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ON CERTAIN SEQUENCES OF LEAST SQUARES APPROXIMANTS

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A sequence of certain rational functions is determined where each member solves an ℓ_2-minimisation problem on a "large" set of roots of unity. The sequence is compared with another sequence of L_2-rational approximants. Our main result extends a result of Rivlin on Walsh type equiconvergence.

1. INTRODUCTION

Let A_ρ, $1 < \rho < \infty$, be the set of functions $f(z)$ analytic in $|z| < \rho$ and having a singularity on the circle $|z| = \rho$. Let π_n denote the class of all polynomials of degree $\leqslant n$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then we put

$$S_{n-1}(z, f) \doteq \sum_{j=0}^{n-1} a_j z^j. \tag{1.1}$$

Let $p_{n-1,m}(z, f)$, $m \geqslant n$, denote the polynomial of degree $n-1$ of least square approximation to $f(z)$ on the mth roots of unity. Then an extension of a result of Walsh [**3**, p. 153] due to Rivlin [**1**, Theorem 1] can be stated as follows:

THEOREM. *Let q be a fixed positive integer and $m = qn + c$ with $0 \leqslant c \leqslant q-1$. If $f \in A_\rho$, $1 < \rho < \infty$ then,*

$$\lim_{n\to\infty} \{p_{n-1,m}(z, f) - S_{n-1}(z, f)\} = 0, \qquad |z| < \rho^{q+1}, \tag{1.2}$$

the convergence being uniform and geometric in $|z| \leqslant r < \rho^{q+1}$. Moreover, the result is sharp in the sense that (1.2) fails for every z satisfying $|z| = \rho^{q+1}$ for any $f \in A_\rho$.

Recently, Saff and Sharma [**2**] discussed the behaviour of certain sequences of rational interpolants. Walsh's theorem [**3**, p. 153] is again a special case of their result. In the present paper we discuss equiconvergence of certain rational sequences of the form used by Saff and Sharma [**2**].

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Section 2 deals with the ℓ_2-minimisation problem on the set $\{\omega^k\}_0^{qn-1}$, where ω is a primitive (qn)th root of unity. Its solution will replace the polynomial $p_{n-1,m}(z, f)$ in (1.2) as stated in our main result (see Theorem 2.1). Similarly, in place of the Taylor polynomial $S_{n-1}(z, f)$ in (1.2), we shall consider the unique rational function $r_{n+m,n}(z, f)$ [**2**, (1.4)] where

$$(1.3) \qquad r_{n+m,n}(z, f) = \frac{P_{n+m,n}(z,f)}{z^n - \sigma^n},\ P_{n+m,n}(z, f) \in \pi_{n+m},\ m \geqslant -1,\ \sigma > 1,$$

which minimises the integral

$$\int_{|z|=1} |f(z) - r(z)|^2\,|dz|$$

over all rational functions of the form $r(z) = \frac{p(z)}{z^n - \sigma^n}$, $p(z) \in \pi_{n+m}$. The minimisation problem will be solved in Section 3. We shall prove our main result in Section 4.

2. A minimisation problem and statement of result

Consider the following problem:

(P1). *Let $m \geqslant -1$ and $q \geqslant 2$ be fixed integers and let $\omega = \exp(2\pi i/qn)$. For $f \in A_\rho$, we want to minimise*

$$(2.1) \qquad \sum_{k=0}^{qn-1} |f(\omega^k) - R(\omega^k, f)|^2$$

over all ratioinal functions of the form

$$R(z, f) = \frac{p(z)}{z^n - \sigma^n}, \quad p(z) \in \pi_{n+m}.$$

If the solution of the problem (P1) is given by

$$(2.2) \qquad R^*_{n+m,n}(z, f) = \frac{P^*_{n+m,n}(z, f)}{z^n - \sigma^n}, \quad P^*_{n+m,n}(z, f) \in \pi_{n+m},$$

then we can state our main result as:

Theorem 2.1. *Let $m \geqslant -1$ and $q \geqslant 2$ be two fixed integers and let $\sigma > 1$. If $f \in A_\rho$, $1 < \rho < \infty$, then*

$$\lim_{n\to\infty} \{R^*_{n+m,n}(z, f) - r_{n+m,n}(z,f)\} = 0 \begin{cases} |z| < \rho^{1+q} & \text{if } \sigma \geqslant \rho^{1+q}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{1+q}, \end{cases}$$

the convergence being uniform and geometric in any compact subset of the regions described above. Moreover, the result is sharp in the sense that for each $|z| = \rho^{1+q}$ with $\sigma \geqslant \rho^{1+q}$, there is an $\hat{f} \in A_\rho$ for which (2.3) does not hold.

Proof of the preceding theorem will be given in Section 4. First we turn to the reformulation of the minimisation problem (P1) which requires the following lemma:

LEMMA 2.1. *Let* $m \geqslant -1$ *and* $q \geqslant 2$ *be fixed integers and let* $g(z) = P(z)/(z^n - \sigma^n)$, *where* $P(z) = \sum_{j=0}^{n+m} c_j z^j \in \pi_{n+m}$ *is given. Then the Lagrange interpolant of* $g(z)$ *on the* (qn)*th roots of unity is given by*

$$L_{qn-1}(z, g) = \sum_{\nu=0}^{qn-1} A_\nu z^\nu$$

where

$$(2.4) \qquad A_{\nu n+j} = \begin{cases} \lambda_1 c_j + \lambda_q c_{j+n}, & \nu = 0, \quad 0 \leqslant j \leqslant m \\ \lambda_{\nu+1} c_j, & 0 \leqslant \nu \leqslant q-1, \quad m+1 \leqslant j \leqslant n-1 \\ \lambda_{\nu+1} c_j + \lambda_\nu c_{j+n}, & 1 \leqslant \nu \leqslant q-1, \quad 0 \leqslant j \leqslant m \end{cases}$$

with

$$(2.5) \qquad \lambda_\nu = \sigma^{(q-\nu)n}/(1 - \sigma^{qn}), \quad \nu = 1, 2, \ldots, q.$$

PROOF: The Lagrange interpolant of the function $(z^n - \sigma^n)^{-1}$ on the (qn)th roots of unity is given by

$$L_{qn-1}\left(z, (z^n - \sigma^n)^{-1}\right) = \frac{z^{qn} - \sigma^{qn}}{(z^n - \sigma^n)(1 - \sigma^{qn})} = \sum_{\nu=1}^{q} \lambda_\nu z^{(\nu-1)n}.$$

For $z = \omega^k$, we have

$$(2.6) \qquad \left(\omega^{kn} - \sigma^n\right)^{-1} = \sum_{\nu=1}^{q} \lambda_\nu \omega^{(\nu-1)nk}, \quad k = 0, 1, 2, \ldots, qn-1.$$

If we multiply (2.6) by $P(\omega^k) = \sum_{j=0}^{n+m} c_j \omega^{kj}$ then a simple calculation leads to

$$\frac{P(\omega^k)}{\omega^{kn} - \sigma^n} = \lambda_1 \sum_{j=0}^{n+m} c_j \omega^{kj} + \lambda_2 \sum_{j=n}^{2n+m} c_{j-n} \omega^{kj} + \ldots. + \lambda_q \sum_{j=(q-1)n}^{qn+m} c_{j-(q-1)n} \omega^{kj}.$$

Notice that in the above, each of the q summations has $m+1$ distinct powers of ω^k which also appear in the preceding sum. If we group the terms involving identical powers of ω^k in separate summations and then rearrange them, we obtain for each $k = 0, 1, 2, \ldots, qn-1$,

$$\frac{P(\omega^k)}{\omega^{kn} - \sigma^n} = \sum_{j=0}^{m} [\lambda_1 c_j + \lambda_q c_{j+n}] \omega^{jk} + \sum_{\nu=0}^{q-1} \sum_{j=m+1}^{n-1} \lambda_{\nu+1} c_j \omega^{(\nu n+j)k}$$
$$+ \sum_{\nu=1}^{q-1} \sum_{j=0}^{m} [\lambda_{\nu+1} c_j + \lambda_\nu c_{j+n}] \omega^{(\nu n+j)k}$$

which reduces to $L_{qn-1}(\omega^k, g) = \sum_{j=0}^{qn-1} A_j \omega^{jk}$ on recalling that ω is a primitive (qn)th root of unity, where the A_j's are given by (2.4). ∎

REMARK 2.1: If the a_ν's are given complex numbers, then from the properties of the roots of unity, it follows that

$$\sum_{k=0}^{qn-1} \Big| \sum_{\nu=0}^{qn-1} a_\nu \omega^{k\nu} \Big|^2 = qn \sum_{\nu=0}^{qn-1} |a_\nu|^2 . \tag{2.7}$$

REMARK 2.2: If $L_{qn-1}(z, f) = \sum_{j=0}^{qn-1} b_j z^j$ is the Lagrange interpolant of degree $qn-1$ to $f(z) \in A_\rho$ in the (qn)th roots of unity, then we recall that

$$b_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(t) t^{qn-j-1}}{t^{qn} - 1} dt, \quad 0, 1, 2, \ldots, qn-1, \tag{2.8}$$

where Γ is the circle $|t| = R$, $1 < R < \rho$.

REMARK 2.3: Since $L_{qn-1}(\omega^k, f) = \sum_{\nu=0}^{qn-1} b_\nu \omega^{k\nu} = f(\omega^k)$ and $L_{qn-1}(\omega^k, g) = \sum_{\nu=0}^{qn-1} A_\nu \omega^{k\nu}$, $k = 0, 1, 2, \ldots, qn-1$, it follows from (2.7) that the minimisation problem (P1) is equivalent to the problem (P2):

(P2). *Minimise*

$$G = \sum_{\nu=0}^{qn-1} |b_\nu - A_\nu|^2 \tag{2.9}$$

over the c_j's, $j = 0, 1, 2, \ldots, n+m$, where the A_ν's are given by (2.4).

It may be noted that the b_ν's are well-defined from (2.8) and that the A_ν's are functions of the c_j's.

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3. Solution of the minimisation problem (P1)

The following proposition determines the solution of the problem (P1):

Proposition 2.1. *The polynomial $P^*_{n+m,n}(z, f)$ (see (2.2)) which solves the minimisation problem (P1) is given by*

$$P^*_{n+m,n}(z, f) = \sum_{\nu=0}^{n+m} p_\nu z^\nu$$

where

$$(3.1)\qquad p_\nu = \begin{cases} -b_\nu \sigma^n + \frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \sum_{j=1}^{q-1} \sigma^{-jn} b_{\nu+jn}, & 0 \leqslant \nu \leqslant m \\ \frac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}} \sum_{j=0}^{q-1} \sigma^{-jn} b_{\nu+jn}, & m+1 \leqslant \nu \leqslant n-1 \\ b_{\nu-n} - \frac{\sigma^{2(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \sum_{j=1}^{q-1} \sigma^{-jn} b_{\nu+(j-1)n}, & n \leqslant \nu \leqslant n+m, \end{cases}$$

and the b_ν, $\nu = 0, 1, \dots, qn-1$ are given by (2.8).

Proof: Since (P1) is equivalent to (P2), it is sufficient to solve the system of equations $\frac{\partial G}{\partial \bar{c}_\nu} = 0$, $\nu = 0, 1, 2, \dots, n+m$, in terms of the λ_ν's and b_j's. Using (2.4), we can rewrite (2.9) as:

$$G = \sum_{j=0}^{m} |b_j - \lambda_1 c_j - \lambda_q c_{j+n}|^2 + \sum_{\nu=0}^{q-1} \sum_{j=m+1}^{n-1} |b_{j+\nu n} - \lambda_{\nu+1} c_j|^2 + \sum_{\nu=1}^{q-1} \sum_{j=0}^{m} |b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n}|^2.$$

Therefore, for $0 \leqslant j \leqslant m$, we have

(3.2)

$$\frac{\partial G}{\partial \bar{c}_j} = -(b_j - \lambda_1 c_j - \lambda_q c_{j+n})\lambda_1 - \sum_{\nu=1}^{q-1} (b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n})\lambda_{\nu+1}$$
$$= \sum_{\nu=1}^{q} \lambda_\nu^2 c_j + (\lambda_q \lambda_1 + \lambda_1 \lambda_2 + \dots + \lambda_{q-1}\lambda_q) c_{j+n} - \sum_{\nu=0}^{q-1} \lambda_{\nu+1} b_{j+\nu n},$$

and

(3.3)

$$\frac{\partial G}{\partial \bar{c}_{n+j}} = -(b_j - \lambda_1 c_j - \lambda_q c_{j+n})\lambda_q - \sum_{\nu=1}^{q-1} (b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n})\lambda_\nu$$
$$= \sum_{\nu=1}^{q} \lambda_\nu^2 c_{j+n} + (\lambda_q \lambda_1 + \lambda_1 \lambda_2 + \dots + \lambda_{q-1}\lambda_q) c_j - \lambda_q b_j - \sum_{\nu=1}^{q-1} \lambda_\nu b_{j+\nu n}.$$

Also, for $m+1 \leqslant j \leqslant n-1$, we have

$$(3.4) \qquad \frac{\partial G}{\partial \bar{c}_j} = -\sum_{\nu=0}^{q-1}(b_{j+\nu n} - \lambda_{\nu+1}c_j)\lambda_{\nu+1} = \sum_{\nu=0}^{q-1}\lambda_{\nu+1}^2 c_j - \sum_{\nu=0}^{q-1}\lambda_{\nu+1}b_{j+\nu n}.$$

For the sake of simplicity, we set

$$(3.5) \qquad \begin{cases} \alpha = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_q^2 = \left(\frac{1+\sigma^{qn}}{1-\sigma^{2n}}\right)\lambda_q, \\ \beta = \lambda_q\lambda_1 + \lambda_1\lambda_2 + \cdots + \lambda_{q-1}\lambda_q = \left(\frac{\sigma^n + \sigma^{(q-1)n}}{1-\sigma^{2n}}\right)\lambda_q. \end{cases}$$

Then from (3.2)–(3.4) we obtain, for $j = 0, 1, \ldots, m$:

$$(3.6) \qquad \begin{cases} \frac{\partial G}{\partial \bar{c}_j} = \alpha c_j + \beta c_{n+j} - \sum\limits_{\nu=0}^{q-1}\lambda_{\nu+1}b_{\nu n+j}, & (0 \leqslant j \leqslant m) \\ \frac{\partial G}{\partial \bar{c}_{n+j}} = \alpha c_{n+j} + \beta c_j - \lambda_q b_j - \sum\limits_{\nu=1}^{q-1}\lambda_\nu b_{\nu n+j}, & \end{cases}$$

and for $m+1 \leqslant j \leqslant n-1$, we see that

$$(3.7) \qquad \frac{\partial G}{\partial \bar{c}_j} = \alpha c_j - \sum_{\nu=0}^{q-1}\lambda_{\nu+1}b_{\nu n+j}.$$

Setting the partial derivatives $\frac{\partial G}{\partial \bar{c}_\nu}$ equal to zero and then solving the system of equations so obtained simultaneously for c_ν and $c_{\nu+n}$, $\nu = 0, 1, \ldots, m$, we obtain

$$(3.8) \qquad c_\nu = \left(\alpha^2 - \beta^2\right)^{-1}\{\alpha\sum_{j=0}^{q-1}\lambda_{j+1}b_{jn+\nu} - \beta\sum_{j=1}^{q-1}\lambda_j b_{jn+\nu} - \beta\lambda_q b_\nu\},$$

and

$$(3.9) \qquad c_{n+\nu} = \left(\alpha^2 - \beta^2\right)^{-1}\{\alpha\sum_{j=1}^{q-1}\lambda_j b_{jn+\nu} + \alpha\lambda_q b_\nu - \beta\sum_{j=0}^{q-1}\lambda_{j+1}b_{jn+\nu}\}.$$

Also from (3.7) we have

$$(3.10) \qquad c_\nu = \alpha^{-1}\sum_{j=1}^{q}\lambda_j b_{(j-1)n+\nu}, \qquad m+1 \leqslant \nu \leqslant n-1.$$

We recall that c_ν, $0 \leqslant \nu \leqslant n+m$, as determined in (3.8)–(3.10), are the coefficients of the polynomial $P^*_{n+m,n}(z, f)$ (see (2.2)) which minimises the expression given by (2.1).

Finally, replacing α, β and λ_j's in c_ν, $(\nu = 0, 1, \ldots, n+m)$, by their respective values from (3.5) and (2.5) we obtain the relation (3.1) after some simple arithmetic. ∎

REMARK 3.1: When $q = 1$, the solution of the problem (P1) is not unique for all $m > -1$. However, $R^*_{n-1,n}(z, f)$ is uniquely defined for $q = 1$. It turns out that $R^*_{n-1,n}(z, f)$, in this case, interpolates the function $f(z)$ in the nth roots of unity.

4. Proof of Theorem 2.1

It is known [**2**] that an integral representation of the rational function $r_{n+m,n}(z, f)$ is given by

$$(4.1)\qquad r_{n+m,n}(z, f) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^n - \sigma^n) f(t)}{(z^n - \sigma^n)(t - z)} \sum_{j=1}^{3} A_j(t, z)dt,$$

where Γ is the circle $|t| = R$, $1 < R < \rho$ and $A_j(t, z)$, $j = 1, 2, 3$ and given by

$$(4.2)\qquad \begin{cases} A_1(t, z) = \frac{t^{m+1} - z^{m+1}}{t^{m+1}}, \quad A_2(t, z) = \frac{z^{m+1}\left(t^{n-m-1} - z^{n-m-1}\right)}{t^n - \sigma^{-1}}, \\ A_3(t, z) = z^n\left(t^{m+1} - z^{m+1}\right)/t^{m+1}(t^n - \sigma^{-n}). \end{cases}$$

Also, from (2.2), (2.8) and (3.1), we obtain after some algebraic operations

$$(4.3)\qquad R^*_{n+m,n}(z, f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(z^n - \sigma^n)(t - z)(t^{qn} - 1)} \sum_{j=1}^{3} A_j(t, z)B_j(t, \sigma)dt$$

with

$$(4.4)\qquad \begin{cases} B_1(t, \sigma) = \sigma^{-(q-2)n}B(t, \sigma) - t^{qn}\sigma^n, \\ B_2(t, \sigma) = (t^{qn} - \sigma^{-qn})(\sigma^{-n} - \sigma^n)/(1 + \sigma^{-qn}), \\ B_3(t, \sigma) = (t^n - \sigma^{-n})\{t^{qn} - \sigma^n B(t, \sigma)\}, \end{cases}$$

where

$$B(t, \sigma) = \frac{t^n\left(t^{(q-1)n} - \sigma^{-(q-1)n}\right)\left(1 - \sigma^{-2n}\right)}{\left(t^n - \sigma^{-n}\right)\left(1 - \sigma^{-2(q-1)n}\right)}.$$

Therefore,

$$(4.5)\qquad R^*_{n+m,n}(z, f) - r_{n+m,n}(z, f) = \frac{1}{2\pi i}\int_\Gamma \sum_{j=1}^{3} \frac{A_j(t, z)K_j(t, \sigma)}{(z^n - \sigma^n)(t - z)(t^{qn} - 1)} f(t)dt.$$

where $K_j(t, \sigma) = B_j(t, \sigma) - (t^{qn} - 1)(t^n - \sigma^n)$, $j = 1, 2, 3$ can be explicitly rewritten after some simplification as

$$(4.6)\qquad \begin{cases} K_1(t, \sigma) = \sigma^{-(q-2)n}B(t, \sigma) - t^{(q+1)n} + t^n - \sigma^n, \\ K_2(t, \sigma) = \frac{t^{qn}\sigma^{-n}\left(1 - \sigma^{-qn}\right) - \sigma^{-qn}\left(\sigma^{-n} - \sigma^n\right)}{1 + \sigma^{-qn}} - t^{(q+1)n} + t^n - \sigma^n, \\ K_3(t, \sigma) = \frac{t^n\sigma^{-qn}\left(\sigma^{2n} - 1\right)\left(t^{(q-1)n}\sigma^{-(q-1)n} - 1\right)}{1 + \sigma^{-2(q-1)n}} + t^n - \sigma^n. \end{cases}$$

An analysis of the kernels $A_j(t, z)$ and $K_j(t, \sigma)$, $j = 1, 2, 3$ from (4.2) and (4.6) yields (2.3).

To prove that the result is sharp, first we consider the point $z^* = \rho^{1+q}$ and the corresponding function $\hat{f}(z) = (z-\rho)^{-1}$. A direct computation from (3.1), (2.2) and (4.1) for $f = \hat{f}$ shows that

$$(4.7) \qquad R^*_{n+m,n}\left(z, \hat{f}\right) - r_{n+m,n}\left(z, \hat{f}\right) = \sum_{j=1}^{3} \frac{A_j(\rho, z)K_j(\rho, \sigma)}{(z^n - \sigma^n)(z-\rho)(\rho^{qn} - 1)}.$$

If $\sigma > \rho^{1+q}$, we get after some simple calculations

$$\lim_{n\to\infty} \{R^*_{n+m,n}\left(\rho^{1+q}, \hat{f}\right) - r_{n+m,n}\left(\rho^{1+q}, \hat{f}\right)\} = \frac{\rho^q}{\rho^q - 1},$$

whereas for $\sigma = \rho^{1+q}$, the sequence (4.7) is undefined for infinitely many n's for $z = \rho^{1+q}\omega_0$ where ω_0 is any primitive root of unity. ∎

REMARK 4.1: If we fix $m = -1$ and $\sigma \to \infty$ in Theorem 2.1, we get a result of Rivlin [**1**, Theorem 1] in the special case when $m(n) = qn$.

REMARK 4.2: According to Remark 3.1, Saff–Sharma's result [**2**, Theorem 2.3] for the special case $m = -1$ can be retrieved from Theorem 2.1.

REFERENCES

[1] T.J. Rivlin, 'On Walsh Equiconvergence', *J. Approx. Theory* **36** (1982), 334–345.

[2] E.B. Saff and A. Sharma, 'On equiconvergence of certain sequence of rational Interpolants', *Rational Approximation and Interpolation Lecture Notes in Math* **1105** (Springer, 1984).

[3] J.L. Walsh, 'Interpolation and Approximation by Rational Functions in the complex Domain', Colloq. Publ. 5th ed.,, *Amer. Math. Soc.* (1969).

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