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Linear Groups Generated by Reflection Tori

Dedicated to Professor Coxeter

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Abstract. A reflection is an invertible linear transformation of a vector space fixing a given hyperplane, its axis, vectorwise and a given complement to this hyperplane, its center, setwise. A reflection torus is a one-dimensional group generated by all reflections with fixed axis and center.

In this paper we classify subgroups of general linear groups (in arbitrary dimension and defined over arbitrary fields) generated by reflection tori.

1 Introduction

Let V be a left vector space over a (possibly commutative) skew field k.

For $g \in \mathrm{GL}(V)$, we set

$$[V,g] = \{vg - v \mid v \in V\} \quad \text{and} \quad C_V(g) = \{v \in V \mid vg - v = 0\},$$

and call these subspaces the *center* and *axis* of g. A transformation $g \in \mathrm{GL}(V)$ satisfying $\dim([V,g]) = 1$ is called a *transvection* if $[V,g] \subseteq C_V(g)$, and a *reflection* or pseudo-reflection otherwise. Observe that $C_V(g)$ is a hyperplane if g is a transvection or a reflection.

Let V^* be the dual of V, considered as a right vector space over k^{opp}, acting from the right on V. For every $v \in V$ and $\phi \in V^*$, we define the map $r_{v,\phi} \colon V \to V$ by

$$xr_{v,\phi} = x - x\phi v.$$

The map $r_{v,\phi}$ is a *reflection* with *center* $\langle v\rangle$ and *axis* $\ker\phi$ (or $\langle\phi\rangle$), provided $v\phi \neq 0, 1$. If both v and ϕ are nonzero but $v\phi = 0$, the element $r_{v,\phi}$ is a transvection. In fact, every transvection or reflection in $\mathrm{GL}(V)$ is equal to $r_{v,\phi}$ for some $v \in V$ and $\phi \in V^*$.

If we specify a hyperplane H and a one-dimensional subspace, that is, a *projective point*, p of V, then by $T_{p,H}$ we denote the subgroup of $\mathrm{GL}(V)$ generated by all $g \in \mathrm{GL}(V)$ with $p = [V,g]$ and $H = C_V(g)$. If $p \in H$, the subgroup $T_{p,H}$ consists of the identity and all transvections with center p and axis H. Such groups are often referred to as *transvection subgroups* of $\mathrm{GL}(V)$. They are isomorphic to the additive group of k. Groups generated by transvection subgroups have been studied extensively by McLaughlin, *cf.* [8], [9], and later, by Cameron and Hall, *cf.* [2]. An analogous study of the subgroups of $\mathrm{GL}(V)$ generated by groups of the form $T_{p,H}$ with $p \notin H$ is the purpose of this paper. In fact, this paper is inspired by the general and geometric setup of Cameron and Hall. In particular, the study of the geometry of points and hyperplanes appearing as centers and axes, respectively, as well as various elementary observations parallels their approach. Moreover, the generic results

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are comparable to those of Cameron and Hall; for small fields, however, more exceptional cases arise, which require special treatment. Note that in our situation $T_{p,H}$ is isomorphic to the multiplicative group k^* of k. Pursuing the analogy, we might have called such groups reflection subgroups, but, in order to avoid confusion with existing terminology and to emphasize that $T_{p,H}$ is a torus (*i.e.*, an Abelian diagonizable subgroup of $\mathrm{GL}(V)$), we prefer the name *reflection torus*.

All reflection tori in $\mathrm{GL}(V)$ generate the full finitary general group $\mathrm{FGL}(V)$ of V, *i.e.*, the subgroup of $\mathrm{GL}(V)$ consisting of all elements $g \in \mathrm{GL}(V)$ with $[V,g]$ finite dimensional. Below we will describe more examples of groups generated by reflection tori, examples that will reoccur in our classification.

Let W^* be a subspace of V^*. The *annihilator* $\mathrm{Ann}(W^*)$ of W^* is defined to be the subspace $\{v \in V \mid v\phi = 0 \text{ for all } \phi \in W^*\}$ of V.

By $R(V, W^*)$ we denote the subgroup G of $\mathrm{GL}(V)$ generated by the reflections $r_{v,\phi}$ with $v \in V$ and $\phi \in W^*$. If $W^* = V^*$, then G is equal to the full finitary general linear group $\mathrm{FGL}(V)$. If $W^* \neq V^*$ but $\mathrm{Ann}(W^*) = 0$, then $R(V, W^*)$ still acts irreducibly on V. Indeed, each point in $P(V)$ is the center of some reflection in G. So, $[U, G] := \langle [U, g] \mid g \in G\rangle = V$ for any nontrivial subspace U of V. Hence G is irreducible on V.

The groups $R(V, W^*)$ are the generic examples of groups generated by reflection tori. But over small fields there are a few other classes of examples.

First consider the case where $k = \mathbb{F}_3$. Then a reflection torus consists of just two elements, the identity and a unique reflection. The orthogonal group $\mathrm{O}(V, Q)$ where Q is some nondegenerate orthogonal form on V contains reflections. Indeed, if $v \in V$ with $Q(v) \neq 0$, then

$$r_v \colon w \in V \mapsto w + b(w, v)Q(v)v$$

is an *orthogonal reflection* in $\mathrm{O}(V, Q)$ with center v and axis $v^\perp = \{w \in V \mid b(w, v) = 0\}$. Here b is the bilinear form associated to Q.

The orthogonal reflections of $\mathrm{O}(V, Q)$ fall into two conjugacy classes, according as the center of a representative reflection contains a vector v with $Q(v) = 1$ or $Q(v) = -1$, respectively. The reflections in each of these classes generate a subgroup of index at most two in the finitary orthogonal group $\mathrm{FO}(V, q)$, *i.e.*, the subgroup of finitary elements in $\mathrm{O}(V, q)$. See for example [6], [7].

For $k = \mathbb{F}_3$, the group $\mathrm{GL}(V)$ contains more subgroups generated by reflections. The following examples are all closely related to the real reflection groups as studied by Coxeter [5].

Let $\mathcal{B}$ be a basis of the vector space V over the field $k = \mathbb{F}_3$. For distinct $b, b' \in \mathcal{B}$ we define the reflection $r_{b\pm b'}$ to be the reflection with center $\langle b \pm b'\rangle$ and axis $\langle b \mp b', \mathcal{B} \setminus \{b, b'\}\rangle$ of V. Furthermore, by r_b we denote the reflection with center $\langle b\rangle$ and axis $\langle \mathcal{B} \setminus \{b\}\rangle$. Now consider the following groups G:

- $G = W_3(A_{\mathcal{B}})$ is the group generated by the reflections $r_{b-b'}$, $b, b' \in \mathcal{B}$.
- $G = W_3(D_{\mathcal{B}})$ is the group generated by the reflections $r_{b\pm b'}$, $b, b' \in \mathcal{B}$.
- $G = W_3(B_{\mathcal{B}})$ is the group generated by the reflections r_b and $r_{b\pm b'}$, $b, b' \in \mathcal{B}$.

These groups will be called *Weyl groups mod* 3 of type A, D, B, respectively. If $\mathcal{B}$ has finite order n, then $W_3(A_{\mathcal{B}}) \simeq W(A_{n-1})$, $W_3(D_{\mathcal{B}}) \simeq W(D_n)$ and $W_3(B_{\mathcal{B}}) \simeq W(B_n)$. In this case we also use the following notation: $W_3(A_{\mathcal{B}}) = W_3(A_{n-1})$, $W_3(D_{\mathcal{B}}) = W_3(D_n)$ and $W_3(B_{\mathcal{B}}) = W_3(B_n)$.

The subspace $[V, G]$ of V is an irreducible module for G, except when $G = W_3(A_n)$ and $3 \mid n + 1$. If G is of type A, it is a hyperplane of V, in all other cases it equals V. We call $[V, G]$ the *natural* $\mathbb{F}_3$ *reflection module* for G. When $G = W_3(A_n)$ and $3 \mid n + 1$, then the G-invariant 1-space $K = \langle b_1 + \cdots + b_{n+1} \rangle$ is contained in $[V, G]$. The (irreducible) module $[V, G]/K$ is called the ($n - 1$-dimensional) *quotient* of the natural $\mathbb{F}_3$ reflection module for G.

Not only the classical Weyl groups appear as groups generated by reflection tori, also the exceptional ones do. Let Λ be a root lattice of type X, with X equal to G_2, F_4, or E_n, $n = 6, 7, 8$, and consider $\overline{\Lambda} := \Lambda/3\Lambda$, as a vector space over $\mathbb{F}_3$. The reflections in the Weyl group $W(X)$ induce reflections on $\overline{\Lambda}$. So the Weyl group $W(X)$ induces a group $W_3(X)$ on $\overline{\Lambda}$ generated by reflection tori. As above, the group $W_3(X)$ is called a *Weyl group mod* 3 of type X and the module $\overline{\Lambda}$ the natural reflection module for G. It is irreducible when G is of type F_4, E_7 or E_8. If $G = W_3(E_6)$, then the module is reducible and admits a 5-dimensional irreducible quotient, called the *quotient* of the natural module. In fact, $W_3(E_6)$ is the group generated by one class of orthogonal reflections. Among the orthogonal groups and Weyl groups mod 3 we find the following inclusions.

$W_3(A_1) < W_3(B_2) = \mathrm{O}(\mathbb{F}_3^2, Q)$, for Q nondegenerate of Witt index $-$.
$W_3(A_1 \times A_1) = \mathrm{O}(\mathbb{F}_3^2, Q)$, for Q nondegenerate of Witt index $+$.
$W_3(A_2) < W_3(G_2) = \mathrm{O}(\mathbb{F}_3^2, Q)$, for Q nonzero, degenerate.
$W_3(D_3) = W_3(A_3) < W_3(B_3) = \mathrm{O}(\mathbb{F}_3^3, Q)$ for Q nondegenerate.
$W_3(D_4) < W_3(B_4) < W_3(F_4) = \mathrm{O}(\mathbb{F}_3^4, Q)$, for Q nondegenerate of Witt index $+$.
$W_3(A_4) < W_3(A_5) < \mathrm{O}(\mathbb{F}_3^4, Q)$, for Q nondegenerate of Witt index $-$.
$W_3(E_6) < \mathrm{O}(\mathbb{F}_3^5, Q)$ for Q nondegenerate.

Now suppose that k is the field $\mathbb{F}_4$. Let h be a Hermitian form on V. For each vector $v \in V$ with $h(v, v) = 1$ and $\alpha \in k$, $\alpha \neq 0, 1$, the map

$$r_v \colon w \in V \mapsto w + \alpha h(w, v)v$$

is a reflection with center v and axis $v^\perp = \{w \in V \mid h(w, v) = 0\}$. This reflection is an element of the finitary unitary group $\mathrm{FU}(V, h) = \{g \in \mathrm{FGL}(V) \mid \forall_{x,y \in V} \quad h(xg, yg)) = h(x, y)\}$.

The reflection r_v is of order 3 and generates the *unitary reflection torus* $T_{\langle v\rangle, v^\perp}$. If the Hermitian form h is nondegenerate, then $\mathrm{FU}(V, h)$ is generated by all reflection tori $T_{\langle v\rangle, v^\perp}$ with $h(v, v) = 1$.

Finally, when $k = \mathbb{F}_5$, we find one more exceptional example of a group generated by reflection tori. Consider the following reflections in $\mathrm{GL}_2(5)$ (with respect to some basis of $\mathbb{F}_5^2$):

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} (\alpha+1)/2 & (\alpha-1)/2 \\ (\alpha-1)/2 & (\alpha+1)/2 \end{pmatrix}$$

with $\alpha \in k^*$. Let G be the subgroup of $\mathrm{GL}_2(5)$ generated by these reflections. Then G acts on the projective line over k with kernel of order 4. Moreover, the partition of the 6 points of this projective line into the pairs $\big(\langle(1,0)\rangle, \langle(0,1)\rangle\big)$, $\big(\langle(1,1)\rangle, \langle(1,-1)\rangle\big)$ and

$\big(\langle(1,2)\rangle, \langle(1,-2)\rangle\big)$ is G-invariant. From this we easily deduce that G induces the group S_4 on the 6 points. Hence $G \simeq 4 \cdot S_4$. The group G will be denoted by $\mathcal{S}$. The group is isomorphic to a finite complex reflection group in $\mathrm{GL}\big(2, \mathbb{Q}(i)\big)$ generated by reflections of order 4 (the group with Shephard-Todd number 8, see [10]).

Our first result is the following theorem.

Theorem 1.1 *Suppose that G is an irreducible subgroup of $\mathrm{GL}(V)$ generated by reflection tori. Then we have one of the following cases.*

1. *$G = R(V, W^*)$, where W^* is a subspace of V^* with $\mathrm{Ann}(W^*) = 0$; the group G contains a unique conjugacy class of reflection tori.*
2. *$k = \mathbb{F}_3$ and G is a Weyl group mod 3 of type A, B, D, F_4 or E_n, $n = 6, 7, 8$. The space V is the natural $\mathbb{F}_3$ reflection module for G, or its irreducible quotient when $G = W_3(A_n)$ and $3 \mid n+1$, or $G = W_3(E_6)$.*
3. *$k = \mathbb{F}_3$ and G is a subgroup of index at most 2 in $\mathrm{FO}(V, Q)$ for some nondegenerate orthogonal form Q generated by one or both classes of orthogonal reflections.*
4. *$k = \mathbb{F}_4$ and $G = \mathrm{FU}(V, h)$ for some nondegenerate Hermitian form h. The group G is generated by its unique class of unitary reflection tori.*
5. *$k = \mathbb{F}_5$, $\dim(V) = 2$, and G is isomorphic to the group $\mathcal{S}$.*

Notice that there is some overlap between the cases 2 and 3 in the above theorem.

As in Cameron and Hall [2] we find exceptional cases of groups generated by reflection tori for small fields.

Vavilov [11] also studied irreducible linear groups generated by reflection tori. He considered the case where V is a finite dimensional vector space over a field with at least 7 elements, and (only) found the groups $\mathrm{GL}(V)$. Vavilov's proof consists of the construction of full transvection subgroups. This enables him to quote the results of McLaughlin [8], [9] (or Cameron and Hall [2]). It also explains why the exceptional cases in our theorem only occur for small fields.

In this paper we do not restrict our attention to irreducible subgroups of $\mathrm{GL}(V)$ generated by reflection tori, but also consider reducible ones. Reducible groups can of course be obtained by taking the direct product of irreducible groups acting on the direct sum of the corresponding modules. But there are other ways to produce reducible examples, as we will now explain.

As before let V be a left vector space over a (skew) field k and let V^* be its dual acting from the right on V. (We consider V^* as a right vector space over k^{opp}.) The elements of $\mathrm{GL}(V)$ act on V from the right. If G is a subgroup of $\mathrm{GL}(V)$, then it also acts on V^*. Indeed, if for $g \in G$ and $\phi \in V^*$ we define ϕg to be the linear map

$$\phi g \colon v \in V \mapsto (vg^{-1})\phi,$$

then $\phi \mapsto \phi g$, $\phi \in V^*$ and $g \in G$, defines an action of G on V^*.

A reflection or transvection $r = r_{v,\phi}$ induces a homomorphism $w \in V \mapsto -[w, r] = (w\phi)v$ from V to V. So we can identify $r_{v,\phi}$ with $v \otimes \phi \in V \otimes V^* \big(\simeq \mathrm{Hom}(V, V)\big)$.

Now suppose $G_0 \leq \mathrm{GL}(V_0)$ is generated by its normal (*i.e.*, invariant under conjugation) set $\mathcal{R}_0$ of reflection tori. Let U_0 be a nontrivial k-vector space different from V_0 and consider the direct sum $V = U_0 \oplus V_0$. We extend the action of G_0 to V trivially, *i.e.*, G_0 centralizes

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U_0. Then G_0 also acts on the tensor product $U_0 \otimes V_0^*$ and we can form the split extension $G = (U_0 \otimes V_0^*) : G_0$ of $U_0 \otimes V_0^*$ by G_0.

As explained above, a nonzero element $u \otimes \phi \in U_0 \otimes V_0^*$ can be identified with a transvection in $\mathrm{GL}(V)$ with center $\langle u \rangle$ and axis $U_0 + \ker(\phi)$. In this way the split extension G acts naturally on $V = U_0 \oplus V_0$. It is straightforward to check that the normal set of reflection tori in this split extension containing $\mathcal{R}_0$ generates the subgroup $(U_0 \otimes [V_0^*, G_0]) : G_0$ of G. This split extension is clearly reducible as U_0 is an invariant subspace of V. The spirit of Theorem 1.2 (see below) is to show that, at least for a single orbit of centers, the above split extension is the only way to construct reducible examples of groups generated by reflection tori.

Before stating this theorem, we need some more notation. If a group $G \leq \mathrm{GL}(V)$ is generated by a normal set $\mathcal{R}$ of reflection tori, the set of all centers of elements from $\mathcal{R}$, called the *center set*, is a union of G-orbits. For any subset Σ of this set of centers we denote by $G(\Sigma)$ the subgroup of G generated by those elements from $\mathcal{R}$ that have their centers in Σ. The subspace $[V, G(\Sigma)]$ of V equals $\langle \Sigma \rangle$ and is $G(\Sigma)$ invariant. By $G\big((\Sigma)\big)$ we denote the group induced by $G(\Sigma)$ on $\langle \Sigma \rangle$. If U is a subspace of V, and Δ a subset of $P(V)$, then $\Delta \cap U$ denotes, by abuse of notation, the set $\{d \in \Delta \mid d \subseteq U\}$. With this notation our most general result reads as follows.

Theorem 1.2 *Suppose that G is a subgroup of $\mathrm{GL}(V)$ generated by reflection tori. If Σ is a G-orbit of centers, then the subspace $\langle \Sigma \rangle$ can be written as $U_0 \oplus V_0$ for some G-invariant space $U_0 \leq C_V\big(G(\Sigma)\big)$, such that one of the following holds.*

1. *$G_0 := G\big((\Sigma \cap V_0)\big)$ is irreducible on V_0, as listed in Theorem 1.1, and acts transitively on $\Sigma \cap V_0$.*
2. *$G_0 := G\big((\Sigma \cap V_0)\big)$ is isomorphic to $W_3(A_n)$ with $3 \mid n+1$, or $W_3(E_6)$, and V_0 is the reducible natural $\mathbb{F}_3$ reflection module for G of dimension n or 6, respectively. Moreover, G_0 is transitive on $\Sigma \cap V_0$.*
3. *$G(\Sigma)$ is not transitive on Σ and there is a unique second G-orbit Σ' of centers in $\langle \Sigma \rangle$ such that $G_0 := G\Big(\big((\Sigma \cup \Sigma') \cap V_0\big)\Big)$ is a Weyl group mod 3 of type B acting on its natural $\mathbb{F}_3$ reflection module V_0.*

Moreover, the group $G\big((\Sigma)\big)$, respectively, $G\big((\Sigma \cup \Sigma')\big)$ in case 3, is isomorphic to the split extension of $U_0 \otimes [V_0^*, G_0]$ by G_0 acting naturally on $U_0 \oplus V_0$, where G_0 is as defined in the appropriate case.

Now suppose $G \leq \mathrm{GL}(V)$ is a group generated by reflection tori. For each orbit Σ of centers the group $G(\Sigma)$ is a normal subgroup of G. If, for some orbit Σ the space U_0 as defined in Theorem 1.2 is nontrivial, then another set of centers could be contained in U_0. More details on several orbits can be found in the last section, notably Theorem 11.1.

The main topic of the remainder of this paper is the proof of Theorem 1.2. It is divided into the following parts. In Section 2 we derive some general properties of reflections and reflection tori. Section 3 is devoted to the case where the dimension of V is equal to 2. The purpose of Section 4 is to set the case leading to the third part of the conclusion in Theorem 1.2 apart. Then in Sections 5 and 6 we handle the generic case. In particular, it will become clear that the non-generic cases can only occur when the field k contains at most 5 elements. Section 7 handles the unitary groups over $\mathbb{F}_4$, Section 8 the orthogonal groups over $\mathbb{F}_3$ and Section 9 the Weyl groups mod 3. The completion of the proof of Theorem 1.2 can be found in Section 10. In Section 11 we discuss the case where G has more than one orbit on the centers of the reflections of elements in $\mathcal{R}$. In particular, we prove Theorem 1.1. We end this paper with an appendix on graphs and trees, relevant to Section 9.

2 Reflections

We will study the right action of reflections and reflection tori on a left vector space V over the (skew) field k and its dual V^*.

Lemma 2.1 *For every $v \in V$ and $\phi \in V^*$ with $v\phi \neq 0, 1$, the reflection $r_{v,\phi}$ belongs to* $\mathrm{GL}(V)$ *and satisfies the following properties.*

1. *The inverse of $r_{v,\phi}$ equals $r_{v,\phi(v\phi-1)^{-1}}$.*
2. *For each $\lambda \in k$, $r_{v,\phi\lambda} = r_{\lambda v,\phi}$.*
3. *If $g \in \mathrm{GL}(V)$, then $g^{-1} r_{v,\phi} g = r_{vg,\phi g}$.*
4. *Let $r_{w,\psi}$ be a second reflection with $\langle v\rangle \neq \langle w\rangle$ or $\langle \phi\rangle \neq \langle \psi\rangle$. Then $r_{v,\phi}$ and $r_{w,\psi}$ commute if and only if $v \in \ker\psi$ and $w \in \ker\psi$.*

The proof can be obtained by straightforward calculations.

The lemma implies that the reflection $r_{v,\phi}$ (with $v\phi \neq 0, 1$) is contained in a unique reflection torus of $\mathrm{GL}(V)$, which we shall denote by $T_{p,H}$, where $p = \langle v\rangle$ and $H = \ker\phi$.

For a geometric approach, it is convenient to consider the action of reflections on the projectivised spaces $P(V)$ and, if necessary, $P(V^*)$. A center determines a point in $P(V)$, an axis a hyperplane in $P(V)$ (and a point in $P(V^*)$). It depends on the context whether we consider p as a point in $P(V)$ or a line in V, *etc.*

The next lemma deals with subspaces invariant under a reflection. Its proof is elementary and is omitted.

Lemma 2.2 *Let r be a reflection with center p and axis H. A subspace of V is invariant under r if and only if it contains p or is contained in H.*

Proposition 2.3 *Let $T_{p,H}$ be a reflection torus. Then:*

1. *For every $q \in H$, the torus $T_{p,H}$ is transitive on the points of $pq \setminus \{p, q\}$.*
2. *Dually, for every hyperplane K containing p, the torus $T_{p,H}$ is transitive on the hyperplanes of $HK \setminus \{H, K\}$, i.e., the set of hyperplanes containing $H \cap K$ but different from H and K.*

Proof Straightforward. ■

Let G be a group containing a *normal* (that is, G-conjugation invariant) set $\mathcal{R}$ of reflection tori. Let $A_p = A_p^{\mathcal{R}}$ be the intersection of all axes of reflections in the reflection tori in $\mathcal{R}$ with center p. It is evident that any element of G that fixes p, leaves A_p invariant; similar considerations hold for the dual situation, but we refrain from setting up notation for it.

Corollary 2.4 *Let $T = T_{p,H}$ be a reflection torus in G with center $p = \langle v \rangle$. There exists a subspace W^* of V^* with $\mathrm{Ann}(W^*) = A_p$ such that for each $\psi \in W^*$ with $v\psi \neq 0, 1$, there is a reflection torus $T_{p,\ker(\psi)}$ in $\mathcal{R}$.*

Proof Proposition 2.3 implies that the set of hyperplanes that occur as axis for some reflection torus in $\mathcal{R}$ with center p is an affine subspace A of the affine space we obtain by removing all hyperplanes on p from $P(V^*)$. Let W^* be the subspace of V^* spanned by A. Then for every element $\psi \in W^*$ with $v\psi \neq 0, 1$, the kernel $\ker(\psi)$ is in A and appears as an axis of some reflection torus in $\mathcal{R}$. Since the intersection of all hyperplanes in A equals A_p, we find $\mathrm{Ann}(W^*) = A_p$. ■

Let G be a subgroup of $\mathrm{GL}(V)$ generated by a normal set $\mathcal{R}$ of reflection tori. We recall some notation from the introduction. Let $\Sigma = \Sigma_{\mathcal{R}}$ be the set of centers of the reflections belonging to reflection tori in $\mathcal{R}$. If $p \in \Sigma$, we shall also say that p is a center of $\mathcal{R}$. For any subset Δ of $P(V)$, we denote by $G(\Delta) = G^{\mathcal{R}}(\Delta)$ the subgroup of G generated by those reflection tori in $\mathcal{R}$ whose centers are in Δ. By $G\big((\Delta)\big) = G^{\mathcal{R}}\big((\Delta)\big)$ we denote the quotient group of $G(\Delta)$ induced on the subspace of V spanned by Δ. (Notice that this subspace is invariant under $G(\Delta)$.) Recall that $A_p = A_p^{\mathcal{R}}$ is the intersection of all axes of reflections of reflection tori in $\mathcal{R}$ with center p. For a subset Δ of Σ, we shall write $A_\Delta = \bigcap_{x \in \Delta} A_x$.

Lemma 2.5 *For each $g \in G$ and subspace W of V we have: $Wg \leq W \Rightarrow Wg = W$.*

Proof As $g \in G$ is a product of a finite number of reflections, the subspace $C_V(g)$ has finite codimension. It is obvious that $g \in G$ induces a bijective map $W/W \cap C_V(g) \to Wg/C_V(g) \cap W$ of finite dimensional spaces. So the dimensions are equal and, since $Wg \subset W$, this implies that $W = Wg$. ■

Lemma 2.6 *Let $a, b \in \Sigma$ with $a \in A_b$. Then $b \in A_a$ provided at least one of the following conditions holds:*

i. A_a is a hyperplane;
ii. a and b are in one G-orbit on Σ.

Moreover, if $a \in A_b$ and $b \in A_a$, and if $r \in \mathcal{R}$ has center a and $s \in \mathcal{R}$ has center b, then $rs = sr$.

Proof Suppose $a \in A_b$. Then, for any reflection r of $\mathcal{R}$ with center b, we have $ar = a$ and thus $A_a r = A_a$. By Lemma 2.2, either $b \in A_a$ or $A_a \subset A_b$, where the inclusion is strict as $a \in A_b \setminus A_a$.

The latter possibility cannot occur if A_a is a hyperplane. So in case (i) we do have $b \in A_a$.

If a and b are in the same G-orbit, then there is an element $g \in G$ with $a = bg$ and hence $A_a = A_b g$. But then $A_b g \leq A_b$, and by Lemma 2.5 this implies that $A_b g = A_b$. So again we have $b \in A_a$.

The last statement follows from Lemma 2.1. ■

3 Subgroups of GL_2 Generated by Reflection Tori

We retain the setting in which G is a subgroup of $\mathrm{GL}(V)$ generated by a normal set $\mathcal{R}$ of reflection tori and Σ is the set of centers of the reflections belonging to reflection tori in $\mathcal{R}$.

Let l be a line of $P(V)$. If l meets Σ in at least 3 points, then it is called a *thick* line. If all points on l are in Σ, then l is called *full*. A line l meeting Σ in just two points is called *thin*. A Σ-line is the intersection of a thick line with Σ. The set of all Σ-lines is denoted by $\mathcal{L}(\Sigma)$. Clearly, the pair $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is a partial linear space. A Σ-*plane* is a subspace of $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ spanned by two intersecting Σ-lines. Clearly each point of a Σ-plane lies in the projective plane spanned by any two of its thick lines.

In this section we concentrate on the groups $G(l)$ and $G\big((l)\big)$. First we analyze the action on lines in general.

Lemma 3.1 *Suppose that l is a thin line with $\Sigma \cap l = \{a, b\}$. Then $A_b \cap l = \{a\}$ and $A_a \cap l = \{b\}$. In particular, $G(a)$ and $G(b)$ commute.*

Proof Let H be a hyperplane such that $T_{a,H} \in \mathcal{R}$. Then, as b is the only center of l distinct from a, we have $T_{a,H} b = b$, so $b \in H$. Hence $b \in A_a$. Similarly, $a \in A_b$. The last statement of the lemma then follows from Lemma 2.1. ■

Lemma 3.2 *Suppose that l is a thick line. Then we have one of the following:*

i. the line l is full;

ii. there is a unique point $p \in l$ which is not in Σ; this point p is contained in A_a for all points $a \in l \cap \Sigma$. The group $G\big((l)\big)$ is doubly transitive on $l \setminus \{p\}$.

Proof Suppose that l is thick but not full. Then there is a point $p \in l \setminus \Sigma$, and $l \cap \Sigma$ has at least three points. For each $x \in l \cap \Sigma$, the group $G(x)$ stabilizes l and preserves $l \cap \Sigma$. Since $l \cap \Sigma$ contains at least 3 points, we find that $G(x)$ fixes p and acts transitively on $l \setminus (\{x, p\})$, which is contained in Σ. It follows that $G\big((l)\big)$ is doubly transitive on $l \setminus \{p\} = l \cap \Sigma$. Moreover, $p \in \bigcap_{x \in l} A_x$. ■

Lemma 3.3 *If, for two distinct centers $a, b \in \Sigma$, we have $b \in A_a$ and $a \notin A_b$, then $l = ab$ is a full line with $A_b \cap l = \varnothing$ and $A_x \cap l = \{b\}$ for all $x \in l \setminus \{b\}$; moreover, the group $G\big((l)\big)$ fixes $\{b\}$ and is doubly transitive on $l \setminus \{b\}$.*

Proof As $a \notin A_b$, there is a hyperplane H such that $p = H \cap l$ is a point distinct from a and b and $T_{b,H} \in \mathcal{R}$. Now $T_{b,H}$ acts transitively on $l \setminus \{p, b\}$, and, as $b \in A_a$, $G(a)$ acts transitively on $l \setminus \{a, b\}$. So all points $x \in l \setminus \{b\}$ are centers in Σ with $A_x = A_a = \{b\}$. In particular, l is a full line. The rest follows easily. ■

We can now completely classify the subgroups of GL_2 generated by a normal set of reflection tori.

Theorem 3.4 *Let G be a subgroup of $\mathrm{GL}(V)$ generated by a normal set $\mathcal{R}$ of reflection tori and let Σ be the set of centers of the reflections belonging to reflection tori in $\mathcal{R}$. Let W be a 2-dimensional subspace of V and $l = P(W)$. If l meets Σ in at least two points then we are in one of the following cases.*

i. *l is thin and $G\big((l)\big)$ is a direct product of two commuting reflection tori.*
ii. *l is thick but not full and with respect to a suitable basis of W, the group $G\big((l)\big)$ is isomorphic to the subgroup*
$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} \;\middle|\; \beta \in k, \gamma \in k^* \right\}$$
of $\mathrm{GL}(W)$.
iii. *l is full and misses the set A_x for each point $x \in l$. The group $G\big((l)\big)$ coincides with* $\mathrm{GL}(W)$.
iv. *l is full and there is a center $p \in l$ with $A_x \cap l = \{p\}$ for all $x \in l \setminus \{p\}$. With respect to a suitable basis of W, the group $G\big((l)\big)$ is isomorphic to*
$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \;\middle|\; \beta \in k, \quad \alpha, \gamma \in k^* \right\}$$
of $\mathrm{GL}(W)$.
v. *l is full, $k = \mathbb{F}_3$ and the induced group $G\big((l)\big)$ is isomorphic to the group $W_3(B_2)$ in its natural action on W.*
vi. *l is full, $k = \mathbb{F}_5$ and $G\big((l)\big)$ is isomorphic to $\mathcal{S}$ acting naturally on W.*

Proof If the line l is thin, then we are in case (i) as follows from 3.1.

If the line is thick but not full, then Lemma 3.2(ii) implies that we are in case (ii).

Now suppose the line l is full. If there are points a and b on l with $a \in A_b$ but $b \notin A_a$, then we can apply the previous lemma to find that we are in case (iii) of the statement of the theorem.

So now we can assume that $a \in A_b$ implies $b \in A_a$. Suppose that on l there is a point x such that l does not meet A_x. Then, by Proposition 2.3, $G(x)$ acts transitively on $l \setminus \{x\}$. If y is a second point on l, then $y \notin A_x$ and thus, by our assumption, also $x \notin A_y$. So in $G(y)$ we find a reflection moving x. Hence $G\big((l)\big)$ is 2-transitive on l, and consequently l misses A_z for all $z \in l$.

Fix a point $x \in l$. The above implies that $G\big((l)\big)$ contains two distinct reflection tori with center x. But then Corollary 2.4 proves that $G\big((l)\big)$ contains all reflection tori in $\mathrm{GL}(W)$ with center x. As $G\big((l)\big)$ is transitive on l, it contains all possible reflection tori of $\mathrm{GL}(W)$ and thus equals $\mathrm{GL}(W)$.

For the remainder of the proof, we may assume that, for each $z \in l$, there is a unique point in $A_z \cap l$, which will be denoted by $z^\perp$.

Now suppose z is a point on $l \setminus \{x, x^\perp\}$. Inside $G\big((l)\big)$ there is a reflection r with center x such that $zr = z^\perp$. But then $(z^\perp)r = (zr)^\perp = z^{\perp\perp}$ which by Lemma 2.6 is equal to z. Thus r is the unique reflection of order 2 in $G\big((l)\big)$ with center x. In particular, the characteristic of k is 0 or odd. Moreover, since z is an arbitrarily chosen point different from x and $x^\perp$, we find $zr = z^\perp$ for all $z \in l \setminus \{x, x^\perp\}$.

Now fix a point y on l distinct from x and $x^\perp$. Then, by the same arguments as above, we find a reflection s with center y in $G\big((l)\big)$ that interchanges all points $z \in l \setminus \{y, y^\perp\}$ with $z^\perp$. So rs fixes all points of $l \setminus \{x, x^\perp, y, y^\perp\}$ and interchanges x with $x^\perp$ and y with $y^\perp$. But as only the identity in $\mathrm{PGL}(W)$ fixes more than 2 points, we find that l contains at most 6 points.

Hence, k is $\mathbb{F}_3$ or $\mathbb{F}_5$. If k is $\mathbb{F}_3$, then $G\big((l)\big)$ is equal to the group $W_3(B_2)$ acting naturally on W. If $k = \mathbb{F}_5$, then with $x = \langle(1,0)\rangle$ and $x^\perp = \langle(0,1)\rangle$ we find for $z = \langle(1,\alpha)\rangle$ that $z^\perp$

is equal to $\langle(-1,\alpha)\rangle$. Now it is easily checked that all reflections in $G\big((l)\big)$ are as described in Section 2. Hence $G\big((l)\big)$ is isomorphic to $\mathcal{S}$, acting naturally on W. ■

Remark 3.5 Consider the setting of the above theorem. In cases (ii), (iii) and (vi) the group $G(l)$ is transitive on $\Sigma \cap l$. In case (iv) the group $G(l)$ fixes one point of $\Sigma \cap l$ and is transitive on the remaining points. We notice that, as a consequence of Lemma 2.6, the unique point fixed by $G(l)$ is in another G-orbit on Σ than the remaining points of $\Sigma \cap l$.

In case (v) the group $G(l)$ has two orbits on l, both containing two points. If a and b are in the same $G(l)$-orbit then $a \in A_b$. We call such a line a B_2-*line*, as the group $G\big((l)\big)$ induced on the line is the Weyl group mod 3 of type B_2.

4 Some Reduction Theorems

Let $G \leq \mathrm{GL}(V)$ be generated by a normal set of reflection tori $\mathcal{R}$. By Σ we denote the set of centers of the tori in $\mathcal{R}$.

Lemma 4.1 *Suppose that $\Sigma_1 \subseteq \Sigma$ is a G-orbit. Then Σ_1 is contained in a connected component of $\big(\Sigma, \mathcal{L}(\Sigma)\big)$. In particular, if Σ is a G-orbit, then $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is connected.*

Proof Let r be a reflection of a reflection torus in $\mathcal{R}$ with center c. If a is an element in Σ_1 and $ar \neq a$, then the line ac contains at least three points from Σ (*viz.*, c, a, ar, as every reflection in G with center c moves the point a to a point on ac). Thus a and ar are on a Σ-line.

Since every $g \in G$ is the product of a finite number of reflections of reflection tori in $\mathcal{R}$, the centers a and ag are connected by a finite number of Σ-lines, which proves the first part of the lemma. The second part is now trivial. ■

Lemma 4.2 *If Σ_1 and Σ_2 are subsets of distinct connected components of $\big(\Sigma, \mathcal{L}(\Sigma)\big)$, then $G(\Sigma_1)$ and $G(\Sigma_2)$ centralize each other.*

Proof This is immediate, as the result is true for any two points on a thin line. ■

The above two lemmas are in fact our first reduction results. They allow us to restrict attention to the case where $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is connected. We will say that Σ is *connected* if the space $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is connected. For the remainder of this section we will assume Σ to be connected.

For our second reduction result we need the following definition. Let l be a thick line meeting $C_V\big(G(\Sigma)\big)$ nontrivially. Then, as $G(\Sigma) = G$, l meets $C_V(G)$ in a unique point. Such a line is called *degenerate* (with respect to Σ). It is as in case (ii) or (iv) of Theorem 3.4, where in the latter case there is a unique point on l not in Σ. Denote by U the subspace of V spanned by the intersection points of degenerate lines with $C_V(G)$. So

$$U := \langle l \cap C_V\big(G(\Sigma)\big) \mid l \text{ is a degenerate line}\rangle.$$

The subspace U of V will be called the *radical* of Σ and is denoted by $\mathrm{Rad}(\Sigma)$. We will call Σ *degenerate* if it does contain degenerate lines and *nondegenerate* otherwise.

We consider the action of $G = G(\Sigma)$ on V/U and denote by N the kernel of this action. A reflection r in G induces a reflection on V/U. Two reflection tori of $\mathcal{R}$ will induce the same reflection torus on V/U if and only if their axes are the same and their centers are the same modulo U.

Lemma 4.3 *Suppose that Σ spans V. Let l be a degenerate line and R a reflection torus in $\mathcal{R}$ with center x on l and axis H. For each $y \in l \cap \Sigma$ there is a reflection torus $R_y \in \mathcal{R}$ with center y and axis H. In particular, $RN = R_yN$.*

Proof Let R be a reflection torus with center x and axis H. Let $R' \in \mathcal{R}$ be a reflection torus with center $y \in l$ different from x and axis H'. Suppose that $H \neq H'$. Then, since V is spanned by Σ, there is a point $z \in \Sigma \setminus l$ such that the line xz does not meet $H \cap H'$. Let π be the plane spanned by l and z and denote by u the unique point on l in $C_V(G)$. The group $\langle R, R'\rangle$ acts doubly transitively on the lines in π through u but different from l. It easily follows from Theorem 3.4 that any line on u meets Σ in at most one point or is thick but not full.

The point z is either not in H or not in H'. In particular, one of the lines xz and yz is thick. If xz is thick, then $(xz)r'$ is thick for every $r' \in R'$. Similarly, if yz is thick, then $(yz)r$ is also thick for every $r \in R$. So, inside the plane π there are at least two thick, hence full, lines not through u. But that implies that at least two lines and hence all lines in π on u are thick. So we can assume the point z to be neither in H nor in H'.

If there is a reflection torus $S \leq G(z)$ that centralizes l, then there is an element $s \in S$ that centralizes l but maps $H' \cap zx$ to $H \cap zx$. The reflection torus $s^{-1}R's$ has center y and axis $H's$. As $H's \cap H'$ contains $H \cap H' \cap H_s$, where H_s is the axis of s, as well as the point $H \cap zx$, we find $H's \cap H'$ to be contained in H. Now Corollary 2.4 implies that there is a reflection torus $R_y \in \mathcal{R}$ with center y and axis H.

It remains to consider the case where there is no reflection torus $S \leq G\big((z)\big)$ that centralizes l. This is only possible when both xz and yz are as in case (iv), (v) or (vi) of Theorem 3.4. If xz is as in case (iv) of Theorem 3.4, then each reflection torus $S \leq G(z)$ centralizes $H \cap \pi$. Similarly, if yz is as in case (iv) of Theorem 3.4, then each reflection torus $S \leq G(z)$ centralizes $H' \cap \pi$. But, since $H' \cap \pi \neq H \cap \pi$, we find that at least one of xz or yz is as in case (v) or (vi) of Theorem 3.4.

Suppose that xz is as in case (v) or (vi) of Theorem 3.4. Let v be the intersection point of xz and H. Then inside $G(v)$ we find a reflection r_v centralizing l. The reflection torus $r_v^{-1}R'r_v$ has center y and axis $H'r_v$. As $H'r_v$ meets yz in a point distinct from $H' \cap yz$, we find yz to be of type (iv), but not every torus in $G(z)$ fixes $H' \cap \pi$. We have reached a contradiction.

This proves that, indeed, there is a reflection torus R_y with center y and axis H in $\mathcal{R}$. Clearly R and R_y induce the same reflection torus on V/U. ∎

Suppose that l is a degenerate line and that R and S are two reflection tori in $\mathcal{R}$ with distinct centers x and y, respectively, on l but with the same axis H. Then for each reflection $r \in R$ there is a (unique) reflection $s \in S$ such that r and s induce the same reflection on V/U. So $rN = sN$. The element $1 \neq rs^{-1} \in N$ centralizes H. In particular, $[V, rs^{-1}]$ is 1-dimensional and rs^{-1} is a reflection or transvection with center on l. Since rs^{-1} is trivial on V/U, we conclude that rs^{-1} is a transvection with center $l \cap H$. As explained in the introduction, we can identify this transvection with an element $u \otimes \phi \in U \otimes V^*$, where $\langle u \rangle$ is its center and $\ker(\phi)$ its axis.

Lemma 4.3 implies that the normal subgroup N of G contains $U \otimes [V^*, G]$. Let V_0 be a complement to U in V and define Σ_0 to be $\Sigma \cap V_0$. Then, Σ_0 is nondegenerate and $G_0 \cap N = 1$, where $G_0 := G(\Sigma_0)$. So G contains the semi-direct product $(U \otimes [V^*, G_0]) : G_0$. Moreover, as each degenerate line meets V_0 in a point, the above implies that $(U \otimes [V^*, G_0]) : G_0$ contains $\mathcal{R}$ and therefore equals G. Hence, we have proved the following.

Theorem 4.4 *Let $G \leq \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning V. Suppose that Σ is connected. Let V_0 be a complement in V to the radical U of Σ. Then the set $\Sigma_0 := \Sigma \cap V_0$ is nondegenerate and $G = (U \otimes [V_0^*, G_0]) : G_0$, where $G_0 = G(\Sigma_0)$.*

We notice that the space $P(U)$ equals $\{l \cap C_V(G) \mid l \text{ is a degenerate } \Sigma \text{ line}\}$. Indeed, if $x \in \Sigma_0$ and $u \otimes \phi \in U \otimes [V_0^*, G_0]$ with $u \neq 0$ and $x\phi \neq 0$ (notice that such a ϕ exists), then the line through x and $\langle u \rangle$ is thick and degenerate.

Let Σ_1 be a G-orbit on Σ. Our final reduction result is concerned with the action of $G(\Sigma_1)$ on Σ_1. Since the normal subgroup N as defined above acts trivially on the set of degenerate lines of Σ we find that, in the notation of the above theorem, $\Sigma_1 \cap \Sigma_0$ is a G_0-orbit. This will allow us to restrict our attention to the case where Σ is nondegenerate.

Lemma 4.5 *If no two points of Σ_1 are on a B_2-line in $\mathcal{L}(\Sigma)$, then $G(\Sigma_1)$ is transitive on Σ_1.*

Proof Suppose $y, z \in \Sigma_1$ and $yg = z$, for some $g \in G$. By assumption, $g = r_1 \cdots r_t$ where each r_i is a reflection with center, say, $c_i \in \Sigma$. By induction on t, it suffices to show that $y_1 = yr_1$ is in the $G(\Sigma_1)$-orbit of y. This is obvious if $y = y_1$ or $c_1 \in \Sigma_1$, so assume not. Then y and y_1 span a thick Σ-line, say l, containing the distinct centers yr ($r \in G(c_1)$) and c_1. Hence we are in case (iv) or (v) of Theorem 3.4. In case (iv) we find that $G(l \cap \Sigma_1)$, which is contained in $G(\Sigma_1) \cap G(l)$, is transitive on the Σ_1-line $l \cap \Sigma_1$. Case (v) is ruled out by assumption. ■

Now we consider the case where the G-orbit Σ_1 decomposes into several $G(\Sigma_1)$-orbits. So, by the above lemma, $k = \mathbb{F}_3$.

Lemma 4.6 *Let Δ_1 and Δ_2 be distinct $G(\Sigma_1)$-orbits in Σ_1. Then we have the following.*

(i) *Any line meeting both Δ_1 and Δ_2 is either thin or a B_2-line.*
(ii) $\Delta_2 \subseteq A_{\Delta_1}$.
(iii) $G(\Delta_1)$ *is transitive on* Δ_1.

Proof Let $a \in \Delta_1$ and $b \in \Delta_2$. Remark 3.5 implies that the line spanned by a and b is either thin or a B_2-line, whence (i). In either case we have $a \in A_b$ and $b \in A_a$. This proves (ii). Moreover, as Δ_1 is a $G(\Sigma_1)$-orbit, it also is a $G(\Delta_1)$-orbit, which proves (iii). ■

Let Σ_2 be the set of all centers x in $\Sigma \setminus \Sigma_1$ with $\Sigma_1 \not\subseteq A_x$. Any element of Σ_2 is on some thick Σ-line with an element of Σ_1. This line is then a B_2-line meeting A_x, $x \in \Sigma \setminus (\Sigma_1 \cup \Sigma_2)$

in at least 2 and hence all points. In particular, $\Sigma_1 \cup \Sigma_2$ is centralized by all reflection tori in $\mathcal{R}$ with center outside $\Sigma_1 \cup \Sigma_2$.

We can now prove the following result.

Theorem 4.7 *Let $G \leq \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ. If $\Sigma_1 \subseteq \Sigma$ is a G-orbit, then either*

(i) $G(\Sigma_1)$ is transitive on Σ_1, or
(ii) there is a unique second G-orbit $\Sigma_2 \subseteq \Sigma$ with $\mathrm{Rad}(\Sigma_1) = \mathrm{Rad}(\Sigma_1 \cup \Sigma_2)$ and $\langle \Sigma_1 \rangle = \langle \Sigma_1 \cup \Sigma_2 \rangle$, such that, for each complement V_0 in $[V, G(\Sigma_1)]$ to $\mathrm{Rad}(\Sigma_1)$, the group $G\big((\Sigma_1 \cup \Sigma_2) \cap V_0\big)$ equals $W_3(B_{\mathcal{B}})$ in its natural action on V_0. Here $\mathcal{B}$ is a basis for V_0.

Proof Let Σ_2 be the set of centers as constructed above. Since $\Sigma_1 \cup \Sigma_2$ is contained in A_x for each $x \in \Sigma \setminus (\Sigma_1 \cup \Sigma_2)$, we find that Σ_1 is a $G(\Sigma_1 \cup \Sigma_2)$-orbit, and Σ_2 is $G(\Sigma_1 \cup \Sigma_2)$-invariant. In particular, Lemma 4.1 implies that $\Sigma_1 \cup \Sigma_2$ is connected.

Suppose Δ_1, Δ_2 and Δ_3 are three orbits such that there exist B_2-lines meeting Δ_i and Δ_{i+1}, with $i = 1, 2$. Let d_i be a point in Δ_i and consider the plane spanned by d_1, d_2 and d_3.

Let $e \in d_1 d_2 \cap \Sigma_2$ and r a reflection in G with center e. Then $(d_2 d_3)r$ is a B_2-line through d_1. It is easily seen inside $\mathrm{GL}_3(3)$ that the reflections with center in the plane spanned by d_1, d_2 and d_3 induce the group $W \simeq W_3(B_3)$ on the plane. So, the line $(d_2 d_3)r$ equals the line $d_1 d_3$.

By the connectedness of $\Sigma_1 \cup \Sigma_2$ any two $G(\Sigma_1)$-orbits Δ and Δ' on Σ_1 are connected by a B_2-line. Moreover, it follows from Lemma 4.6 that $G(\Delta \cup \Delta')$ is transitive on $\Delta \times \Delta'$. This implies that every line meeting both Δ and Δ' is a B_2-line.

The group W intersects $G(\Sigma_2)$ in a Weyl group mod 3 of type D_3 which is transitive on the six points of Σ_2 in the plane spanned by d_1, d_2 and d_3. Each reflection in G with center in $\Sigma_2 \cap d_1 d_2$ centralizes d_3. Hence, these two reflections induce a transposition on the set of all $G(\Sigma_1)$-orbits switching Δ_1 and Δ_2. The above implies that Σ_2 is a single G-orbit.

Let l be a thick Σ_1 or Σ_2-line and m a B_2-line meeting both Σ_1 and Σ_2 in two points. Suppose that l and m meet nontrivially. Knowledge of reflection subgroups of $\mathrm{GL}_3(3)$ shows that the group generated by the reflections with center on l and m induces $3^2 : W_3(B_2)$ or $W_3(B_3)$ on the 3-space spanned by l and m. In the latter case l is a nondegenerate Σ_2-line. In the first case the plane spanned by l and m contains exactly two thick lines only meeting Σ_1 and two only meeting Σ_2. These 4 lines are not full and intersect in the unique point of the plane outside $\Sigma_1 \cup \Sigma_2$. This intersection point is centralized by all reflections in $G(l, m)$ and is the radical of the plane.

This has the following consequences. First, if l is a degenerate Σ_i-line, then there is a degenerate Σ_j-line l', $\{i, j\} = \{1, 2\}$, with $l \cap l'$ in the radical of both Σ_1 and Σ_2. So, $\mathrm{Rad}(\Sigma_1) = \mathrm{Rad}(\Sigma_2) = \mathrm{Rad}(\Sigma_1 \cup \Sigma_2)$.

Secondly, if Δ is a $G(\Sigma_1)$-orbit on Σ_1, then each thick Δ-line is not full, and the subspace Δ of $\big(\Sigma_1, \mathcal{L}(\Sigma_1)\big)$ is an affine space consisting of all points outside the hyperplane $U = \mathrm{Rad}(\Delta)$ of $\langle \Delta \rangle$. Notice that U is centralized by all reflections with center in Σ_1 and hence contained in $\mathrm{Rad}(\Sigma_1) = \mathrm{Rad}(\Sigma_1 \cup \Sigma_2)$. Theorem 4.4 now allows us to assume that Σ_1 is nondegenerate, so that each $G(\Sigma_1)$-orbit on Σ_1 consists of a single point.

Fix, in each point of Σ_1, a vector b spanning it and denote by $\mathcal{B}$ the set of all such vectors. We notice that $\mathcal{B}$ is a linearly independent set. Indeed, if $\lambda_1 b_1 + \cdots + \lambda_n b_n$, $\lambda_i \neq 0$, is a

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linear combination of elements $b_1, \dots, b_n$ from $\mathcal{B}$ which is 0, then applying the reflection in G with center b_1 will yield that also $-\lambda_1 b_1 + \cdots + \lambda_n b_n = 0$. Combining these two equations yields that $\lambda_1 = 0$ which is against our assumptions. As $[V, G(\Sigma_1)]$ is spanned by the centers in Σ_1, $\mathcal{B}$ is a basis of $[V, G(\Sigma_1)]$. Now let $b, b' \in \mathcal{B}$. On the B_2-line spanned by b and b' and in the corresponding reflection group we can check that the reflections r_b with center $\langle b\rangle$ and axis $\langle \mathcal{B} \setminus \{b\}\rangle$ and $r_{b\pm b'}$ with center $\langle b \pm b'\rangle$ and axis $\langle \mathcal{B} \setminus \{b, b'\}, b \mp b'\rangle$ are in G. Thus $G(\Sigma_1 \cup \Sigma_2)$ induces the group $W_3(B_{\mathcal{B}})$ on $[V, G(\Sigma_1)]$. ■

5 Planes in the Geometry of Reflection Tori

We keep the notation of the previous section. We will now assume that $G \leq \mathrm{GL}(V)$ is generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning V. Moreover, we assume Σ to be a single G-orbit and hence connected, *cf.* Lemma 4.1. The case where Σ consists of more than one G-orbit will be studied in Section 11. Moreover, the reduction results of the previous section allow us to assume that Σ is nondegenerate. Note, however, that this assumption need not be valid for $G\big((\Delta)\big)$, where Δ is any subset of Σ.

The following proposition plays a very important rôle in the sequel. The section is almost entirely devoted to its proof.

Proposition 5.1 *Let π be a Σ-plane and W the 3-dimensional subspace of V containing π. Then we have one of the following.*

i. *$P(W) \setminus \pi$ is the subspace $\bigcap_{x\in\pi} A_x$ of $P(W)$.*
ii. *$k = \mathbb{F}_3$, the group $G\big((\pi)\big)$ is the group $W_3(A_3) \simeq W_3(D_3)$ in its natural action on the root lattice modulo 3.*
iii. *$k = \mathbb{F}_4$ and there is a nondegenerate Hermitian form h on W such that the points of π are those subspaces $\langle w\rangle$ of W for which $h(w, w) = 1$. The group $G\big((\pi)\big)$ is $U(W, h)$.*

The Σ-planes as in case (i) of Proposition 5.1 are called *generic* planes. The planes in the other cases are called *exceptional planes*. The exceptional planes are *dual affine planes* of order 2 in case (ii) and of order 3 in case (iii).

Lemma 5.2 *If $|k| \geq 5$, then all planes are generic.*

Proof Let l and m be two thick lines intersecting in a point $x \in \Sigma$. Let y be a point on $l \cap \Sigma$ not in A_x. Then $x \notin A_y$ and we can find a second point $z \neq x$ in $m \cap \Sigma$ which is also not in A_y. So yz is also thick. On yz there are at least $|k| - 1$ points in Σ but not in A_x. Together with x, each of these points generates a thick line. Thus x is on at least $|k| - 1$ thick lines.

Now consider the plane $P(W)$. If there is at most one point of $P(W)$ not in Σ, then we are in case (i). Thus suppose r and s are distinct points in $P(W)$ but not in Σ. The line rs contains at most 2 points from Σ. Since we can replace x by y which is also on two thick lines of π, we may assume that x is not on rs.

If all points of $P(W)$ that are not in Σ are inside rs, we find two thick lines on x meeting rs in points not in Σ. These intersection points are points of A_x, see 3.2, and we have $rs \subseteq A_x$. If t is a point on $rs \cap \Sigma$, then the line xt is full and contains a point $u \notin A_t$. But then by 2.6 also $t \notin A_u$. However, with u in the rôle of x, the above implies that rs and thus also t is

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in A_u. We have found a contradiction. Thus rs contains no point of Σ and we are again in case (i) of the lemma.

So finally we can assume that there is a third point t of π not on rs that is also not in Σ. But that implies that among rs, rt and st we can find at least one non-thick line, say n, in $P(W)$ which is not contained in A_x nor contains x. Any thick line on x meets n either in a point from Σ or in $A_x \cap n$. As there are at most 2 points from Σ on n, there are at most three thick lines on x. Hence $|k| - 1 \leq 3$. This proves the lemma. ■

Suppose π is a Σ-plane in the projective plane of order 4. Using knowledge of $\mathrm{GL}_3(\mathbb{F}_4)$, see for example the Atlas [4], we easily find that $G\big((\pi)\big)$ is one of the following subgroups of $\mathrm{GL}_3(\mathbb{F}_4)$: the group $\mathrm{GL}_3(\mathbb{F}_4)$ itself, $4^2 : \mathrm{GL}_2(\mathbb{F}_4)$, $4^2 : 3$, or $\mathrm{U}_3(\mathbb{F}_4)$. The first 3 examples correspond to generic planes, the last one to the exceptional dual affine plane of order 3 as in case (iii) of the proposition.

If π is embedded in a plane of order 3, then the subgroups of $\mathrm{GL}_3(\mathbb{F}_3)$ that can occur as $G\big((\pi)\big)$ are $\mathrm{GL}_3(\mathbb{F}_3)$, $3^2 : \mathrm{GL}_2(\mathbb{F}_3)$, $3^2 : W_3(B_2)$, $3^2 : 2$, all leading to a generic plane, $W_3(A_3) = W_3(D_3)$ corresponding to the dual affine plane in case (ii) of the proposition, or $W_3(B_3)$. In the latter case the plane π consists of 9 centers falling apart in 2 orbits of length 6 and 3, respectively. We will call such a plane a *B_3-plane*.

To prove the proposition it remains to rule out B_3-planes. This can be done with the following lemma.

Lemma 5.3 *If Σ contains B_3-planes, then it is not a single G-orbit.*

Proof Assume that Σ is a G-orbit but contains B_3-planes. Consider the geometry Π with point set Σ and as lines the thick, but not full Σ-lines. Then, by the above, any two intersecting lines in this geometry generate an affine plane as in case (i) or a dual affine plane as in case (ii) of Proposition 5.1. The diameter of any connected component of Π is at most 2. Moreover, two reflections whose centers are at distance 2 in Π commute. As, by assumption, there are B_3-planes, Σ splits into at least two connected components of Π. As G is transitive on the points of Π, each component of Π has diameter 2.

We show that for any $x \in \Sigma$ and reflection r in $\mathcal{R}$, the point xr is in the same connected component of Π as x. This is clearly true if $x = xr$ or if the line on x and xr is thick but not full. Hence, let us assume that x and $y := xr$ are in distinct components. Then x and y span a full line, l say.

The existence of B_3-planes and the transitivity of G on Σ imply that there is a B_2-line m on x such that x and $z = A_x \cap m$ are in the same component of Π. Clearly $m \neq l$.

Now consider the Σ-plane π spanned by l and m. If it is a B_3-plane, then the reflection r centralizes z. Hence also $z = zr$ and $y = xr$ are in the same component, a contradiction. Thus the plane is generic. If the line on yz is not full, then we have found a path from x to y via z. Hence yz is full, and the plane π is projective. We easily find that $G\big((\pi)\big)$ is transitive on the points of π. As the intersection point p of xz and $(xz)r$ is on two B_2-lines, $A_p \cap \pi$ is a line. Hence, the map $q \mapsto A_q$ induces a polarity on π without absolute points. Such polarities, however, do not exist. This final contradiction proves the lemma. ■

This finishes the proof of Proposition 5.1.

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6 The Generic Case

We now embark upon the proof of Theorem 1.2. By the results of Section 4 we may and do assume that G is generated by a normal set $\mathcal{R}$ of reflection tori with Σ a nondegenerate G-orbit spanning V. In view of Theorem 3.4, we assume that $\dim(V)$ is at least 3. Lemma 4.1 implies connectedness of Σ and the existence of Σ-planes.

Lemma 6.1 *If there exists a full line, then all Σ-planes are generic.*

Proof All Σ-planes are either generic or exceptional. Suppose x is a point of Σ. Then, as Σ is a single G-orbit, there is a full line on x, say l. By 5.1 we see that any other thick line m on x spans together with l a generic Σ-plane.

Now suppose m and n are thick lines on x spanning an exceptional plane. Inside the generic planes $\langle l, m\rangle$ and $\langle l, n\rangle$, respectively, we see that at most one line on y or z, respectively, is not full. So, as $|l| \geq 4$, we find a point $u \in l \setminus \{x\}$ with both uy and uz being full. However, then the Σ-plane on u, y and z contains both full and thin lines, which is not possible. This contradiction shows that there are only generic planes on x and hence on any point of Σ, which proves the lemma. ∎

Theorem 6.2 *Let $G \leq \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning V of dimension at least* 3*. Suppose that Σ is a nondegenerate G-orbit. If all Σ-planes are generic, then $G = R(V, W^*)$ for some $W^* \leq V^*$ with $\mathrm{Ann}(W^*) = 0$ and $\mathcal{R}$ consists of all reflection tori with axis in W^*.*

Proof Since all generic planes are linear, the space $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is linear. Let l be a thick but not full line, and let x be the unique point of l not in Σ. Then, as Σ is assumed to be nondegenerate, there is a reflection torus R not centralizing x. Let y be the center of R. Inside the generic plane spanned by l and y, however, we see that $x \in A_y$. This contradicts the fact that R does not centralize x. Thus all lines are full and all Σ-planes are projective, which implies that Σ consists of all points of $P(V)$.

As every point $p \in P(V)$ is contained in a projective Σ-plane with $x \in \Sigma$, we find, by Proposition 5.1, that $p \notin A_x$. So, $A_x = \varnothing$. The theorem now follows from Corollary 2.4. ∎

7 Unitary Groups over $\mathbb{F}_4$

We keep the notation and hypothesis of the previous section. By Lemma 6.1 and Theorem 6.2 we can concentrate on the case where there are no full Σ-lines and where there exist exceptional planes. In particular, $k = \mathbb{F}_3$ or $\mathbb{F}_4$, and every Σ-plane is either affine or dual affine. This easily implies that the diameter of $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is equal to 2.

In this section we handle the case where $k = \mathbb{F}_4$.

Lemma 7.1 *Let x be a point in Σ. Then A_x is a hyperplane.*

Proof Suppose that R_1 and R_2 are two tori in $\mathcal{R}$ with the same center x but distinct axes H_1 and H_2, respectively. As $V = \langle\Sigma\rangle$, there is an element in $y \in \Sigma$ such that the line xy does not meet $H_1 \cap H_2$. But then xy is full, which is against our assumptions. ∎

Lemma 7.2 *Let l be a thick line. Then $G(l) \simeq \mathrm{SL}_2(3)$. The central involution τ of $G(l)$ is a transvection on V with center the point p of l not in Σ. The transvection τ is the unique transvection with center p fixing a point $y \in \Sigma$ if and only if $p \in A_y$.*

Proof Let l be a thick line. Then, as Σ is nondegenerate, there is an element $R \in \mathcal{R}$ that does not centralize the unique point of $l \setminus \Sigma$. But then the Σ-plane π spanned by the center of R and $l \cap \Sigma$ is dual affine. Inside $G(\pi)$ we find $G(l)$ isomorphic to $2 \cdot A_4 \simeq \mathrm{SL}_2(3)$. Its central involution τ acts as a transvection on the projective plane spanned by π. Indeed, let p be the unique point of $\langle l \rangle$ outside Σ. Then τ fixes all points of l, and interchanges the two points of Σ as well as the two points outside Σ different from p, on the 4 thin lines of π on p. If π' is an affine Σ-plane on l, then τ is in the kernel of the action on π', see 5.1.

Let y be an element in Σ. If l is contained in A_y, then $y \in A_x$ for each $x \in l$, and thus $y\tau = y$. If l is not contained in A_y, then the plane $\langle y, l \rangle$ is either affine or dual affine. In both cases we find that $[y, \tau] \leq p$.

Since $V = [V, G]$ is spanned by Σ, we find that $[V, \tau] = p$, and τ is indeed a transvection on V with center p.

The action of τ on Σ and hence on V is uniquely determined by its center. ∎

By $\mathcal{T}$ we denote the set of transvections that we obtain as centers of subgroups $G(l)$ of G where l is a thick line.

Clearly $\mathcal{T}$ is a union of (possibly a single) conjugacy class(es) of transvections. Let Δ be the set of centers for the transvections in $\mathcal{T}$. For each $d \in \Delta$, there is a unique transvection in $\mathcal{T}$ with center d, see Lemma 7.2. This transvection is denoted by τ_d, its axis by A_d.

Lemma 7.3 *Let l be a line meeting $\Sigma \cup \Delta$ in at least two points. Then we have one of the following.*

(i) l is a thick line meeting Δ in a unique point d. The transvection τ_d centralizes l.
(ii) l is a thin line meeting Δ in 3 points. A transvection of $\mathcal{T}$ with center on l switches the two points of $\Sigma \cap l$.
(iii) l is contained in Δ. The transvections τ_d, $d \in l$, all centralize l.

Proof If l is a thick line then clearly we are in case (i). Suppose l is thin and x and y are the two points of $l \cap \Sigma$. Then, since the diameter of $\big(\Sigma, \mathcal{L}(\Sigma)\big)$ is 2, there is a point $z \in \Sigma$ with both xz and yz thick. The plane spanned by x, y and z is nondegenerate, and inside this plane we see that the three points of l different from x and y are centers of transvections switching x and y.

Suppose l is a line containing a point $x \in \Sigma$ and a point $y \in \Delta$. Let m be a thick line on y. If m is not contained in A_x, then there is a thick line on x meeting l nontrivially. But then $\pi = \langle x, l \rangle$ is a plane as in the conclusion of 5.1. Inside these planes all lines meeting Σ nontrivially contain at least two points from Σ. Thus assume that $l \subseteq A_x$. Let z be a point of m different from y. The line xz is thin. Thus by the above, there is a transvection $\tau \in \mathcal{T}$ with center on xz and $x\tau = z$. But then $m\tau = A_x\tau \cap \pi = A_z \cap \pi = xy$ and xy is thick.

Let l be a line meeting Δ in two points, d and e say, but disjoint from Σ. Let m be a thick line meeting l in d. Then by the above, every line through e meeting l in a point of Σ contains at least two points from Σ. Hence the intersection of Σ with the plane spanned by

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l and m contains a Σ-plane. If this Σ plane is affine, then l is contained in Δ and any two transvections t_d, $d \in l$, centralizes l. If the plane is dual affine, then l meets Σ nontrivially, which is against our assumptions. ■

The above lemma implies that every line of $P(V)$ which meets $\Sigma \cup \Delta$ in two points is contained in $\Sigma \cup \Delta$. Hence $\Sigma \cup \Delta$ is the complete point set of $P(V)$.

Now we are able to conclude with the following theorem.

Theorem 7.4 *Let $G \leq \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning V. Suppose that Σ is connected, nondegenerate and a G-orbit. If there exists a dual affine Σ-plane of order 3, then $G = \mathrm{FU}(V, h)$ for some nondegenerate unitary form h on the $\mathbb{F}_4$ vector space V and $\mathcal{R}$ is the unique class of unitary reflection tori in G.*

Proof Clearly V is an $\mathbb{F}_4$-vector space. We claim that the map

$$\perp\colon p \in P(V) \mapsto p^{\perp},$$

where $p^{\perp} = A_p$ for $p \in \Delta \cup \Sigma$ is a nondegenerate unitary polarity on $P(V)$. For that purpose we have to show that for each $p \in P(V)$ the subspace $p^{\perp}$ of $P(V)$ is a hyperplane, and that for all $p, q \in P(V)$ we have:

$$q \in p^{\perp} \text{ implies } p \in q^{\perp}.$$

If $p \in \Delta$, then by definition $p^{\perp}$ is a hyperplane. If $p \in \Sigma$, then $p^{\perp}$ is a hyperplane by Lemma 7.1.

Now suppose that $p, q \in P(V)$ and $q \in p^{\perp}$. If $p, q \in \Sigma$, then Lemma 2.6 implies $q \in p^{\perp}$. If $p, q \in \Delta$, then Lemma 7.3 implies $q \in p^{\perp}$. If $p \in \Sigma$ and $q \in \Delta$, or $q \in \Sigma$ and $p \in \Delta$, then by Lemma 7.3 and Theorem 3.4 the line pq is thick and again $p \in q^{\perp}$.

So indeed, $\perp$ is a polarity. Since $\perp$ restricted to a dual affine plane is a unitary polarity, $\perp$ itself is a nondegenerate unitary polarity. So, with h being a unitary form inducing this polarity, we find G and $\mathcal{R}$ to be as stated in the theorem. ■

8 Orthogonal Reflection Groups over $\mathbb{F}_3$

In this section we consider the case where G is generated by a normal set $\mathcal{R}$ of reflection tori inside the group $\mathrm{GL}(V)$, where V is a vector space over the field $\mathbb{F}_3$. Since each reflection torus is of order 2, we will often identify the tori with their nontrivial elements. We assume that the set Σ of centers of elements in $\mathcal{R}$ forms a single G-orbit, is connected and nondegenerate. As the case where all Σ-planes are generic is handled by Theorem 6.2, we can and do assume that there are exceptional planes around. Lemma 6.1 implies that there are no full lines in $\big(\Sigma, \mathcal{L}(\Sigma)\big)$, so that the reflections in the elements of $\mathcal{R}$ form a class of 3-transpositions, *i.e.*, the product of two elements of $\mathcal{R}$ has order 1, 2 or 3. (Many results of this and the following section can be deduced from the theory of groups generated by 3-transpositions, see for example [6], [7]. However, for the sake of completeness we prefer the tailor-made approach given below.) Moreover, any Σ-plane is either affine or dual affine. From Proposition 5.1 we easily deduce that in the 3-dimensional case there is always a quadratic form left invariant by the reflections. This is true in general as follows from the following result of [6].

Proposition 8.1 *There exists a G-invariant quadratic form Q on V such that the reflections in $\mathcal{R}$ are orthogonal reflections with respect to Q. The form is unique up to scalar multiplication.*

Since Σ is a G-orbit, we can choose the form Q in such a way that the points in Σ are $+$*-points*, *i.e.*, projective points $\langle v\rangle$, $v \in V$, with $Q(v) = 1$.

In the remainder of this section we will consider the case where there exist affine Σ-planes.

Theorem 8.2 *Let $G \leq \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning V. Suppose that Σ is connected, nondegenerate and a G-orbit. If there exists a dual affine Σ-plane of order 2 and an affine Σ-plane of order 3, then $G = \mathrm{FO}(V, Q)$ for some nondegenerate orthogonal form Q on the $\mathbb{F}_3$-vector space V and $\mathcal{R}$ is the class of reflection tori in G whose center is a $+$-point.*

The proof is divided into a number of steps. But first we need some notation and observations.

Fix a point x in Σ. The *tangent lines* on x, *i.e.*, the lines meeting the quadric of Q in a single point, form a polar space P_x isomorphic with the polar space of Q restricted to A_x. The Σ-lines on x are (part of) tangent lines and can therefore be considered to be points of P_x. The affine Σ-planes correspond to lines of P_x. The Σ-lines and affine Σ-planes on x actually form a subspace of P_x denoted by Σ_x. The assumption that there is an affine Σ-plane π on x, but no degenerate line, implies that the (polar) rank of the subspace Σ_x is at least 2.

Step 1 There is a 5-dimensional subspace W of V containing x, such that $Q|_W$ is nondegenerate and $\Sigma \cap P(W)$ consists of all $+$-points of $P(W)$. Moreover, the hyperplane $A_x \cap W$ contains singular lines.

Proof Since the polar space P_x has rank at least 2, we can find a grid in Σ_x. The subspace of V spanned by all lines on x in this grid is the space W we are looking for. ■

Step 2 Let X be a finite subset of Σ which is the union of tangents through x. If X contains $\Sigma_x \cap P(W)$, then all $+$-points of $\langle X\rangle$ are in Σ.

Proof Suppose X is a minimal counter example to the statement. Consider P_x^X, the polar subspace of P_x of those tangents that are in $\langle X\rangle$. Then P_x^X has rank at least 2 as $\langle X\rangle$ contains W. The Σ-lines on x that are in $\langle X\rangle$ form a subspace Σ_x^X of P_x^X. By our assumption, this subspace has to be proper. Indeed, if $\Sigma_x^X = P_x^X$, then, as every $+$-point from $\langle X\rangle$ is the sum of at most two $+$-points on tangents through x, we would find that all $+$-points of $\langle X\rangle$ are in Σ.

Fix a tangent line l on x in X but not in $P(W)$. Then, by minimality of X, we find that $X - \{l\}$ does not generate $\langle X\rangle$. So $Y := \langle X - \{l\}\rangle$ is a hyperplane of $\langle X\rangle$. Moreover, all $+$-points of Y are in Σ. The tangent lines on x that are in Y form a geometric hyperplane H of P_x^X contained in Σ_x^X. Now the line $l \in \Sigma_x$ is a point of P_x^X outside this hyperplane. But, as P_x^X has rank at least 2, it is generated by H and any 'point' outside this geometric hyperplane, see [3]. But this leads to the contradiction that Σ_x^X equals P_x^X. ■

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Step 3 Σ consists of all points $\langle w\rangle \in V$ with $Q(w) = 1$.

Proof Let $p = \langle w\rangle \in V$ with $Q(w) = 1$. Then, as $V = [V, G] = \langle\Sigma\rangle = \langle\Sigma_x\rangle$, there is a finite subset X' of Σ_x such that $p \in \langle X'\rangle$. But then p is also contained in $\langle X\rangle$, where $X = X' \cup \big(P(W) \cap \Sigma_x\big)$, and Step 2 implies that $p \in \Sigma$. ∎

Step 4 Q is nondegenerate.

Proof If Q has a nontrivial radical, then a line through a +-point and a point of the radical is a degenerate Σ-line. But Σ is nondegenerate, so Q is also nondegenerate.

9 Weyl Groups Acting on their Natural Module Mod 3

In this section we consider the case where G is generated by a normal set $\mathcal{R}$ of reflection tori in $\mathrm{GL}(V)$, with V a vector space over $\mathbb{F}_3$ of dimension at least 3. By Σ we denote the set of centers of the reflections in $\mathcal{R}$. As before we assume that Σ is a single G-orbit and nondegenerate. By Lemma 6.1 and Theorem 8.2 we can and do restrict our attention to the case where the reflections in $\mathcal{R}$ form a conjugacy class of 3-transpositions and that all Σ-planes are dual affine. As the reflection tori of $\mathcal{R}$ contain just one nontrivial reflection, we identify these tori with these nontrivial reflections. Moreover, by Proposition 8.1 there is a unique quadratic form Q on V such that the reflections in $\mathcal{R}$ leave Q invariant and have as centers a +-point with respect to Q. So, if B denotes the bilinear form associated to Q, then a reflection $r \in \mathcal{R}$ with center $\langle v\rangle$, for some $v \in V$, has axis $v^\perp = \{w \in V \mid B(w, v) = 0\}$. This reflection will be denoted by r_v.

The *diagram* $\Delta(R)$ of a subset R of reflections in $\mathcal{R}$ is the graph whose vertices are the elements of R and whose edges consists of the pairs of non-commuting elements from R.

Lemma 9.1 *Let $R \subseteq \mathcal{R}$ be finite. If $\Delta(R)$ contains a subdiagram of type $\widetilde{D_4}$, then there is a proper subset R' of R with $\langle R'\rangle = \langle R\rangle$.*

Proof It is enough to consider the case where R is a subset of 5 reflections in $\mathcal{R}$ with diagram $\Delta(R)$ of type $\widetilde{D_4}$.

Let $r \in R$ such that $R' = R - \{r\}$ is a subset of R with diagram D_4. So $\langle R'\rangle$ is isomorphic to the Weyl group $W_3(D_4)$ and there is a basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ of $[V, \langle R'\rangle]$ such that the reflections in R' have as centers the 1-spaces spanned by $b_1 - b_2$, $b_1 + b_2$, $b_2 - b_3$ and $b_3 - b_4$. Moreover, we can assume that the reflection r centralizes the hyperplane $\langle b_1 - b_2, b_1 + b_2, b_3 - b_4\rangle$ of $[V, \langle R'\rangle]$, but not $b_2 - b_3$. But then r does not centralize the three vectors $b_2 - b_3$, $b_3 + b_4$ and $b_2 + b_4$ which all span centers in Σ. So, either r has center $b_3 + b_4$ and $r \in \langle R'\rangle$ or the Σ-plane spanned by the center of r and $\langle b_3 + b_4\rangle$ and $\langle b_2 + b_4\rangle$ is an affine plane, which is against our assumptions. ∎

This lemma together with the results from the appendix yield:

Proposition 9.2 *Let R be a finite subset of reflections of $\mathcal{R}$ with connected diagram. Then $\langle R\rangle$ is isomorphic to a Weyl group mod 3 of type A_n, D_n, E_6, E_7 or E_8 and $[V, \langle R\rangle]$ is the natural $\mathbb{F}_3$-reflection module for $\langle R\rangle$, or, $\langle R\rangle$ is of type A_n, $3 \mid n{+}1$, or E_6 and $[V, \langle R\rangle]$ is the irreducible quotient of the natural $\mathbb{F}_3$-reflection module of dimension $n - 1$ or 6, respectively.*

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Proof Without loss of generality we can assume that R is a minimal generating set for $\langle R\rangle$. Consider $\Delta(R)$. Now consider any pair s, r of non-commuting elements from R. We can transform R into the set $R' = \{r, s\} \cup \{r^{-1}tr \mid t \in R \setminus \{r, s\}\}$. We notice that R' is also a minimal generating set of $\langle R\rangle$. The diagram $\Delta(R')$ is isomorphic to the diagram $\Delta(R)^{(r,s)}$ as defined in the appendix. Now Theorem 12.4 implies that we can transform R into a tree. Without loss of generality we can assume that $\Delta(R)$ is a tree. In the equivalence classes of the diagrams of type $\widetilde{D}_n$ and $\widetilde{E}_n$ there are diagrams with an induced subdiagram of type $\widetilde{D}_4$. Then the previous lemma and minimality of R, however, imply that $\Delta(R)$ does not contain these diagrams as a subdiagram. But that implies that $\Delta(R)$ is of type A_n, D_n, E_6, E_7 or E_8, and it is clear that $\langle R\rangle$ is isomorphic to a Weyl group mod 3 of type A_n, D_n, E_6, E_7 or E_8.

The module $[V, \langle R\rangle]$ is easily seen to be a quotient of the natural $\mathbb{F}_3$ reflection module for $\langle R\rangle$. But, as proper quotients only exist when G is of type A_n where $3 \mid n + 1$, or E_6, the proposition follows. ∎

Proposition 9.3 *If there is a finite subset R of $\mathcal{R}$ such that $\langle R\rangle \simeq W_3(E_6)$, then G is isomorphic to $W_3(E_n)$, $n = 6$, 7 or 8 and G acts naturally on V which is isomorphic to the $\mathbb{F}_3$ reflection module for G or its irreducible quotient when $G = W_3(E_6)$.*

Proof Since a Weyl group mod 3 of type A_n or D_n does not contain a subgroup generated by reflections isomorphic to $W_3(E_6)$, the proposition is clear from the previous result. ∎

Now suppose that all finite subsets of $\mathcal{R}$ with connected diagram generate Weyl groups mod 3 of type A_n or D_n. We will use our knowledge of the finite Weyl groups of type A_n and D_n and their natural $\mathbb{F}_3$-reflection modules to finish our classification in the case where $\mathcal{R}$ is infinite.

Suppose that $\mathcal{R}$ is infinite. If R is a finite subset of $\mathcal{R}$, generating a Weyl group mod 3 of type A_n or D_n, $n \geq 4$, then by the previous result we see that the subspace $[V, \langle R\rangle]$ is the natural n-dimensional reflection module for $\langle R\rangle$. (Indeed, if $\langle R\rangle$ is isomorphic to $W_3(A_n)$, with $3 \mid n + 1$, then we can replace R by a larger subset of $\mathcal{R}$ so that it generates a (larger) subgroup isomorphic to $W_3(A_{n+1})$ or $W_3(D_{n+1})$ and check inside this group that $[V, \langle R\rangle]$ is at least n-dimensional.) In particular, there is a unique point in $[V, \langle R\rangle]$ (which is not in Σ) fixed by a parabolic subgroup of type A_{n-1} or D_{n-1}, respectively, of $\langle R\rangle$. Such a point is called a *base point* for the set R (or the group it generates). The group $\langle R\rangle$ induces the symmetric group S_{n+1} on the $n + 1$ base points of R. By Ω we denote the set of all those points that appear as base point for some finite subset of $\mathcal{R}$. For each point $p \in \Omega$ we fix a vector b spanning p. The set of all such vectors is denoted by $\mathcal{B}$.

If p is a base point for some set R of reflections and S is a finite set of reflections in $\mathcal{R}$ containing R and with connected diagram, then p is also a base point for S. This can be checked easily within the group $\langle S\rangle$.

Proposition 9.4 *Suppose that $\mathcal{R}$ is infinite. If all finite subsets R of $\mathcal{R}$ with connected diagram generate a Weyl group mod 3 of type A_n or D_n, then G is isomorphic to $W_3(A_{\mathcal{B}})$ or $W_3(D_{\mathcal{B}})$. The space V is the natural $\mathbb{F}_3$-reflection module for G.*

Proof Let r be a reflection in $\mathcal{R}$. Then r induces a transposition on the set Ω of base points. Indeed, let R be a set of reflections with $r \in R$ generating a Weyl group mod 3 of type A_4,

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then we see that there are 2 base points that are switched by r. Let b_1 and b_2 be two vectors in $\mathcal{B}$ spanning these two base points. For any third point p of Ω we can find a subset R' of $\mathcal{R}$ such that this point is a base point for R'. But then $R \cup R'$ is contained in a finite subset S of reflections with connected diagram. The three base points are all base points for S and, as r induces a transposition on base points, p is fixed by r. This proves that r indeed induces a transposition on Ω.

So, r induces the reflection $r_{b_1-b_2}$ or $r_{b_1+b_2}$ on V. But then it is clear that G is isomorphic to $W_3(A_{\mathcal{B}})$ or $W_3(D_{\mathcal{B}})$ acting on a quotient of the natural module. As the natural module is irreducible, we find that V is a natural module for G. ■

Combining the results from this section we obtain the following classification.

Theorem 9.5 *Let $G \le \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ spanning the $\mathbb{F}_3$-vector space V of dimension at least 3. Suppose that Σ is connected and a G-orbit. If all Σ-planes are dual affine of order 2, then G is a Weyl group mod 3 of type A, D, or E. The module V is the natural $\mathbb{F}_3$ reflection module for G or, $G = W_3(A_n)$ (with $3 \mid n+1$) or $W_3(E_6)$, and V is the irreducible quotient of the natural $\mathbb{F}_3$ reflection module for G of dimension $n-1$ or 5, respectively.*

10 Proof of Theorem 1.2

In this short section we will show how Theorem 1.2 follows from the results obtained so far. So, suppose that G is a subgroup of $\mathrm{GL}(V)$ generated by a normal set $\mathcal{R}$ of reflection tori. Let Σ be a G-orbit on the set of centers of $\mathcal{R}$. If the group $G(\Sigma)$ is not transitive on Σ, Theorem 4.7 shows that assertion 3 of the conclusion of Theorem 1.2 holds.

Therefore, we can assume that $G(\Sigma)$ is transitive on Σ. But then Theorem 4.4 implies that we can restrict attention to the case where Σ is nondegenerate. Without loss of generality, we assume from now on that $G = G(\Sigma)$, where Σ is nondegenerate and spans V.

If $\dim(V) = 2$, then Theorem 3.4, together with nondegeneracy clearly implies irreducibility of G on V. We obtain cases 1 and 5 of Theorem 1.1.

Thus we can assume that the dimension of V is at least 3. The connectedness of Σ (Lemma 4.1) then implies that there are Σ-planes. If all Σ-planes are generic, then Theorem 6.2 implies that the group G acts irreducibly, leading to case 1 of Theorem 1.1. If there are exceptional planes, then Theorem 7.4, Theorem 8.2 and Theorem 9.5 imply that the group G is either irreducible on V, or $G = W_3(A_n)$ ($n = 2 \bmod 3$) or $W_3(E_6)$, and V is the natural $\mathbb{F}_3$ reflection module for G, as listed under cases 2, 3, 4 of Theorem 1.1.

11 Proof of Theorem 1.1

As in the previous sections we consider a group $G \le \mathrm{GL}(V)$ generated by its normal set of reflection tori $\mathcal{R}$ with center set Σ. Up till now we have considered the case where Σ is a single G-orbit. But how can two distinct orbits of centers fit together? In this section we try to answer this question. In particular, we prove the following.

Theorem 11.1 *Let $G \le \mathrm{GL}(V)$ be generated by a normal set $\mathcal{R}$ of reflection tori. Denote by Σ the set of all centers of elements in $\mathcal{R}$. Then Σ is a union of G-orbits. Let Σ_1 and Σ_2 be two* *distinct G-orbits on Σ. Then one of the following holds.*

i. *Σ_1 and Σ_2 are not connected by thick $\Sigma_1 \cup \Sigma_2$-lines. Then $G(\Sigma_1)$ and $G(\Sigma_2)$ commute and $\langle \Sigma_i \rangle \leq C_V\big(G(\Sigma_j)\big)$, for $\{i, j\} = \{1, 2\}$.*
ii. *Up to a permutation of indices, Σ_2 is contained in $\mathrm{Rad}(\Sigma_1)$ but not conversely.*
iii. *Σ_2 does not meet the radical of Σ_1. Moreover, if Σ_1 is nondegenerate then, with $V_1 = \langle \Sigma_1 \rangle$, we have the following.*
   (a) *$G(\Sigma_1 \cup \Sigma_2)$ is an orthogonal group $O(V_1, Q)$, where V_1 is an $\mathbb{F}_3$ vector space and Q a nondegenerate quadratic form on V_1. The orbits Σ_1 and Σ_2 are the two orbits on the non-isotropic points of the orthogonal form Q.*
   (b) *$G(\Sigma_1 \cup \Sigma_2)$ is a Weyl group mod 3 of type B acting naturally on V_1, and Σ_1, Σ_2 are the two classes of centers of G.*

Proof Let Σ_1 and Σ_2 be two distinct G-orbits on Σ. For $i = 1, 2$, set $V_i := \langle \Sigma_i \rangle$ and let $\mathcal{R}_i$ be the subset of $\mathcal{R}$ consisting of those tori with center in Σ_i.

Suppose that $\Sigma_2 \cap V_1$ is empty. Then each reflection in a reflection torus from $\mathcal{R}_2$ centralizes V_1 and $\Sigma_1 \leq A_{\Sigma_2}$. If also Σ_1 intersects V_2 trivially, then $G(\Sigma_1)$ and $G(\Sigma_2)$ commute and $V_i \leq C_V\big(G(\Sigma_j)\big)$, $i \neq j$. If Σ_1 meets V_2 nontrivially, then there is an element $R \in \mathcal{R}_1$ acting nontrivially on Σ_2. In particular, there is a degenerate Σ_2-line meeting Σ_1 in a point of $\mathrm{Rad}(\Sigma_2)$, see case (iv) of Theorem 3.4. This implies that Σ_1 is contained in the radical of Σ_2.

Thus, now assume that $\Sigma_1 \cap V_2$ and $\Sigma_2 \cap V_1$ are nonempty. Then, since G stabilizes Σ_1 and Σ_2, we find $V_1 = V_2$. Let $\{i, j\} = \{1, 2\}$. Then there is an element $x \in \Sigma_i$ with A_x not containing Σ_j. So, if $G(\Sigma_i)$ is not transitive on Σ_i, then by Theorem 4.7 Σ_j is the unique G-orbit on Σ with Σ_i not in A_x for some $x \in \Sigma_j$ and $G(\Sigma_1 \cup \Sigma_2)$ induces a Weyl group mod 3 of type B on the space $V_1/U = V_2/U$, where $U = \mathrm{Rad}(\Sigma_i)$.

So, assume that both $G(\Sigma_1)$ and $G(\Sigma_2)$ are transitive on Σ_1 and Σ_2, respectively. We can restrict attention to the case where Σ_1 is nondegenerate.

Theorem 3.4 implies that the dimension of V has to be at least 3. So, there exist Σ_1-planes. If all Σ_1-planes are generic, then Theorem 6.2 implies that all points of $P(V_1)$ are in Σ_1, so there is no room for Σ_2. If there is a dual affine Σ_1-plane of order 3, then Theorem 7.4 implies that there is a nondegenerate Hermitian form h on V_1 such that Σ_1 consists of all non-singular points with respect to h. As no reflection with center a singular point will leave Σ_1 invariant, we again have no room for Σ_2. So we can assume that $k = \mathbb{F}_3$. If there are both affine and dual affine Σ_1-planes, then $\dim(V_1) > 4$ and there is a nondegenerate orthogonal form Q on V_1, such that Σ_1 consists of the $+$-points of Q. Since there is no reflection in $\mathrm{GL}(V_1)$ with center singular with respect to Q that leaves Σ_1 invariant, Σ_2 consists of all $-$-points of Q.

By Theorem 9.5, it remains to consider the case where $G(\Sigma_1)$ is a Weyl group mod 3 of type A, D or E. There is a unique $G(\Sigma_1)$-invariant quadratic form Q on V_1 such that the elements of Σ_1 are (part of the set of) $+$-points with respect to Q. As $G(\Sigma_2)$ normalizes $G(\Sigma_1)$, also $G(\Sigma_2)$ is contained in the orthogonal group $\mathrm{O}(V_1, Q)$.

Let $a \in \Sigma_1$ and $p \in \Sigma_2$ not orthogonal to a. Then, as a is a $+$-point, the line ap is a B_2-line containing two $+$-points from Σ_1, a and b say, and two $-$-points from Σ_2, see Theorem 3.4. In particular, Σ_2 consists of $-$-points.

Let π be a dual affine Σ_1-plane containing a and b. Denote by Π the projective plane spanned by π. Then $G(\pi)$ is transitive on the 3 $-$-points in Π, which are therefore all in Σ_2. Clearly, $G\Big(\big((\Sigma_1 \cup \Sigma_2) \cap \Pi\big)\Big) = W_3(B_3)$. In particular, the assumption of $G(\Sigma_2)$ being transitive on Σ_2 implies that $\dim V \geq 4$.

Let q be in Σ_2 but not orthogonal to p. Thus q lies outside Π. Denote by Δ the span of Π and q. There are at least two centers in $\Sigma_1 \cap \Pi$ nonorthogonal to q, and hence B_2-lines on q with two points of each of Σ_1 and Σ_2. In particular, $G\big((\Sigma_1 \cap \Delta)\big)$ must be $W_3(A_4)$, $W_3(A_5)$ or $W_3(D_4)$ according as the restriction of Q to Δ has Witt index $-$ or $+$.

In the latter case, *i.e.*, $G\big((\Sigma_1 \cap \Delta)\big) \cong W_3(D_4)$, the $+$-points of Δ form a single orbit, which coincides with $\Sigma_1 \cap \Delta$. Moreover, the $-$-points come into three $G\big((\Sigma_1 \cap \Delta)\big)$-orbits, each of size 4. Each consists of an orthonormal basis of $-$-points. As p and q are not orthogononal, they are in distinct orbits under $G\big((\Sigma_1 \cap \Delta)\big)$. But the reflections of any two of these three orbits generate a group isomorphic to $W_3(D_4)$ which is transitive on the $-$-points. Hence $G\big((\Delta)\big) = \mathrm{O}(\Delta, Q|_\Delta)$ which is isomorphic to $W_3(F_4)$, and we are in case (iii)(a).

If $G\big((\Sigma_1 \cap \Delta)\big) = W_3(A_4)$ or $W_3(A_5)$, then it has a single orbit of size 15 on $-$-points of Δ. Thus, $\Sigma_2 \cap \Delta$ consists of all $-$-points of Δ. But then $G\big((\Sigma_1 \cap \Delta)\big) = W_3(A_5)$ and $G\big((\Delta)\big) = \mathrm{O}(\Delta, Q|_\Delta)$.

If $\dim V \geq 5$, then we can choose the point q in such a way that $G\big((\Sigma_1 \cap \Delta)\big)$ contains a subgroup $W_3(A_4)$ and hence equals $W_3(A_5)$.

But a Weyl group mod 3 of type A, D or E, different from A_5 or E_6, does not contain a subgroup generated by reflections of type A_5 acting irreducibly on the space spanned by its centers. If $G = W_3(E_6)$, then G acts as an orthogonal group on a 5-dimensional space, and Σ_2 is the orbit of $-$-points. This proves the theorem. ■

We are now able to complete the proof of Theorem 1.1. Suppose that the group $G \leq \mathrm{GL}(V)$ is generated by a normal set $\mathcal{R}$ of reflection tori with center set Σ. Suppose that G acts irreducibly.

If Σ_1 is a G-orbit on Σ, then the radical of Σ_1 is G-invariant, and hence trivial. So Σ_1 is nondegenerate. If $\dim(V) = 2$, then Theorem 1.1 follows easily from Theorem 3.4. So, we assume $\dim(V) \geq 3$.

If $G(\Sigma_1) = G$ for some orbit Σ_1, then $\big(\Sigma_1, \mathcal{L}(\Sigma_1)\big)$ contains planes and Theorem 1.1 follows from the results of Sections 6–9.

If $G(\Sigma_1) \neq G$ then for some G-orbit Σ_1, then the above theorem implies that G is as described in case (iii) of Theorem 11.1 (note that there is no room for a third orbit). This finishes the proof of Theorem 1.1.

12 Appendix: Graphs and Trees

The theorem below is used in the proof of Proposition 9.2. The special case treated in Theorem 3.3 of [1] suffices for this proof, but we prefer to give Theorem 12.4 with proof as it was part of the first author's PhD thesis defense 25 years ago.

Graphs are finite, without loops and without multiple edges.

Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. Adjacency is denoted by $\sim$. For vertices γ, δ of Γ we write $\Gamma^{(\gamma,\delta)}$ to denote the graph on the same vertex set as Γ and with

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edge set

$$E\Gamma^{(\gamma,\delta)} = \begin{cases} \big(E\Gamma \cup \{\{\zeta,\delta\} \mid \zeta \nsim \delta, \zeta \sim \gamma\}\big) \setminus \{\{\zeta,\delta\} \mid \zeta \sim \delta, \gamma\} & \text{if } \gamma \sim \delta \\ E\Gamma & \text{otherwise.} \end{cases}$$

Lemma 12.1 *If Γ is connected, then so is $\Gamma^{(\gamma,\delta)}$.*

We say that $\Gamma^{(\gamma,\delta)}$ can be obtained from Γ by an elementary transformation with respect to (γ,δ). Note that $\Gamma^{(\gamma,\delta)(\gamma,\delta)} = \Gamma$. Let X, Y be subsets of $V\Gamma$. We say that Γ is (X,Y)-*equivalent* to Δ if it can be obtained from Δ by a series of elementary transformations with respect to pairs (γ,δ) with $\gamma \in X$ and $\delta \in Y$.

Lemma 12.2 *If Z is a set of vertices of Γ separated from X by Y (that is, X and Z are in distinct connected components of $\Gamma \setminus Y$), then for any graph Δ that is (X,Y)-equivalent to Γ, the subgraphs of Δ and of Γ induced on Z coincide and so do the sets of edges in Δ and in Γ between Z and Y.*

In particular, the subgraph induced on Z is the same for all graphs in the (X,Y)-equivalence class of Γ.

Lemma 12.3 *If γ is a vertex of Γ and there is a circuit on γ in Γ, then there is a graph Δ which is $(V\Gamma \setminus \{\gamma\}, V\Gamma)$-equivalent to Γ and satisfies*

$$\mathrm{valency}_\Delta(\gamma) = \mathrm{valency}_\Gamma(\gamma) - 1.$$

Proof Let $C : \gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$ be the distinct vertices along a minimal induced circuit on γ in Γ. Suppose $\Gamma \setminus C = \{\zeta_1, \ldots, \zeta_l\}$. Then

$$\Delta = \Gamma^{(\gamma_2,\gamma)\cdots(\gamma_{n-1},\gamma)(\gamma_2,\zeta_1)\cdots(\gamma_{n-1},\zeta_1)\cdots(\gamma_2,\zeta_l)\cdots(\gamma_{n-1},\zeta_l)}$$

is as required. To see this, note first that the circuit C has changed into the tree determined by the adjacencies

$$\gamma \sim \gamma_{n-1} \sim \gamma_n, \quad \gamma_{n-1} \sim \gamma_{n-2} \sim \cdots \sim \gamma_2.$$

Second, for $\zeta \notin C$, the adjacency between ζ and γ has not changed. So, the valency of γ in Δ is the same as the one for Γ except that the valency of γ in C has dropped from 2 to 1. ∎

Theorem 12.4 *Every connected graph is equivalent to a tree.*

Proof Consider induced subtrees of Γ not containing two vertices of any circuit. Notice that, as a vertex is a tree not containing two vertices of a circuit, such subtrees exist. Fix a maximal subtree T not containing two vertices of any circuit. Let γ be an endpoint of T having a neighbor outside T. Then γ separates $T \setminus \{\gamma\}$ from $V\Gamma \setminus T$. Suppose γ lies on a circuit. Then, by the above separation property, the circuit lies entirely in $(V\Gamma \setminus T) \cup \{\gamma\}$. Apply Lemma 12.3 to the induced graph on $(V\Gamma \setminus T) \cup \{\gamma\}$ to find a $\big((V\Gamma \setminus T), (V\Gamma \setminus T) \cup \{\gamma\}\big)$-equivalent graph Δ of Γ with $\mathrm{valency}_\Delta(\gamma) < \mathrm{valency}_\Gamma(\gamma)$. (Use Lemma 12.2 to see

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that the transformations on $(V\Gamma \setminus T) \cup \{\gamma\}$ can be extended to all of Γ without effecting the graph structure on T.)

Continue until there are no more circuits on γ. As the valency of γ decreases, this loop terminates. Pick a neighbor δ of γ in $(\Delta \setminus T) \cup \{\gamma\}$. Now the induced graph on $T \cup \{\delta\}$ is a tree. Repeat the above procedure, beginning with the choice of a vertex of T separating the rest of T from a nonempty remainder in Γ, until there are no such end nodes. (At each step, the induced subtree becomes bigger, so the loop terminates.) Then $T = \Gamma$ is a tree. ■

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