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GENERALIZED DE LA VALLÉE POUSSIN DISCONJUGACY TESTS FOR LINEAR DIFFERENTIAL EQUATIONS([1])

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1. **Introduction.** In this paper, we study the oscillatory behavior of the solutions of the linear differential equation

$$Ly = r_1(t)y^{(n-1)} + \cdots + r_n(t)y, \tag{1.1}$$

where

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y \tag{1.2}$$

and all functions are assumed to be continuous on a bounded interval $[a, b)$. An nth-order linear equation is said to be disconjugate on an interval I provided it has no nontrivial solution with more than $n-1$ zeros, counting multiplicities, in I. We assume that $Ly=0$ is disconjugate on $[a, b)$ and derive a disconjugacy criterion for (1.1) of the form

$$\sum_{k=1}^{n} \int_a^b |r_k(t)| \nu_{n-k+1}(t)\, dt \le 1. \tag{1.3}$$

The function ν_k is determined in terms of fundamental principal systems of solutions of $Ly=0$, which is a concept defined in Willett [19] and further described in §2.

Condition (1.3) is a generalization of the multitude of disconjugacy tests which are called de la Vallée Poussin tests. Such tests were originated by de la Vallée Poussin [17]. They are of the form

$$\sum_{k=1}^{n} (b-a)^k \|r_k\| A_k \le 1, \tag{1.4}$$

where A_k is a constant and $\|r\|$ is some norm of r, and apply to equation (1.1) for the special case

$$L = D^n.$$

Recent surveys which include results of this type have been carried out by Aramă and Ripianu [0], Richard [12], and A. Yu. Levin [10]. Other results have been obtained by Martelli [11] and Hartman [3]. There is also a series of papers in Japanese by Hukuhara [4], Satô [13], and Tumura [16], which are not available to me. However, Hukuhara [5] (cf. also, Math. Reviews 29, No. 3704) lists some of the results in these papers.

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Most of the past results involving conditions of the type (1.4) are derived directly from the differential equation by using inequalities relating the growth of functions to the growth of their derivatives, or by using differential inequalities. An alternate approach is suggested by the nature of the de la Vallée Poussin condition, which is a "smallness condition" associated with considering the equation

$$y^{(n)} = r_1(t)y^{(n-1)}+\cdots+r_n(t)y \tag{1.5}$$

a perturbation of the equation

$$y^{(n)} = 0. \tag{1.6}$$

In §2, we derive our disconjugacy condition (1.3) from this viewpoint, that is, we consider (1.1) a perturbation of a disconjugate equation $Ly=0$ with Ly defined by (1.2). The main result is Theorem 2.3.

In §3, we give some applications of Theorem 2.3. We list the main application here in order to give a comparison with known results.

Let $[x]$ denote the greatest integer contained in x.

THEOREM 1.1. *If*

$$\frac{1}{[(n-1)/2]![n/2]!}\int_a^b |r_n(t)|\,\frac{(t-a)^{n-1}(b-t)^{n-1}}{(b-a)^{n-1}}\,dt+2^{n-1}\int_a^b |r_1(t)|\,dt + \sum_{k=2}^{n-1}\frac{(2^{n-1}-1)}{(k-1)!}\int_a^b |r_k(t)|\,\frac{(t-a)^{k-1}(b-t)^{k-1}}{(b-a)^{k-1}}\,dt \le 1, \tag{1.7}$$

then (1.5) *is disconjugate on* $[a, b)$.

For the proof of Theorem 1.1, see §3.

Theorem 1.1 is a generalization to (1.3) of a very precise result of Levin [9] for equations of the form

$$y^{(n)} = r_n(t)y. \tag{1.8}$$

In the case of (1.8), condition (1.7) falls short (in generality) of reproducing Levin's result by a factor of 2, when n is odd, and a factor of $2(n-1)/n$, when n is even. In the case of equation (1.8), condition (1.7) can be replaced by the two conditions which are formed from (1.7) by replacing $|r_n|$ by $r_n^+=(|r_n|+r_n)/2$ and $r_n^-=(|r_n|-r_n)/2$, because of the comparison theorem of Kondrat'ev [6].

Condition (1.7) immediately implies disconjugacy conditions of the form

$$\frac{(b-a)^n\|r_n\|_p}{[(n-1)/2]![n/2]!}A_n+(b-a)\|r_1\|_p 2^{n-1}A_1 + \sum_{k=2}^{n-1}\frac{(b-a)^k\|r_k\|_p}{(k-1)!}(2^{n-1}-1)A_k \le 1 \tag{1.9}$$

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where A_k, $k=1,\ldots,n$, are constants, $1\le p\le\infty$, and

$$\|r\|_p = \left(\int_a^b |r(t)|^p\,dt/(b-a)\right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|r\|_\infty = \sup\{|r(t)| : a \le t < b\}.$$

We obtain by applying Hölder's inequality to (1.7) the following values for the coefficients A_k, $k=1,\ldots,n$, in (1.9):

$$A_k = \frac{(k-1)!(k-1)!}{(2k-1)!} \quad \text{for } p = \infty, \tag{1.10}$$

$$A_k = 2^{2-2k} \quad \text{for } p = 1, \tag{1.11}$$

$$A_k = \left(\frac{\pi^{1/2}\Gamma(\lambda)}{2^{2\lambda-1}\Gamma(\lambda+1/2)}\right)^{1-(1/p)} \quad \text{for } 1 < p < \infty,\ \lambda = \frac{pk-1}{p-1}. \tag{1.12}$$

To see how the Gamma Functions in (1.12) arise, consult formulas (5) and (13) in [2, pp. 9–10].

For $p=\infty$, the best values of A_k in (1.9), except for $n=3$ or 4, that have appeared in the literature seem to be

$$\begin{gathered} A_1 = \frac{1}{2^n}, \qquad A_n = \frac{1}{n2^n}, \\ A_k = \frac{(k-1)!}{[(k-1)/2]!\,[k/2]!\,2^k k(2^{n-1}-1)}, \quad k = 2,\ldots,n-1, \end{gathered} \tag{1.13}$$

which were obtained by Levin [8] and Hukuhara [4], and

$$\begin{gathered} A_1 = \frac{n-1}{n2^{n-1}}, \qquad A_n = \frac{(n-1)^{n-1}}{n!\,n^n}\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!, \\ A_k = \frac{n-k}{nk(2^{n-1}-1)}, \quad k = 2,\ldots,n-1, \end{gathered} \tag{1.14}$$

which were obtained by Tumura [16] and Bessmertnykh and Levin [1]. One can easily show that for values of k sufficiently close to n and n sufficiently large, the coefficient A_k in (1.10) is smaller than either of its counterparts in (1.13) or (1.14). For example, A_n in (1.10) is always smaller than A_n in (1.13) or (1.14) for all $n\ge 4$.

For $p=1$, the best values of A_k that have appeared in the literature seem to be

$$\begin{gathered} A_n = \frac{[n/2]}{(n-1)2^{n-1}}, \qquad A_1 = 2^{1-n}, \\ A_k = \frac{(k-2)!}{[(k-1)/2]!\,[(k-2)/2]!\,2^{k-1}(2^{n-1}-1)}, \quad k = 2,\ldots,n-1, \end{gathered} \tag{1.15}$$

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which were obtained by Hukuhara [4] and Hartman [3], and

$$(1.16)\qquad \begin{aligned} A_n &= \left(\frac{n-1}{n}\right)^{n-3}\left[\frac{n}{2}\right]\frac{1}{4n}, \qquad A_1 = 2^{-n},\\ A_k &= \frac{(k-1)!}{[(k-1)/2]![(k-2)/2]!4k(2^{n-1}-1)}\left(\frac{k-1}{k}\right)^{k-3},\\ &\qquad\qquad k = 2,\ldots,n-1, \end{aligned}$$

which were obtained by Hartman [3]. The same remark again applies, that is, (1.11) is substantially better than (1.15) and (1.16) for values of k close to n and n sufficiently large. However, in all fairness to Hartman, we point out that Hartman further generalizes the basic disconjugacy condition by replacing (1.9) with two such conditions with the first having

$$\|r_k\|_1 = \int_a^{(a+b)/2} |r_k(t)|\, dt/(b-a), \quad k = 1,\ldots,n,$$

and the second having

$$\|r_k\|_1 = \int_{(a+b)/2}^{b} |r_k(t)|\, dt/(b-a), \quad k = 1,\ldots,n.$$

In this case, the coefficients A_k are as defined in (1.15). Hartman also is able to incorporate the coefficient r_1 into an exponential function in a worthwhile manner, which successfully completes one of the generalizations attempted in [20]. The latter generalization is not particularly important in practice, however, since the coefficient r_1 can be always eliminated from a given equation by well-known transformations.

For $1<p<\infty$, the best values of A_k that have appeared in the literature seem to be

$$(1.17)\qquad \begin{aligned} A_n &= \frac{[(n-1)/2]![n/2]!}{(n-1)!}, \qquad A_1 = \frac{1}{2^{n-1}},\\ A_k &= \frac{1}{2^{n-1}-1}, \quad k = 2,\ldots,n-1, \end{aligned}$$

which were obtained by Martelli [11]. Here, one can easily show that for at least the values of k when $2k>n$, (1.12) produces a smaller value for A_k than (1.17).

The cases $n=2$ and $n=3$ are treated further in §3, and refinements of Theorem 1.1 are obtained for these cases. A comparison with known results is made there.

Finally, all the known results mentioned above and in §3 actually imply disconjugacy on the closed interval $[a, b]$ provided $Ly=0$ is nonsingular at b. We will show in a future paper that (1.9) and the corresponding condition in Theorem 2.3 of the next section are each sufficient to imply disconjugacy on $[a, b]$ in this case, and even in some cases when $Ly=0$ is singular at a or b, provided strict inequality holds in (1.9) and (2.11). $Ly=0$ is singular at a or b in this case means $a=-\infty$, $b=\infty$, or one of the coefficients p_k is not improperly integrable on (a, b).

2. **Disconjugate perturbations of disconjugate equations.** In what follows, by solution we shall always mean a nontrivial solution. A function φ is said to have a zero at c of order k if $\varphi(c)=\cdots=\varphi^{(k-1)}(c)=0$ and $\varphi^{(k)}(c)\neq 0$. The usual notation for open, closed, and half-open intervals is used.

This section is concerned with determining an effective criteria for deciding whether equation (1.2) is disconjugate on a given bounded interval I, given that $Ly=0$ is disconjugate on I. We start with the following lemma:

LEMMA 2.1. (Sherman). *If (1.2) is not disconjugate on* $[a, b)$, *then* $\exists[c, d]\subset(a, b)$ *and a solution* φ *of (1.2) which has a total of at least n zeros at c and d and does not vanish in* (c, d).

Proof. This lemma is an immediate consequence of Theorem 2 of Sherman [14], which states that the first conjugate point function $\eta(x)$ is continuous, and Theorem 5 of Sherman [15], which states that there always exists a solution with n zeros concentrated at c and d if $d=\eta(c)$.

Lemma 2.1 can be used to establish the following "ineffective" characterization of disconjugacy for (1.2).

THEOREM 2.1. *Equation (1.2) is disconjugate on* $[a, b)$, *if and only if, for any* $[c, d]\subset(a, b)$, $\exists$ *a system* $(y_1, \ldots, y_n)$ *of solutions of (1.2) such that* y_k *has a zero at c of order* $k-1$ *and a zero at d of order* $n-k$.

Proof. Assume (1.2) is not disconjugate on $[a, b)$. By Lemma 2.1, there exists $[c, d]\subset(a, b)$ and a solution φ with a total of at least n zeros at c and d. Let $(y_1, \ldots y_n)$ be the system of solutions for the interval $[c, d]$ that exist by hypothesis, that is, y_k has a zero at c of order $k-1$ and a zero at d of order $n-k$, $k=1, \ldots, n$. Clearly, $(y_1, \ldots, y_n)$ is a linearly independent set. Hence, there exists constants $c_1, \ldots, c_n$ such that

$$\varphi^{(k)}(t) = c_1 y_1^{(k)}(t)+\cdots+c_n y_n^{(k)}(t), \quad c \leq t \leq d, \quad k = 0, \ldots, n-1. \tag{2.1}$$

If φ has a zero of order m at c, then letting $t=c$ and $k=0, \ldots, m-1$ in (2.1) implies $c_1=\cdots=c_m=0$. Next, letting $t=d$ and $k=0, \ldots, n-m-1$ implies $c_{m+1}=\cdots=c_n=0$, since φ has a zero of order at least $n-m$ at d. But then φ is identically zero, which is a contradiction. Hence, (1.2) is disconjugate on $[a, b)$.

That the converse statement is true is well known from the equivalence of disconjugacy and the existence-uniqueness of solutions of boundary value problems for linear equations.

In what follows, we assume $Lu=0$ is disconjugate on (a, b). Let $[c, d]\subset(a, b)$.

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We have shown in [19] that $Lu=0$ has a fundamental principal system $(u_1, \ldots, u_n)$ of solutions on $[c, d)$, that is, u_k has a zero of order $k-1$ at c and

$$\lim_{t\to d^-} \frac{u_k(t)}{u_{k+1}(t)} = 0, \quad k = 1, \ldots, n-1. \tag{2.2}$$

Since d is a finite number in the present context, (2.2) implies that u_k must have a zero at d of order at least $n-k$. But (2.2) also implies that $(u_1, \ldots, u_n)$ is a linearly independent set of solutions, hence, the Wronskian $W(u_1, \ldots, u_n)$ does not vanish in $[c, d]$. This implies that u_k has a zero at d of order at most $n-k$, since otherwise $W=0$ at $t=d$. We have just proven the following lemma:

LEMMA 2.2. *If $(u_1, \ldots, u_n)$ is a fundamental principal system on $[c, d]$, then u_k has a zero of order $k-1$ at c and a zero of order $n-k$ at d.*

We also showed in [19] that the formal adjoint equation $L^*v=0$ has a fundamental principal system $(v_n, \ldots, v_1)$ on $[c, d]$. Let $1 \le j \le n$ and define

$$H_j(t, s) = \begin{cases} \sum_{k=1}^{j-1} (1-)^{n-k} u_k(t)[v_k(s)/v_j(s)]', & c \le s < t, \\ \sum_{k=j+1}^{n} (-1)^{n-k+1} u_k(t)[v_k(s)/v_j(s)]', & t \le s < d, \end{cases} \tag{2.3}$$

where $\sum_{k=1}^{0} \equiv 0 \equiv \sum_{k=n+1}^{n}$. Then, the function $H_j(t, s)$ is $(n-1)$-times continuously differentiable in t for $c \le s \le d$, $c \le t \le d$, except for finite jump discontinuities at $t=s$ in the $(n-2)$nd and $(n-1)$st derivatives. Let $\mu_1^j(t)=u_j(t)$ and $\mu_2^j, \ldots, \mu_n^j$ be defined as follows:

$$v_j(t)\mu_k^j(t) - \sum_{m\neq j} |u_m^{(k-1)}(t)| v_m(t) = \begin{cases} 0, & k = 2, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

The main use of the function H_j is described in the following lemma.

LEMMA 2.3 (Willett [19]). *For each $f \in C[c, d)$, the function*

$$w_j(t) = \int_c^d H_j(t, s) \left(\int_s^d v_j(\tau) f(\tau)\, d\tau \right) ds \tag{2.4}$$

exists in $C^n[c, d]$, $Lw_j=f$, and

$$w_j^{(k)}(t) = o(\mu_{k+1}^j(t)), \quad \text{as } t \to d^-, \quad k = 0, \ldots, n-1. \tag{2.5}$$

THEOREM 2.2. *Assume that $Lu=0$ is disconjugate on $[a, b)$, $[c, d] \subset (a, b)$, and H_j is defined by* (2.3). *If for each j, $j=1, \ldots, n$, the equation*

$$y_j(t) = u_j(t) + \int_c^d H_j(t, s) \left(\int_s^d v_j(\tau) F[y_j(\tau)]\, d\tau \right) ds, \tag{2.6}$$

where

$$F[y] \equiv r_1(t) y^{(n-1)} + \cdots + r_n(t) y, \tag{2.7}$$

has a solution $y_j \in C^{n-1}[c, d]$, then y_j has a zero at c of order $j-1$ and a zero at d of der $n-j$.

Proof. Let $y_j = u_j + w_j$ so that w_j is given by (2.4) with $f(\tau) = F[y_j(\tau)]$. Thus, w_j satisfies (2.5). Now, Lemma 2.2 implies u_m has a zero of order $n-m$ at d and v_m has a zero of order $m-1$ at d. Thus, from the definition of μ^j_{k+1}, we conclude that μ^j_{k+1} has a zero of order $n-k-j$ at d, $k=0, \ldots, n-j$. Hence, (2.5), implies $w_j^{(k)}(d)=0$ for $k=0, \ldots, n-j$, that is, w_j has a zero at d of order at least $n-j+1$. Since u_j has a zero at d of order $n-j$ and $y_j = u_j + w_j$, we conclude that y_j has a zero at d of order $n-j$.

Now consider the situation at c. The continuity of $H_j(t, s)$ implies

$$y_j^{(k)}(t) = u_j^{(k)}(t) + \int_c^d \frac{\partial^k H_j}{\partial t^k}(t, s) \left(\int_s^d v_j(\tau) F[y_j(\tau)]\, d\tau \right) ds, \quad k = 0, \ldots, n-2.$$

Since u_j has a zero of order $j-1$ at c by Lemma 2.2, y_j has a zero of order at least $j-1$ at c provided

$$\lim_{t \to c^+} \int_t^d \left| \frac{\partial^k H_j}{\partial t^k}(t, s) \right| ds = 0, \quad k = 0, \ldots, j-2. \tag{2.8}$$

But the integral in (2.8) is bounded by

$$\sum_{m=j+1}^{n} |u_m^{(k)}(t)| [v_m(t)/v_j(t)].$$

Since $u_m^{(k)}$, v_m, and v_j have zeros at c of order $m-k-1$, $n-m$, and $n-j$, respectively, the product $u_m^{(k)} v_m / v_j$ has a zero at c of order $j-k-1$. Thus, (2.8) follows. There remains to show that y_j has a zero at c of order at most $j-1$. We note that Lemma 2.3 implies

$$Ly_j = F[y_j] \equiv r_1 y_j^{(n-1)} + \cdots + r_n y_n, \quad j = 1, \ldots, n,$$

which means that $\{y_1, \ldots, y_n\}$ is a set of solutions of a linear differential equation. From the behavior of y_j at d, it is furthermore clear that $\{y_1, \ldots, y_n\}$ is a linearly independent set on $[c, d]$. Hence, the Wronskian $W(y_1, \ldots, y_n)$ does not vanish at $t=c$, which would not be the case if $y_j^{(j)}(c)=0$ for any j, $1 \le j \le n$.

THEOREM 2.3. *Assume that $Lu=0$ is disconjugate on the bounded interval (a, b). For each $[c, d] \subset (a, b)$, let $(u_1, \ldots, u_n)$ and $(v_n, \ldots, v_1)$ be the fundamental principal systems on $[c, d)$ of $Lu=0$ and $L^*v=0$, respectively, and let $H_j(t, s; c, d)$ be the corresponding function defined by* (2.3). *Let*

$$\nu_k(t) = \sup_{[c,d] \subset (a,b)} \sup_{1 \le j \le n} \rho_k(t; c, d, j), \tag{2.9}$$

where

$$\begin{cases} \rho_1 = v_j u_j, \\ \rho_k = v_j \displaystyle\int_c^d \left| \frac{\partial^{k-1} H_j}{\partial t^{k-1}} \right| ds, \quad k = 2, \ldots, n-1 \\ \rho_n = 1 + v_j \displaystyle\int_c^d \left| \frac{\partial^{n-1} H_j}{\partial t^{n-1}} \right| ds. \end{cases} \tag{2.10}$$

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If

$$\sum_{k=1}^{n} \int_a^b |r_k(t)| \nu_{n-k+1}(t)\, dt \le 1, \tag{2.11}$$

then equation (1.1) *is disconjugate on* $[a, b)$.

Proof. The theorem follows from Theorems 2.1 and 2.2 provided equation (2.6) has a solution y_j for each j, $j=1, \ldots, n$, and each $[c, d] \subset (a, b)$. For a fixed interval $[c, d] \subset (a, b)$ and a fixed j, (2.11) implies

$$\sum_{k=1}^{n} \int_c^d |r_k(t)|\, \rho_{n-k+1}(t) dt < 1. \tag{2.12}$$

But (2.12) is sufficient to imply the usual successive approximations starting with $u_j(t; c, d)$ converges to a solution $y_j(t)$ of (2.6) for $c \le t \le d$.

Condition (2.11) is a generalized de la Vallée Poussin type condition for disconjugacy of (1.1). We shall compute the functions ν_j for some special operators L in the next section.

3. **Applications.** The main application of Theorem 2.3 that we have at this time is Theorem 1.1 of §1.

Proof of Theorem 1.1. Let $L = D^n$ and $[c, d] \subset (a, b)$. The fundamental principal system on $[c, d]$ for $u^{(n)} = 0$ is given by $(u_1, \ldots, u_n)$ with

$$u_m(t) = \left(\frac{d-t}{d-c}\right)^{n-m} \frac{(t-c)^{m-1}}{(m-1)!}.$$

The fundamental principal system on $[c, d]$ for the adjoint equation, which is $v^{(n)} = 0$, is given by $(v_n, \ldots, v_1)$ with $v_m = u_{n-m+1}$. We use the following estimate:

$$|u_m^{(p)}(t)| \le \left(\frac{d-t}{d-c}\right)^{n-m-p} \frac{(t-c)}{(m-1)!}^{m-1-p} (n-1)(n-2)\ldots(n-p). \tag{3.1}$$

Hence, the functions $\rho_k = \rho_k(t; c, d, j)$ defined by (2.10) satisfy

$$\begin{aligned} \rho_k &\le \sum_{m \ne j} |u_m^{(k-1)}(t)| v_m(t) \\ &\le \left(\frac{d-t}{d-c}\right)^{n-k} (t-c)^{n-k} \frac{2^{n-1}-1}{(n-k)!}, \quad k = 2, \ldots, n-1; \end{aligned} \tag{3.2}$$

$$\rho_1 = \left(\frac{d-t}{d-c}\right)^{n-1} \frac{(t-c)^{n-1}}{(j-1)!(n-j)!} \le \left(\frac{d-t}{d-c}\right)^{n-1} \frac{(t-c)^{n-1}}{[(n-1)/2]!\,[n/2]!}, \tag{3.3}$$

$$\rho_n \le 1 + \sum_{m \ne j} |u_m^{(n-1)}(t)| v_m(t) \le 1 + \sum_{m \ne j} \frac{(n-1)!}{(m-1)!(n-m)!} \le 2^{n-1}. \tag{3.4}$$

Since the product $(d-t)(t-c)/(d-c)$ is monotone increasing in d and monotone decreasing in c, it is a trivial matter to compute the supremum with respect to all $[c, d] \subset (a, b)$ of the upper bounds computed for the ρ_k in (3.2)–(3.4). Letting these functions be the ν_k in (2.11) results in (1.7), and the theorem follows.

It is the case that the estimates made in (3.2)–(3.4) can be improved upon for particular values of n, for example, we obtain for $n=3$

COROLLARY 3.1. *If*

$$2\int_a^b |r_1(t)|\,dt+\int_a^b \frac{(t-a)(b-t)}{(b-a)}|r_2(t)|\,dt+\int_a^b \frac{(t-a)^2(b-t)^2}{(b-a)^2}\,|r_3(t)|\,dt \le 1, \tag{3.5}$$

then

$$y'''+r_1(t)y''+r_2(t)y'+r_3(t)y = 0 \tag{3.6}$$

is disconjugate on $[a, b)$.

As in the case of (1.9), condition (3.5) can be considered a generalization in some respects of the known de la Vallée Poussin tests for (3.6). Besides the results mentioned in §1, we note the test of Lasota [7] obtained for just the third-order equation:

$$\frac{(b-a)}{4}\,\|r_1\|_\infty+\frac{(b-a)^2}{\pi^2}\,\|r_2\|_\infty+\frac{(b-a)^3}{2\pi^2}\,\|r_3\|_\infty \le 1. \tag{3.7}$$

Corresponding to the coefficient triplet $(1/4, 1/\pi^2, 1/2\pi^2)$ in (3.7), condition (3.5) generates the triplet (2, 1/6, 1/30). See Richard [12] for a survey of known results for the second- and third-order equations.

We finish by illustrating for the second order case a way to reduce the effect of the coefficient $r_1(t)$. We consider the equation

$$y'' = r_1(t)y'+r_2(t)y \tag{3.8}$$

as a perturbation of

$$Lu = u''-r_1(t)u' = 0. \tag{3.9}$$

The adjoint equation to $Lu=0$ is

$$L^*v = v''+(r_1v)' = 0, \tag{3.10}$$

and it is a simple matter to compute the fundamental principal systems for (3.9) and (3.10) in terms of constants and the function

$$E(t) = \exp\left(\int_a^t r_1(s)\,ds\right).$$

In this case

$$v_1(t) = \left(\int_a^t E(s)\,ds\right)\left(\int_t^b E(s)\,ds\right)/E(t)\left(\int_a^b E(s)\,ds\right),$$

so that Theorem 2.3 implies

COROLLARY 3.2. *If*

$$\int_a^b |r_2(t)|E^{-1}(t)\left(\int_a^t E(s)\,ds\right)\left(\int_t^b E(s)\,ds\right)dt \le \int_a^b E(s)\,ds, \tag{3.11}$$

then (3.8) *is disconjugate on* $[a, b)$.

For a more complete analysis including generalizations of (3.11) for second order linear equations, see Willett [18].

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