Y, K) in the category U.*

Proof. By Theorems 12 and 14 in [1].

Let $U(X, Y)$ be the set of all uniformly continuous maps from (X, J) to (Y, K), and W be the uniform continuous convergence structure on $U(X, Y)$; specifically, W is the set of all filters H on $U(X, Y) \times U(X, Y)$ such that $(e \times e)(H \times F) \in K$ for every $F \in J$.

LEMMA 2. *W is a uniform convergence structure of $U(X, Y)$.*

Proof. $(e \times e)\big((\dot{f} \times \dot{f}) \times F\big) = (f \times f)(F) \in K$ for every $f \in U(X, Y)$ and $F \in J$, f being uniformly continuous. Thus $\dot{f} \times \dot{f} \in W$. Suppose $A \in W$, and $B \leq A$; then for every $F \in J$, $B(F) \leq A(F)$ in K. Hence $B \in W$. If A, B in W, then $(A \cup B)(F) = A(F) \cup B(F)$ in K for every $F \in J$, showing that $A \cup B \in W$. For every $A \in W$,

$$A^{-1}(F) = \big(A\big(F^{-1}\big)\big)^{-1} .$$

$F^{-1} \in J$, therefore $A\big(F^{-1}\big) \in K$ which implies that $\big(A\big(F^{-1}\big)\big)^{-1} \in K$. For A, B in W, $(A \circ B)(F) \leq A(F) \circ B\big(F^{-1} \circ F\big) \in K$, since $(A \circ B)(F) \subseteq A(F) \circ B\big(F^{-1} \circ F\big)$, where $A \in A$, $B \in B$, and $F \in F$. Hence $A \circ B \in W$.

LEMMA 3. *For each (Y, K) in U, the functor $- \times Y : X \mapsto X \times Y$ is a left adjoint to $U(Y, -) : Z \mapsto U(Y, Z)$, whenever $X \times Y$ and $U(Y, Z)$ have the respective structures V and W used in Lemmas 1 and 2.*

Proof. By Lemma 1, $- \times Y$ is an endofunctor of U. The adjoint situation is shown below by an equivalent condition [4], which states that for each uniform convergence space (Z, L) the corresponding evaluation map $e : U(Y, Z) \times Y \to Z$ is a universal arrow from the functor $- \times Y$ to Z. For every uniform convergence space (X, J) and $f : X \times Y \to Z$, define $f^* : X \to U(Y, Z)$ by $\big(f^*(x)\big)(y) = f(x, y)$. At the set-level,

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$f = e(f^* \times \mathrm{Id}\, Y)$, where Id Y is the identity map of Y . For every filter $F \in J$, and $G \in K$,

$$\big((f^* \times f^*)(F)\big)(G) = (e \times e)\big((f^* \times f^*)(F) \times G\big)$$
$$= (e \times e)\big(\big((f^* \times \mathrm{Id}\, Y) \times (f^* \times \mathrm{Id}\, Y)\big)(H)\big) = (f \times f)(H) ,$$

where $H = (p \times p)^{-1}F \cap (q \times q)^{-1}G$ is a filter in V . Since we give $U(Y, Z)$ the uniform continuous convergence structure, e is uniformly continuous. Hence, if f is uniformly continuous, we have $(f^* \times f^*)(F) \in W$ for every $F \in J$, which shows that f^* is uniformly continuous. On the other hand, if f^* is uniformly continuous, clearly so is f .

THEOREM. *The category* U *of uniform convergence spaces and uniformly continuous maps is cartesian closed.*

Proof. U has a terminal object, namely a singleton uniform convergence space, has products by Lemma 1, and satisfies Lemma 3. Hence U is cartesian closed.

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