AN EXTREMAL PROBLEM FOR POLYGONS INSCRIBED IN A CONVEX CURVE

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A. Zirakzadeh **(1)** has determined for $n = 3$ the minimal value of the perimeter length of a polygon $A_1 A_2 \ldots A_n$, where $A_1, A_2, \ldots, A_{n-1}$, and A_n divide the perimeter of a convex curve C, of perimeter length l, into n parts of equal length; further he has stated a conjecture concerning the general case. In the following a simpler proof for the case $n = 3$ is given; the minimum for even values of n, which confirms the conjecture of A. Zirakzadeh, is determined; and a fairly precise estimation for odd values of n, which refutes the conjecture of A. Zirakzadeh, is given. For $n = 3$ we have the following theorem.

THEOREM 1. *If the points P, Q, R divide the perimeter of a convex curve C, of perimeter length l, into three parts of equal length, then the perimeter length of the triangle PQR is at least $\frac{1}{2}l$. Equality holds if and only if C is an equilateral triangle and P, Q, R are the mid-points of the three sides.*

Two lemmas are needed for the proof of Theorem 1.

LEMMA 1. *Let the angles of a triangle XOY at the vertices X, Y, and O be ϕ, ψ, and 2α; let $d > 0$ be a small distance, and X' be a point on the side OX, and Y' be a point on the half-line OY beyond Y, satisfying $XX' = YY' = d$. Then $X'Y' - XY = (\cos\psi - \cos\sigma)d + o(d)$ (Figure 1).*

The very simple proof of this lemma is omitted here.

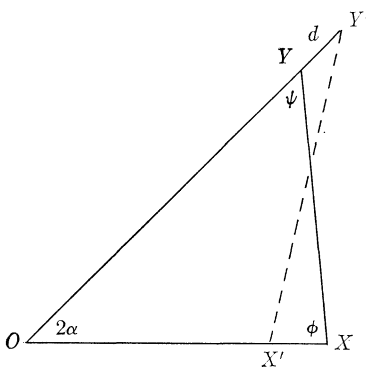

FIGURE 1

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A well-known corollary is that if $X'O + Y'O = c$ is constant, and X', Y' move on the half-lines OX and OY, then $X'Y'$ assumes its minimal value at the position $OX' = OY'$. So the minimal length of $X'Y'$ is $c \sin \alpha$.

For the sake of simplicity the following notation is introduced: $x \in [y, z]$ means that either $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$ holds.

LEMMA 2. *The points P, Q, and R are on the sides $a = BC$, $b = CA$, and $c = AB$, respectively, of the triangle ABC. Let*

$$t = \tfrac{1}{3}(a+b+c), \qquad a^* = \tfrac{1}{2}(b+c), \qquad b^* = \tfrac{1}{2}(c+a), \qquad c^* = \tfrac{1}{2}(a+b),$$
$$p = QA + AR, \qquad q = RB + BP, \qquad \text{and} \quad r = PC + CQ.$$

If $p \in [t, a^*]$, $q \in [t, b^*]$, and $r \in [t, c^*]$, then $PQ + QR + RP \geqslant \frac{1}{2}(a+b+c)$, and equality holds if and only if the points P, Q, and R are the mid-points of the sides of the triangle ABC.

From the continuity of the perimeter length, it follows that there are three points P, Q, R, satisfying the conditions of the lemma, for which

$$PQ + QR + RP$$

is minimal. In the following P, Q, and R will mean an extremal position, and it will be shown that P, Q, and R are the mid-points of the sides. We introduce the notations of Figure 2.

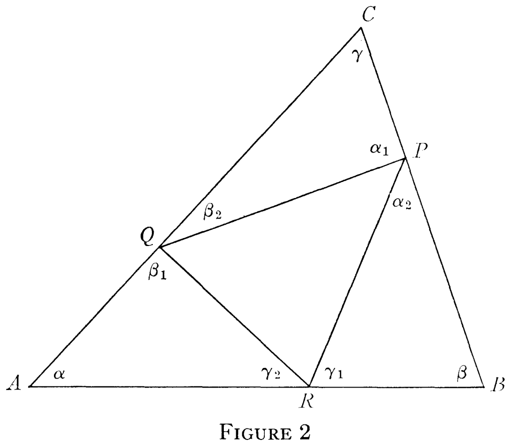

FIGURE 2

Lemma 1 implies that

$$(1) \qquad \cos\alpha_1 + \cos\beta_1 + \cos\gamma_1 = \cos\alpha_2 + \cos\beta_2 + \cos\gamma_2,$$

since otherwise the perimeter length of the triangle would decrease by moving the points in a suitable direction.

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Let K, L, and M denote the mid-points of the sides BC, CA, and AB respectively. After a suitable change in the notation, one of the following two statements is always valid:

(i) P is on the segment BK, Q on CL, and R on AM (Figure 3);

(ii) P is on the segment KC, Q on CL, and R on MB, and at most one of these points is the end point of the corresponding segment (Figure 4).

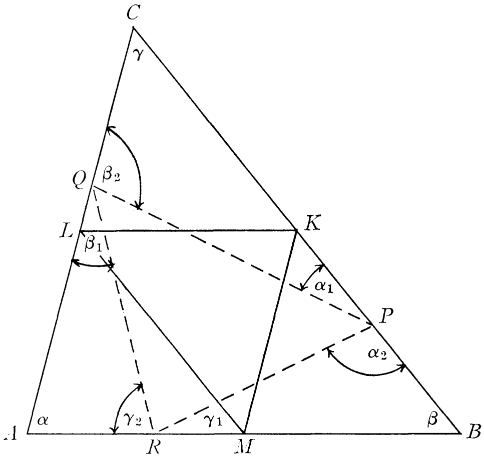

FIGURE 3

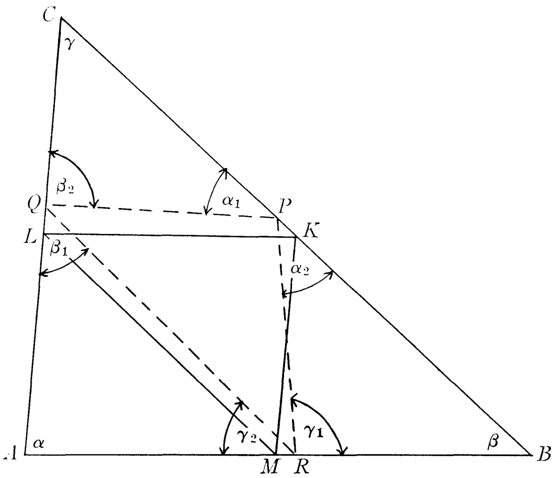

FIGURE 4

These two cases are examined separately.

(i) From Figure 3 the following inequalities are immediate:

$$\alpha_1 \leqslant \beta \leqslant \gamma_2, \qquad \beta_1 \leqslant \gamma \leqslant \alpha_2, \qquad \gamma_1 \leqslant \alpha \leqslant \beta_2.$$

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By (1), equalities must hold everywhere. Naturally this is true if and only if $P = K$, $Q = L$, and $M = R$.

(ii) The position of the points implies that $a^* < p$ and $r < c^*$, so that $c < a$, i.e. $\gamma < \alpha$. It follows from the conditions on the points P,Q,R (see Figure 4) that $\alpha_2 < \gamma < \alpha < \gamma_1$. Consequently, $\beta_1 < \beta_2$; for otherwise $\beta_1 \leqslant \beta_2$ implies that $\alpha_1 = \pi - \beta_2 - \gamma > \pi - \beta_1 - \alpha = \gamma_2$, and from this it follows that $\alpha_1 > \gamma_2$, $\beta_1 \geqslant \beta_2$, and $\gamma_1 > \alpha_2$, contradicting (1).

Since $a^* < p$ and $r < c^*$, Q can be moved towards A into a point Q^* so that the points P, Q^*, and R still satisfy the conditions of the lemma, and so that the inequality $\angle PQ^*C \geqslant \angle RQ^*A$ holds. But then $PQ^* + RQ^* < PQ + QR$ holds trivially, contradicting the extremal property of P, Q, and R.

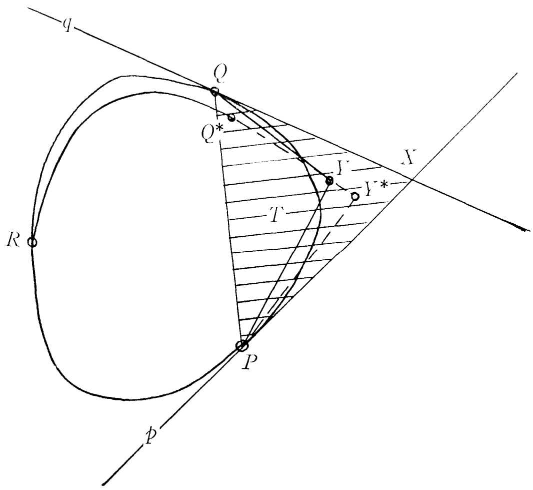

FIGURE 5

Proof of Theorem 1. As in **(1)**, the Blaschke Selection Theorem may be used to ensure that there is a set S (either a segment or a convex curve) of perimeter length l, and three points on its perimeter, P, Q, and R, such that the perimeter length of PQR is the least possible. It will be proved that S is a triangle. In the following, S will denote one of the extremal curves and P, Q, and R the corresponding points.

The mid-points of the sides of an equilateral triangle with sides of length $\frac{1}{3}l$ divide the perimeter into parts of length $\frac{1}{3}l$, and the perimeter length of the triangle formed by the mid-points is $\frac{1}{2}l$. This implies that the perimeter length of the triangle PQR is at most $\frac{1}{2}l$. Hence S cannot be a line segment, since in that case the perimeter length of the (degenerate) triangle PQR is $\frac{2}{3}l$.

Consequently, S is a convex curve.

Let p, q, and r be three arbitrary support lines of S through the points P, Q, and R. First we shall show that the lines p, q, and r form a triangle and S is exactly this triangle.

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Suppose this is not true. Then it can be assumed that the arc PQ of S (not containing R) does not lie entirely on the lines p and q. Let T be the common part of the following three closed half-planes: the half-plane determined by PQ and not containing R, the half-plane determined by p and containing Q, and, lastly, that half-plane determined by q which contains P. T is either a triangle or an infinite part of the plane bounded by three lines. If T is a triangle, say PQX, then the convexity of S implies that the length of the arc PQ is less than $PX + XQ$, so that there is a point Y in T, not on the lines p and q, satisfying $PY + YQ = \frac{1}{3}l$. If T is not a triangle, it is obvious that there is a point Y with this property. Substitute for the arc PQ of S the segments PY and YQ, and denote by S' the new curve. Naturally S' is also an extremal figure. Let us fix the arc PR of S', and rotate the point Q, together with the arc QR, around R towards P with a sufficiently small angle. Denote by Q^* the new position of Q. The perimeter length of the triangle PQ^*R is trivially smaller than the perimeter length of PQR. Let Y be that point which is separated from R by PQ^* and for which $PY^* = PY$ and $Q^*Y^* = QY$. Y was neither on the line p nor on q, so, if the rotation angle was sufficiently small, the curve S^*, constructed from the arcs PR and RQ^*, and from the segments Q^*Y^* and Y^*P, is also convex and of perimeter length l. Its perimeter is divided into three parts of equal length by the points P, Q^*, and R, and, since $PQ + QR + RP > PQ^* + Q^*R + RP$, this contradicts the extremal property of S.

So S is a triangle formed by the lines p, q, and r, and, as p, q, and r were arbitrary support lines, this means that S is a triangle ABC and P, Q, and R are interior points of the sides BC, CA, and AB, respectively. Theorem 1 now follows from Lemma 2, which yields slightly more than is needed here.

Now we deal with the case $n \geqslant 4$. Let h_n denote the infimum of the perimeter lengths of the polygons $A_1 A_2 \ldots A_n$.

THEOREM 2. *If n is even, then $h_n = [(n-2)/n]l$, and the only extremal figure is the segment of length $\frac{1}{2}l$. If n is odd, then*

$$[(n-2)/n]l < h_n = ([n - 2 + o(1))/n]l.$$

Repeating almost word for word the proof of Theorem 1, it can be shown that if C is an extremal figure (which exists by the Blaschke Selection Theorem), then C is either a segment or a convex k-gon: $B_1 B_2 \ldots B_k$, $k \leqslant n$, where each side contains at least one point A_i, and $A_i \neq B_j$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k)$.

When C is a segment and n is even, $h_n \geqslant [(n-2)/n]l$, and since equality can be achieved, $h_n = [(n-2)/n]l$. If n is odd, it is easily seen that $h_n = [(n-1)/n]l$.

In the case $C = B_1 B_2 \ldots B_k$, let A_{j_1} and A_{j_2} be those points A_i which surround B_j. Then

$$h_n = \sum_{j=1}^{j=k} A_{j_1} A_{j_2} + ([n-k)/n]l,$$

since the perimeter length of $B_1 B_2 \ldots B_k$ is l. If the angle at B_j is $\pi - 2\phi_j$,

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then since $A_{j1}B_j + B_jA_{j2} = l/n$, the corollary of Lemma 1 implies that $A_{j1}A_{j2} \geqslant (l/n) \cos \phi_j$. On the other hand,

$$3 \leqslant k, \qquad \sum_1^k \phi_j = \pi, \qquad \text{and } 0 < \phi_j < \pi,$$

and from this it follows that

$$\sum_1^k (1 - \cos \phi_j) < 2, \qquad \text{i.e.} \sum_1^k \cos \phi_j > k - 2.$$

This is obvious geometrically (see Figure 6), but it can be also verified by simple counting. By means of this inequality, we obtain

$$h_n > [(k-2)/n]l + [(n-k)/n]l = [(n-2)/n]l.$$

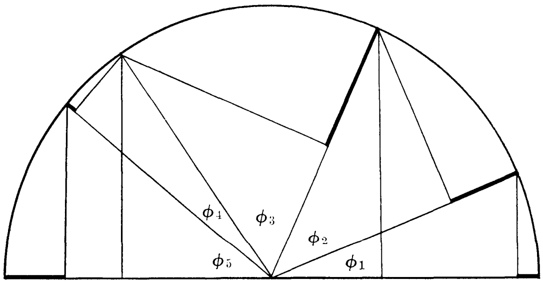

FIGURE 6

By comparing the two cases above, we obtain the theorem for even n: $h_n = [(n-2)/n]l$. But if n is odd we have only the inequality

$$h_n > [(n-2)/n]l.$$

The following example gives a reasonably good upper bound for h_n. If $n = 4m + 1$, take the triangle with sides ml/n, $[(m+1)/n]l$, and $(2m/n)l$. If $n = 4m - 1$, take the isosceles triangle with sides $(m/n)l$ and base $[(2m-1)/n]l$ (if $n \geqslant 11$, this gives a better estimate than the construction of **(1)**). It is easily seen that this cannot be the extremal figure, since, by decreasing the height of the triangle, a better result can be achieved. I suspect that this is the method by which the extremal figure can be obtained. The upper bound given by the construction above can be expressed concisely as follows: $h_n < [(n - 2 + o(1))/n]l$. (Roughly speaking, Zirakzadeh's conjecture was that $h_n = [(n - 3 + \sqrt{2} + o(1))/n]l$.))

This completes the proof of Theorem 2.

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