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ACTIONS OF A LOCALLY COMPACT GROUP WITH ZERO

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Introduction. In [**2**] we find the definition of a locally compact group with zero as a locally compact Hausdorff topological semigroup, S, which contains a non-isolated point, 0, such that $G = S - \{0\}$ is a group. Hofmann shows in [**2**] that 0 is indeed a zero for S, G is a locally compact topological group, and the unit, 1, of G is the unit of S. We are to study actions of S and G on spaces, and the reader is referred to [**4**] for the terminology of actions.

If X is a space (all are assumed Hausdorff) and $A \subset X$, A^* denotes the closure of A. If $\{x_\rho\}$ is a net in X, we say $\lim_\rho x_\rho = \infty$ *in* X if $\{x_\rho\}$ has no subnet which converges in X.

IP Actions. We say a topological semigroup T acts *infinity preserving* (IP) on the space X provided that T acts on X such that whenever $\{x_\rho\}$ is a convergent net in X, and $\{t_\rho\}$ is a net in T such that $\lim_\rho t_\rho = \infty$ in T, then $\lim_\rho (t_\rho x_\rho) = \infty$ in X. We remark that whenever T and X are locally compact, this agrees with the definition given in [**1**].

The strength of a semigroup S acting IP on the space X is shown in

THEOREM 1. *Suppose the semigroup S acts IP on the space X. Then,*

(a) *If $\{s_\rho\}$ and $\{x_\rho\}$ are nets in S and X respectively and $x, y \in X$ such that* $\lim_\rho x_\rho = x$ *and* $\lim_\rho (s_\rho x_\rho) = y$, *then $\{s_\rho\}$ has a subnet converging in S, and there is an $s \in S$ such that $y = sx$.*

(b) *If K is a compact subset of X, SK is closed in X.*

(c) *Suppose $x \in X$ satisfies the condition that whenever $s, t \in S$ and $sx = tx$, then $s = t$. If $\{s_\rho\}$ is a net in S and $\{x_\rho\}$ is a net in X such that* $\lim_\rho x_\rho = x$ *and* $\lim_\rho (s_\rho x_\rho) = sx$ *for some $s \in S$, then* $\lim_\rho s_\rho = s$.

(d) *If $x \in X$ satisfies the condition in* (c), *then the map $\mu_x : S \to Sx$ given by $\mu_x(s) = sx$ is a homeomorphism from S onto Sx.*

The method of proving this theorem is so similar to that employed to obtain some results in [**1**] that the proof of the theorem is not included here. It is a corollary of this theorem that if the topological group G acts IP on the locally compact space X, then the orbit space X/G is a locally compact Hausdorff space. The proof of this may also be found in [**1**].

If the topological group G acts on the space X, a set $C \subset X$ is a *continuous cross-section* to the orbits of G in X if there is a $1 - 1$ continuous function

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$f : X/G \to X$ onto C such that if $Gx \in X/G$, $f(Gx) \in Gx$. It follows that if $\nu : X \to X/G$ is the natural map, then, $f \circ \nu : C \to C$ and $\nu \circ f : X/G \to X/G$ are the identities, so $f : X/G \to C$ is in fact a homeomorphism. This leads to

THEOREM 2. *Suppose the topological group G acts IP on the space X such that for all $x \in X$, $gx = hx$ implies $g = h$. If there is a continuous cross-section, C, to the orbits of G in X, then X is homeomorphic to $G \times X/G$.*

Proof. As mentioned above, there is a homeomorphism $f : X/G \to C$. Define $\psi : G \times C \to X$ by $\psi(g, c) = gc$. Since C is a cross-section and ψ is the restriction of the action of G on X to $G \times C$, it follows that ψ is a continuous function from $G \times C$ onto X. If (g_1, c_1), $(g_2, c_2) \in G \times C$ such that $g_1c_1 = \psi(g_1, c_1) = \psi(g_2, c_2) = g_2c_2$, then $Gc_1 = Gc_2$, so $c_1 = f(Gc_1) = f(Gc_2) = c_2$. Thus, $g_1c_1 = g_2c_2 = g_2c_1$ so, by hypothesis, $g_1 = g_2$. Hence, $(g_1, c_1) = (g_2, c_2)$, and ψ is $1 - 1$.

To show that ψ is a homeomorphism, let $\{(g_\rho, c_\rho)\}$ be a net in $G \times C$ such that $\lim_\rho \psi(g_\rho, c_\rho) = \lim_\rho (g_\rho c_\rho) = gc = \psi(g, c)$ for some $(g, c) \in G \times C$. If U is any neighbourhood of Gc in X/G, then $\nu^{-1}(U)$ is a neighbourhood of gc in X. Thus, there is an index γ such that for all $\rho \geqq \gamma$, $g_\rho c_\rho \in \nu^{-1}(U)$. Hence, there is an index γ such that for all $\rho \geqq \gamma$, $Gc_\rho = \nu(g_\rho c_\rho) \in U$. Since U is an arbitrary neighbourhood of Gc in X/G, $\lim_\rho Gc_\rho = Gc$ in X/G because X/G is Hausdorff, as mentioned above. Then, since $f : X/G \to C$ is continuous, $\lim_\rho c_\rho = c$.

Now, by hypothesis, $c \in C \subset X$ satisfies the condition in Theorem 1(c). Also, $\{g_\rho\}$ is a net in G and $\{c_\rho\}$ is a net in X such that $\lim_\rho c_\rho = c$ and $\lim_\rho (g_\rho c_\rho) = gc$ with $g \in G$. Thus, since G is *IP* on X, $\lim_\rho g_\rho = g$ by Theorem 1(c). Hence, $\lim_\rho (g_\rho, c_\rho) = (g, c)$, and we see that ψ is indeed a homeomorphism.

If we define $\Psi : G \times X/G \to X$ by $\Psi(g, Gx) = \psi(g, f(Gx))$ we see that Ψ is the desired homeomorphism from $G \times X/G$ onto X.

As a corollary to this theorem, we present

COROLLARY 2.1. *Let the solid* [**5**, p. 54] *Lie group G act IP on the locally compact space X such that for all $x \in X$, $gx = hx$ implies $g = h$. If X/G is normal and Lindelöf, then X is homeomorphic to $G \times X/G$.*

Proof. From Theorem I in [**1**] we know that X is a fibre bundle over X/G. The fibre over a point $Gx \in X/G$ is the orbit $Gx \subset X$. By Theorem 1(d), each fibre (orbit) is homeomorphic to G, and is therefore solid. Hence (see [**5**, p. 55]) there is a continuous cross-section to the orbits of G in X, and the Corollary follows from Theorem 2.

Actions of S. Throughout the remainder of this paper, S is to designate a locally compact group with zero, and G is to be its maximal group; $G = S - \{0\}$. Investigating the proof of Lemma 1.8 in [**2**], we see that Hofmann actually proves

LEMMA 1. *If* $\{g_\rho\}$ *is a net in* G, *then* $\lim_\rho g_\rho = 0$ *in* S *if and only if* $\lim_\rho g_\rho^{-1} = \infty$ *in* S.

Along this same vein, we present

LEMMA 2. *Let* $\{g_\rho\}$ *be a net in* G. *Then*, $\lim_\rho g_\rho = \infty$ *in* G *if and only if every subnet of* $\{g_\rho\}$ *which converges in* S *converges to* 0.

Proof. Suppose $\lim_\rho g_\rho = \infty$ in G and let $\{g_\alpha\}$ be a subnet such that $\lim_\alpha g_\alpha = s \in S$. It is immediate that $s \notin G$, so $s = 0$.

Conversely, if $\{g_\alpha\}$ were a subnet converging to $g \in G$ in G, $\lim_\alpha g_\alpha = g$ in S. But, this would be a contradiction since $g \neq 0$. Hence, $\lim_\rho g_\rho = \infty$ in G.

Whenever S acts on X, we set $F_1 = \{x : Sx = x\}$, $F_2 = \{x : Gx = x\}$, and $M = X - 0X$. Since $G \subset S$, it is clear that $F_1 \subset F_2$. In fact,

LEMMA 3. $F_1 = 0X = F_2$, *and is closed in* X.

Proof. Since 0 is not isolated in S, there is a net $\{g_\rho\}$ in G such that $\lim_\rho g_\rho = 0$. If $x \in F_2$, $g_\rho x = x$ for each ρ, so $x = \lim_\rho (g_\rho x) = 0x$, and $x \in 0X$. Hence, $F_2 \subset 0X$. If $0x \in 0X$, $s(0x) = (s0)x = 0x$ for every $s \in S$, so $S(0x) = 0x$. Hence, $0X \subset F_1 \subset F_2 \subset 0X$.

If $y \in (0X)^*$, there is a net $\{x_\rho\}$ in $0X$ such that $\lim_\rho x_\rho = y$. Then $y = \lim_\rho x_\rho = \lim_\rho (0x_\rho) = 0y$, so we conclude that $0X$ is closed.

We now present

THEOREM 3. *Suppose* S *acts on* X. *For every* $x \in X$, $(Gx)^* = Sx = Gx \cup \{0x\}$. *If* $x \in M$, Gx *is closed in* M.

Proof. It is known that $G^*x \subset (Gx)^*$, so $Sx \subset (Gx)^*$. If $y \in (Gx)^*$, there is a net $\{g_\rho\}$ in G such that $\lim_\rho (g_\rho x) = y$. If $\{g_\rho\}$ has no subnet which converges in S, then $\lim_\rho g_\rho = \infty$ in S. Thus, by Lemma 1, $\lim_\rho g_\rho^{-1} = 0$, so $x = \lim_\rho [g_\rho^{-1}(g_\rho x)] = 0y$. Hence, $x \in F_2$, and $x = \lim_\rho (g_\rho x) = y$, and $y \in Sx$. If $\{g_\rho\}$ has a subnet $\{g_\alpha\}$ converging to $s \in S$, we have $y = \lim_\alpha (g_\alpha x) = sx \in Sx$. In any case, $y \in Sx$, so $(Gx)^* \subset Sx$, and $Sx = (Gx)^*$. Since $S = G \cup \{0\}$, the remainder of the Theorem follows.

Next we prove

THEOREM 4. *If* S *acts on* X, G *acts IP on* M.

Proof. G acts on M if $gx \in M$ whenever $g \in G$ and $x \in M$. If, however, $gx = 0y \in 0X$, then $x = g^{-1}(0y) = 0y$, so $x \notin M$. Hence, $gx \in M$ whenever $g \in G$ and $x \in M$.

If G is not *IP* on M, we may find nets $\{x_\rho\}$ and $\{g_\rho\}$ in M and G respectively such that for some $x, y \in M$, $\lim_\rho x_\rho = x$, and $\lim_\rho (g_\rho x_\rho) = y$ while $\lim_\rho g_\rho = \infty$ in G. If $\{g_\alpha\}$ is a subnet of $\{g_\rho\}$ which converges in S, then $\lim_\alpha g_\alpha = 0$ by

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Lemma 2. Hence, $0x = \lim_\alpha(g_\alpha x_\alpha) = y$. This contradiction of $y \in M$ implies that $\lim_\rho g_\rho = \infty$ in S. Then, by Lemma 1, $\lim_\rho g_\rho^{-1} = 0$ in S, so

$$x = \lim_\rho[g_\rho^{-1}(g_\rho x_\rho)] = 0y.$$

However, this contradicts $x \in M$. This final contradiction leads us to conclude that G is indeed IP on M, and we are through.

From Lemma 3, if S acts on X, M is open in X. This together with the above theorem leads us, as mentioned earlier, to

COROLLARY 4.1. *If S acts on the locally compact space X, then M/G is a locally compact Hausdorff space.*

If S acts on X and $x \in X$, we set $S_x = \{s \in S : sx = x\}$ and

$$G_x = \{g \in G : gx = x\}.$$

Clearly $G_x \subset S_x$ for every $x \in X$. We see from Lemma 3 that $x \in 0X$ if and only if $S_x = S$. We further have

LEMMA 4. *$S_x = G_x$ if and only if $x \in M$.*

Proof. If $x \in M$, $0x \neq x$, so $0 \notin S_x$. It follows that $S_x \subset G_x$, so $S_x = G_x$.
If $S_x = G_x$, $0 \notin S_x$ since $0 \notin G$. Thus, $0x \neq x$. It is easy to show that this implies $x \notin 0X$, so $x \in M$, and we are finished.

Another consequence of Theorem 4 is

COROLLARY 4.2. *If $G_x = \{1\}$, then μ_x maps G homeomorphically onto Gx and S homeomorphically onto Sx.*

Proof. From Theorem 4, G is IP on M. If $G_x = \{1\}$, $x \in M$ and μ_x maps G homeomorphically onto Gx by Theorem 1(d).

To show that μ_x maps S homeomorphically onto Sx it is now sufficient to show that if $\{g_\rho\}$ is a net in G such that $\lim_\rho(g_\rho x) = 0x$, then $\lim_\rho g_\rho = 0$.

Let $\{g_\alpha\}$ be a subnet and suppose that no subnet of $\{g_\alpha\}$ converges in S. Then, $\lim_\alpha g_\alpha = \infty$ in S, so $\lim_\alpha g_\alpha^{-1} = 0$ by Lemma 1. Then,

$$x = \lim_\alpha[g_\alpha^{-1}(g_\alpha x)] = 0x.$$

This contradicts $G_x = \{1\}$, so there is a subnet $\{g_\sigma\}$ of $\{g_\alpha\}$ and an $s \in S$ such that $\lim_\sigma g_\sigma = s$. Then, $sx = \lim_\sigma(g_\sigma x) = 0x$. If $s \in G$, then $x = s^{-1}(sx) = s^{-1}(0x) = 0x$. This contradicts $x \in M$, so $s = 0$. Thus, every subnet of $\{g_\rho\}$ has a subnet converging to 0. From this it follows that $\lim_\rho g_\rho = 0$, and we are finished.

Actions of G. In [**3**], Horne constructs the space $X \cup X/P$ when P, the positive reals, acts on X. We can make a similar construction for G acting on X. The points of $\mathscr{X} = X \cup X/G$ are those of the union of the two sets. If $x \in X$, a base for the neighbourhoods of x in $\mathscr{X}$ is to consist of a base for its neighbourhoods in X. If $Gx \in X/G$, let U be a neighbourhood of x in X and V a neighbourhood of 0 in S. Setting $V' = V - \{0\}$, which is then open in G, form the set $[V'; U] = V'U \cup \nu(U)$, where $\nu : X \to X/G$ is the natural map. The collection

$$\{[V'; U] : V \text{ a neighbourhood of } 0,\ U \text{ a neighbourhood of } x\}$$

forms a base for the neighbourhoods of Gx in $\mathscr{X}$. As in [**3**], we see that X is a dense open subset of $\mathscr{X}$. Before discussing $\mathscr{X}$ any further, we present

THEOREM 6. *G acts IP on X if and only if* $\lim_\rho (g_\rho x_\rho) = \infty$ *in X whenever $\{g_\rho\}$ is a net in G such that* $\lim_\rho g_\rho = 0$ *in S and $\{x_\rho\}$ is a convergent net in X.*

Proof. If G is *IP* on X and $\{g_\rho\}$ and $\{x_\rho\}$ are nets satisfying the conditions of the Theorem, then $\lim_\rho g_\rho = \infty$ in G by Lemma 2, so $\lim_\rho (g_\rho x_\rho) = \infty$ in X.

Let us now assume the hypotheses of the converse. If G is not *IP* on X, we may find nets $\{g_\rho\}$ and $\{x_\rho\}$ in G and X such that for some $x, y \in X$, $\lim_\rho x_\rho = x$ and $\lim_\rho (g_\rho x_\rho) = y$, while $\lim_\rho g_\rho = \infty$ in G. If $\{g_\rho\}$ has a subnet $\{g_\alpha\}$ which converges in S, $\lim_\alpha g_\alpha = 0$ from Lemma 2. Thus, $\lim_\alpha (g_\alpha x_\alpha) = \infty$ in X, which contradicts $\lim_\alpha (g_\alpha x_\alpha) = y$. Hence, $\lim_\rho g_\rho = \infty$ in S. Setting $h_\rho = g_\rho^{-1}$ and $y_\rho = g_\rho x_\rho$, we see that $\lim_\rho h_\rho = 0$ in S and $\lim_\rho y_\rho = y$. Thus, by assumption, $\lim_\rho (h_\rho y_\rho) = \infty$ in X. But, $x = \lim_\rho [g_\rho^{-1}(g_\rho x_\rho)] = \lim_\rho (h_\rho y_\rho)$, so we have arrived at a contradiction. We therefore conclude that G is *IP* on X, which completes the proof.

With the aid of Theorem 6, we now have

THEOREM 7. *If G acts on the locally compact space X, G is IP on X if and only if $\mathscr{X}$ is Hausdorff.*

Proof. Suppose $\mathscr{X}$ is Hausdorff. If G is not *IP* on X, we may, by virtue of Theorem 6, find nets $\{g_\rho\}$ and $\{x_\rho\}$ in G and X such that for some $x, y \in X$, $\lim_\rho x_\rho = x$ and $\lim_\rho (g_\rho x_\rho) = y$, while $\lim_\rho g_\rho = 0$ in S. However, if this occurs, Gx and y cannot be separated in $\mathscr{X}$. For, let U be a neighbourhood of y and $[V'; W]$ a neighbourhood of Gx. We may find an index σ such that $g_\sigma \in V'$, $x_\sigma \in W$, and $g_\sigma x_\sigma \in U$. Thus, $g_\sigma x_\sigma \in U \cap [V'; W]$. This contradiction of $\mathscr{X}$ being Hausdorff implies that G is *IP* on X.

Conversely, suppose G acts *IP* on X. Since X is locally compact, X/G is Hausdorff, as mentioned previously. We may clearly separate points in X. Suppose $Gx, Gy \in X/G$ are distinct. Since X/G is Hausdorff, there are neighbourhoods U and V of x and y respectively such that $\nu(U) \cap \nu(V) = \emptyset$. It follows that $GU \cap GV = \emptyset$, and hence $[S'; U] \cap [S'; V] = \emptyset$. Thus, we may separate two points of X/G in $\mathscr{X}$.

Finally, let $x \in X$ and $Gy \in X/G$. If these cannot be separated in $\mathscr{X}$, $U \cap [V'; W] \neq \emptyset$ for all neighbourhoods U, V, and W of x, 0, and y respectively. We may then find nets $\{g_\rho\}$ in G and $\{x_\rho\}$ in X such that $\lim_\rho g_\rho = 0$, $\lim_\rho x_\rho = x$, and $\lim_\rho (g_\rho x_\rho) = y$. However, G is *IP* on X, so, by Theorem 6, we conclude that $\lim_\rho (g_\rho x_\rho) = \infty$ in X, which yields a contradiction. This implies we can separate x and Gy, which gives the conclusion.

THEOREM 8. *Suppose G acts on the locally compact space X such that $G_x = \{1\}$ for all $x \in X$ and there is an open continuous cross-section, C, to the orbits of G in X. Then $\mathscr{X}$ is Hausdorff if and only if $\mathscr{X}$ is homeomorphic to $S \times X/G$.*

Proof. Since X/G is homeomorphic to a subset, C, of X, X/G is Hausdorff. Thus, since S is Hausdorff, $S \times X/G$ is Hausdorff, so $\mathscr{X}$ must be Hausdorff if it is homeomorphic to $S \times X/G$.

Conversely, suppose $\mathscr{X}$ is Hausdorff. Then, G acts IP on X by Theorem 7, so Theorem 2 gives $\psi : G \times C \to X$ by $\psi(g, c) = gc$ as a homeomorphism. Extend ψ to $\phi : S \times C \to \mathscr{X}$ by defining $\phi(0, c) = Gc$. Since C is a cross-section, we see that ϕ is a function from $S \times C$ onto $\mathscr{X}$. Let $(g, c) \in S \times C$, $g \neq 0$, and W a basic neighbourhood of $\phi(g, c) = \psi(g, c) = gc \in X$ in $\mathscr{X}$. Then, W is open in X, so, since ψ is continuous, there is a set U open in $G \times C$ containing (g, c) such that $\psi(U) \subset W$. Since $G \times C$ is open in $S \times C$, U is open in $S \times C$, $(g, c) \in U$, and $\phi(U) = \psi(U) \subset W$. Hence, ϕ is continuous at $(g, c) \in S \times C$, $g \neq 0$. If $(0, c) \in S \times C$ and W is a basic neighbourhood of $\phi(0, c) = Gc$ in $\mathscr{X}$, there is a neighbourhood V of 0 in S and U of c in X such that $W = [V'; U] = V'U \cup \nu(U)$. Now, $U_1 = V \times (U \cap C)$ is a neighbourhood of $(0, c)$ in $S \times C$. Let $(t, k) \in U_1$. If $t \neq 0$, then $t \in V'$ and $\phi(t, k) = \psi(t, k) = tk \in V'U \subset [V'; U] = W$. If $t = 0$, then $\phi(0, k) = Gk \in \nu(U) \subset [V'; U] = W$. Hence, $\phi(U_1) \subset W$, and ϕ is continuous at $(0, c)$. Thus, ϕ is a continuous map from $S \times C$ onto $\mathscr{X}$.

Since ψ is $1-1$ and C is a cross-section, it is clear that ϕ is $1-1$. Hence, to show that ϕ is a homeomorphism, we need only show that ϕ is an open mapping. To this end, let $V \times U$ be open in $S \times C$. If $0 \notin V$, then $\phi(V \times U) = \psi(V \times U)$ is open in X, because $V \times U$ will be open in $G \times C$. Hence, since X is open in $\mathscr{X}$, $\phi(V \times U)$ is open in $\mathscr{X}$. If $0 \in V$, one sees that $\phi(V \times U) = V'U \cup \nu(U)$. Since C is open, U is open in X and since V is a neighbourhood of 0 in S, $\phi(V \times U) = V'U \cup \nu(U) = [V'; U]$ is open in $\mathscr{X}$. Thus, ϕ is an open map, and is therefore a homeomorphism.

If $f : X/G \to C$ is the homeomorphism as in Theorem 2, $\Phi : S \times X/G \to \mathscr{X}$ given by $\Phi(s, Gx) = \phi(s, f(Gx))$ gives the desired homeomorphism from $S \times X/G$ onto $\mathscr{X}$, and the proof is complete.

Conclusion. We wish to conclude the results of this paper with one which combines those of the preceding sections. It is

THEOREM 9. *Suppose S acts on the locally compact space X such that*

(i) $G_x = \{1\}$ *for each* $x \in M$,

(ii) *there is an open continuous cross-section to the orbits of G in M,*

(iii) $0X = 0M$,

(iv) *if $x, y \in M$ such that $0x = 0y$, then $Gx = Gy$,*

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(v) *if U is open in M and V is a neighbourhood of* 0 *in S, then VU is open in X.*
Then, X is homeomorphic to $S \times M/G$.

Proof. We know from Theorem 4 that G acts IP on M. We then apply Theorem 7 to conclude that $\mathscr{M}$ is Hausdorff because M, being open in X by Lemma 3, is locally compact. Conditions (i) and (ii) of the statement permit us to apply Theorem 8 to conclude that $\mathscr{M}$ is homeomorphic to $S \times M/G$. We complete the proof by showing that $\mathscr{M}$ is homeomorphic to X.

Define $T : \mathscr{M} \rightarrow X$ by: if $x \in M$, $T(x) = x$, and if $Gy \in M/G$, $T(Gy) = 0y$. Since $0x = 0y$ whenever $Gx = Gy$, we see that T is indeed a function. Furthermore, since $X = M \cup 0X$, condition (iii) shows that T maps onto X. Condition (iv) allows us to conclude that T is also $1 - 1$. Hence, T is a $1 - 1$ function from $\mathscr{M}$ onto X. We next show that T is continuous.

Let U be an open set in X. Suppose $y \in T^{-1}(U)$. Then $y = T(y) \in U$. We see that $U \cap M$ is an open set in $\mathscr{M}$ containing y and contained in $T^{-1}(U)$. Now assume $Gx \in T^{-1}(U)$. Then $0x = T(Gx) \in U$. Since S acts on X, there is an open set V about 0 in S and an open set W about x in X such that $VW \subset U$. $[V'; W]$ is a neighbourhood of Gx in $\mathscr{M}$. If $z \in [V';W]$, $T(z) = z \in V'W \subset VW \subset U$, so $z \in T^{-1}(U)$. If $Gz \in [V';W]$, there is a $w \in W$ such that $Gw = Gz$. Then, $T(Gz) = T(Gw) = 0w \in 0W \subset VW \subset U$, so $Gz \in T^{-1}(U)$. Hence, $[V';W]$ is a neighbourhood of Gx in $\mathscr{M}$ contained in $T^{-1}(U)$. Combining this with the above, we see that $T^{-1}(U)$ is open in $\mathscr{M}$, and conclude that T is continuous.

The proof of the Theorem will be complete as soon as we have shown that T is an open mapping. We do this by showing that the image of a basic open set in $\mathscr{M}$ is open in X. We must consider two types of basic open sets; those about points of M, and those about points of M/G. If U is a basic set about a point in M, $T(U) = U$ is open in M. Moreover, M is open in X, so $T(U)$ is open in X. If, on the other hand, $[V';W]$ is a basic open set about a point of M/G, one sees that $T([V';W]) = VW$. W is open in M and V is a neighbourhood of 0 in S, so VW is open in X by (v).

This concludes the proof of the Theorem.

Questions. Before terminating this paper, we would like to mention two questions which plague us. First, can the assumption in Theorem 8 that the cross-section is open be removed? If this can be answered in the affirmative, this assumption can also be removed from (ii) of Theorem 9. Secondly, which, if any, of the assumptions in Theorem 9 can be removed and still yield the same result? In particular, can (v) be removed?

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