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INEQUALITIES FOR BAER INVARIANTS OF FINITE GROUPS

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ABSTRACT. In this note we further our investigation of Baer invariants of groups by obtaining, as consequences of an exact sequence of A. S.-T. Lue, some numerical inequalities for their orders, exponents, and generating sets. An interesting group theoretic corollary is an explicit bound for $|\gamma_{c+1}(G)|$ given that $G/Z_c(G)$ is a finite p-group with prescribed order and number of generators.

In a previous paper [3] we investigated groups G of the form $G = H/Z_c(H)$, where $c \geq 1$ and $Z_c(H)$ is the c-th term of the upper central series of some group H. Extending terminology of [9], such groups G were said to be *c-capable*. We proved that a finitely generated abelian group is c-capable if and only if it is 1-capable. Moreover, we showed that this result does not extend to p-groups by exhibiting a 2-group (of order 2^9) which is 1-capable but not 2-capable. Our method for demonstrating that a particular group G is not c-capable involved presenting it as the quotient of a free group F by a normal subgroup R, and then computing the Baer invariant

$$M^{(c)}(G) = \big(R \cap \gamma_{c+1}(F)\big) / \gamma_{c+1}(R, F),$$

where

$$\gamma_1(F) = F, \quad \gamma_{c+1}(F) = \big[\gamma_c(F), F\big],$$
$$\gamma_1(R, F) = R, \quad \gamma_{c+1}(R, F) = \big[\gamma_c(R, F), F\big].$$

The group $M^{(c)}(G)$ is well-known to be an invariant of G (see for instance [8]), and is clearly abelian. In particular, $M^{(1)}(G)$ is the Schur multiplier of G.

A computer program for computing $M^{(c)}(G)$ is listed in [3]. As input data, the program requires a finite presentation of G, and any positive integer q divisible by e^c where e is the exponent of $M^{(c)}(G)$. The main aim of this note is to give a few simple results for helping to choose such an integer q. We obtain these results (as well as results on the order of $M^{(c)}(G)$, and on the number of generators of $M^{(c)}(G)$) as fairly direct consequences of an exact sequence of Lue [13].

When $c = 1$ our results (but not our proofs) reduce to those of M. R. Jones [10, 11] on the Schur multiplier, and Lue's sequence reduces to the exact homology sequence of Stallings and Ganea (*cf.* [8]).

In order to read the rest of this paper, one will need to be familiar with the nonabelian tensor and exterior product of groups; a good introductory account of these can be found in [2].

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Let N be a normal subgroup of G. The main result of Lue [13] can be reformulated as a natural exact sequence

$$(*)\qquad \ker\Big(\big((N\wedge G)\wedge G\big)\wedge\cdots\wedge G\xrightarrow{\mu}\gamma_{c+1}(N,G)\Big)\longrightarrow M^{(c)}(G)\longrightarrow M^{(c)}(G/N)\longrightarrow N/\gamma_{c+1}(N,G)\longrightarrow G/\gamma_{c+1}(G)\longrightarrow G/N\gamma_{c+1}(G)\longrightarrow 1.$$

Here $\wedge$ denotes the nonabelian exterior product of groups, and the group $\big((N\wedge G)\wedge G\big)\wedge\cdots\wedge G$ involves one copy of N and c copies of G; we shall henceforth denote this iterated exterior product by $\wedge^{c+1}(N,G)$.

We write $|G|$ and $e(G)$ for the order and exponent of G. The minimum number of elements needed to generate G is denoted by $d(G)$. The following result is due to M. R. R. Moghaddam [14].

PROPOSITION 1 [14]. *Set* $H=G/N$.
(i) $|M^{(c)}(H)|$ *divides* $|M^{(c)}(G)|\;|N\cap\gamma_{c+1}(G)|/|\gamma_{c+1}(N,G)|$.
(ii) $e\big(M^{(c)}(H)\big)$ *divides* $e\big(M^{(c)}(G)\big)\times e\big(N\cap\gamma_{c+1}(G)/\gamma_{c+1}(N,G)\big)$.
(iii) $d\big(M^{(c)}(H)\big)\le d\big(M^{(c)}(G)\big)+d\big(N\cap\gamma_{c+1}(G)/\gamma_{c+1}(N,G)\big)$.

PROOF. The sequence $(*)$ yields an exact sequence

$$M^{(c)}(G)\longrightarrow M^{(c)}(H)\longrightarrow\big(N\cap\gamma_{c+1}(G)\big)/\gamma_{c+1}(N,G)\longrightarrow 1$$

from which (i), (ii) and (iii) follow. ■

As pointed out in [13], the first five terms of the sequence $(*)$ in fact hold for Baer invariants with respect to an arbitrary variety [8]. Thus Proposition 1 (and several subsequent results) automatically extend to these more general Baer invariants. When $c=1$, Proposition 1 reduces to [10, Theorem 3.1].

The structure of the nonabelian tensor product of groups has been investigated extensively by several authors. (To cite just one instance, paper [7] obtains bounds on the order of $G\otimes H$ when G and H are finite prime-power groups acting compatibly on each other.) Since this structure is fully understood in many instances, it is useful to obtain bounds on $M^{(c)}(G)$ in terms of the tensor product. We obtain such bounds in Propositions 2 and 5 below.

A normal subgroup B in G is said to be *k-central* if $\gamma_{k+1}(B,G)=1$. In this case conjugation yields an action of $G/\gamma_{k+1}(G)$ on B, and an action of B on $G/\gamma_{k+1}(G)$. We can thus form the iterated nonabelian tensor product $\Big(\big(B\otimes G/\gamma_{k+1}(G)\big)\otimes G/\gamma_{k+1}(G)\Big)\otimes\cdots\otimes G/\gamma_{k+1}(G)$ involving c copies of $G/\gamma_{k+1}(G)$. Let us denote this iterated tensor product by $\otimes^{c+1}\big(B,G/\gamma_{k+1}(G)\big)$. (Note that for $k=1$ the tensor product $\otimes$ coincides with the usual tensor product of abelian groups.) We define the group $\wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big)$ by a pushout square in the category of groups (in which α and β are the obvious quotient

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homomorphisms):

$$\begin{array}{ccc} \otimes^{c+1}(B,G) & \xrightarrow{\alpha} & \otimes^{c+1}\big(B,G/\gamma_{k+1}(G)\big) \\ \beta\big\downarrow & \text{pushout} & \big\downarrow \\ \wedge^{c+1}(B,G) & \longrightarrow & \wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big). \end{array}$$

In other words, $\wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big) = \wedge^{c+1}(B,G)/\beta\big(\ker(\alpha)\big)$.

PROPOSITION 2. *Let B be a k-central subgroup of G with $k \le c$. Set $A = G/B$.*

(i) $|M^{(c)}(G)|\,|B\cap\gamma_{c+1}(G)|$ *divides* $|M^{(c)}(A)|\,\big|\wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big)\big|$.

(ii) $e\big(M^{(c)}(G)\big)$ *divides* $e\big(M^{(c)}(A)\big)e\Big(\wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big)\Big)$.

(iii) $d\big(M^{(c)}(G)\big) \le d\big(M^{(c)}(A)\big) + d\Big(\wedge^{c+1}\big(B,G/\gamma_{k+1}(G)\big)\Big)$.

PROOF. The sequence $(*)$ with $N = B$ yields an exact sequence

$$\wedge^{c+1}(B,G) \longrightarrow M^{(c)}(G) \longrightarrow M^{(c)}(A) \longrightarrow B\cap\gamma_{c+1}(G) \longrightarrow 1$$

which, thanks to the commutative triangle of homomorphisms (*cf.* [13])

$$\begin{array}{ccc} \wedge^{c+1}(B,G) & \longrightarrow & M^{(c)}(G) \\ & \searrow \quad \wedge^{c+1}(B,G/\gamma_{k+1}G) \quad \swarrow & \end{array}$$

implies (i), (ii) and (iii). ∎

Proposition 2 reduces, when $c = 1$ and $k = 1$, to [10, Theorem 4.1] since in this case one readily observes an exact sequence

$$B\wedge B \longrightarrow \wedge^2\big(B,G/\gamma_2(G)\big) \longrightarrow B\otimes A^{ab} \longrightarrow 1,$$

and consequently:

(i) $\big|\wedge^2\big(B,G/\gamma_2(G)\big)\big|$ divides $|M^{(1)}(B)|\,|B\otimes A^{ab}|$;

(ii) $e\Big(\wedge^2\big(B,G/\gamma_2(G)\big)\Big)$ divides $e\big(M^{(1)}(B)\big)e(B\otimes A^{ab})$;

(iii) $d\Big(\wedge^2\big(B,G/\gamma_2(G)\big)\Big) \le d\big(M^{(1)}(B)\big) + d(B\otimes A^{ab})$.

For positive integers c and d let $\chi_c(d)$ denote the number of generators in a basis for the free abelian group $\gamma_c(F)/\gamma_{c+1}(F)$ where F is the free group of rank d. There is a well-known formula for $\chi_c(d)$ due to Witt. Let $\mu(m)$ be the Moebius function, defined for all positive integers m by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime number, $\mu(p^k) = 0$ for $k > 1$, and $\mu(bc) = \mu(b)\mu(c)$ if b and c are coprime integers. Witt's formula is

$$\chi_c(d) = \frac{1}{c}\sum_{m|c}\mu(m)d^{(c/m)}$$

where m runs through all divisors of c. Thus, for instance, $\chi_2(d) = (d^2-d)/2$, $\chi_3(d) = (d^3-d)/3$, $\chi_4(d) = (d^4-d^2)/4$.

THEOREM 3. *Suppose that G is a d-generator p-group (for some prime p). Let Φ denote the Frattini subgroup of G, and suppose that $\gamma_i(\Phi, G)$ has order p_i^m for $i \geq 1$. Then $M^{(c)}(G)$ is a p-group, and*

$$p^{\chi_{c+1}(d)} \leq |M^{(c)}(G)|\,|\gamma_{c+1}(G)| \leq p^{\chi_{c+1}(d)+m_c d+m_{c-1}d^2+\cdots+m_1 d^c}.$$

The upper and lower bounds are attained when G is elementary abelian: in this case $M^{(c)}(G)$ is elementary abelian on $\chi_{c+1}(d)$ generators.

PROOF. The sequence $(*)$ with $N = G$ yields $M^{(c)}(G)$ as a quotient of $\ker\big(\mu\colon \wedge^{c+1}(G, G) \longrightarrow G\big)$. It is shown in [4] that the exterior product of p-groups is a p-group. Consequently $M^{(c)}(G)$ is a p-group.

Note that $A = G/\Phi$ is elementary abelian of order p^d. In other words, A is a vector space of dimension d over $\mathbb{Z}_p$. It is observed in [5, Theorem 5] that the free Lie ring $L(A)$ on A is isomorphic to the Lie ring $\bigoplus_{c\geq 0} M^{(c)}(A)$ (with the obvious Lie bracket), and in particular, that $M^{(c)}(A)$ is isomorphic to the $(c+1)$-st term $\gamma_{c+1}\big(L(A)\big)$ of the lower central series of the Lie ring $L(A)$. But $\gamma_{c+1}\big(L(A)\big)$ is a vector space over $\mathbb{Z}_p$ of dimension $\chi_{c+1}(d)$. So the lower bound of the theorem follows from Proposition 1(i) with $H = A$.

To prove the upper bound let us introduce the invariant (*cf.* [8])

$$\gamma^*_{c+1}(G) = \gamma_{c+1}(F)/\gamma_{c+1}(R, F),$$

where $F/R \cong G$ is any free presentation of G. The sequence $(*)$ with $N = \Phi$ and $A = G/\Phi$ implies an exact sequence

$$\otimes^{c+1}(\Phi, G) \xrightarrow{\ \iota\ } \gamma^*_{c+1}(G) \longrightarrow \gamma^*_{c+1}(A) \longrightarrow 1.$$

Thus

$$|M^{(c)}(G)|\,|\gamma_{c+1}(G)| = |\gamma^*_{c+1}(G)| \leq |\gamma^*_{c+1}(A)|\,|\otimes^{c+1}(\Phi, G)|.$$

We know that $M^{(c)}(A) = p^{\chi_{c+1}(d)}$. Given an arbitrary normal subgroup N in G, Corollary 3 in [7] provides an upper bound for $|\otimes^{c+1}(N, G)|$. In particular, it provides the upper bound

$$|\otimes^{c+1}(\Phi, G)| \leq p^{m_c d+m_{c-1}d^2+\cdots+m_1 d^c}$$

which completes the proof. ∎

When $c = 1$, Theorem 3 improves on [10, Corollary 3.2] (which in turn is a generalisation of a result of J. A. Green). The second author has pursued the above methods further for the case $c = 1$, and obtained sharper upper bounds for $|M^{(1)}(G)|\,|\gamma_2(G)|$ in [6].

We remark that the inequalities

$$\Big|M^{(c)}\big((\mathbb{Z}_p)^d\big)\Big| \leq |M^{(c)}(G)|\,|\gamma_{c+1}(G)| \leq \Big|M^{(c)}\big((\mathbb{Z}_p)^n\big)\Big|$$

were proved in [15] and [14].

The final assertion in Theorem 3 leads to the computation of, for instance, the Baer invariants $M^{(c)}(Q_2)$ of the quaternion group Q_2 of order 8. It is well-known that $M^{(1)}(Q_2)$

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is trivial. (Recall from [3] that $Z_c^*(G)$ is the canonical image in G of the c-th term of the upper central series of $F/\gamma_{c+1}(R,F)$. Let us recall two properties of $Z_c^*(G)$ from [3, Lemma 2.1]: (i) $Z_1^*(G)$ lies in $Z_c^*(G)$; (ii) for any normal subgroup N of G which lies in $Z_c^*(G)$, the induced homomorphism $M^{(c)}(G) \rightarrowtail M^{(c)}(G/N)$ is injective.) Now $Z_1^*(Q_2)$ is shown in [3] to be the centre of Q_2. But the centre of Q_2 is equal to the derived subgroup. So for $c \geq 2$ the sequence $(*)$ with $G = Q_2$ and $N = Z_1^*(Q_2)$ yields an isomorphism $M^{(c)}(Q_2) \cong M^{(c)}(Q_2/[Q_2,Q_2])$. Since $Q_2^{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, it follows from Theorem 3 that, for $c \geq 2$, $M^{(c)}(Q_2)$ is elementary abelian of order $2^{\chi_{c+1}(2)}$.

Theorem 3 has a "group-theoretic" corollary.

COROLLARY 4. *(i) Let K be a group. Set $G = K/Z_c(K)$ and let Φ denote the Frattini subgroup of G. If G is a d-generator p-group with $|\gamma_i(\Phi,G)| = p^{m_i}$ for $i \geq 1$, then*

$$|\gamma_{c+1}(K)| \leq p^{\chi_{c+1}(d)+m_c d + m_{c-1}d^2+\cdots+m_1 d^c}.$$

(ii) If G is an elementary abelian p-group of order p^d, then there exists a group K such that $G \cong K/Z_c(K)$ and such that the bound is attained.

PROOF. (i) There is a canonical surjection $\gamma_{c+1}^*(G) \twoheadrightarrow \gamma_{c+1}(K)$. Thus $|\gamma_{c+1}(K)| \leq |\gamma_{c+1}^*(G)| = |M^{(c)}(G)|\,|\gamma_{c+1}(G)|$, and so the bound of the corollary follows from Theorem 3.

(ii) Suppose that G is elementary abelian of order p^d, and that G is freely presented as $G = F/R$. It is shown in [3] that $G \cong K/Z_c(K)$ where $K = F/\gamma_{c+1}(R,F)$. The bound of the corollary is attained since $|\gamma_{c+1}(K)| = |\gamma_{c+1}^*(G)|$. ■

When $c = 1$, Corollary 4 improves on [16, Theorem 2.1].

We remark that for $|K/Z_c(K)| = p^n$ the inequality $|\gamma_{c+1}(K)| \leq \left|M^{(c)}\big((\mathbb{Z}_p)^n\big)\right|$ is the principal result of [15]. (No explicit bound on $|\gamma_{c+1}(K)|$ is given in [15].)

Suppose that G is any nilpotent group of class k. Then conjugation yields trivial actions of $G/Z_{k-1}(G)$ on $\gamma_k(G)$, and of $\gamma_k(G)$ on $G/Z_{k-1}(G)$. We define the group $\wedge^{c+1}(\gamma_k G, G/Z_{k-1}G)$ by a pushout square in the category of groups (in which α and β are the obvious quotient homomorphisms):

$$\begin{array}{ccc} \otimes^{c+1}(\gamma_k G, G) & \xrightarrow{\alpha} & \otimes^{c+1}(\gamma_k G, G/Z_{k-1}G) \\ {\scriptstyle\beta}\big\downarrow & \text{pushout} & \big\downarrow \\ \wedge^{c+1}(\gamma_k G, G) & \longrightarrow & \wedge^{c+1}(\gamma_k G, G/Z_{k-1}G). \end{array}$$

In other words, $\wedge^{c+1}(\gamma_k G, G/\gamma_{k-1}G) = \otimes^{c+1}(\gamma_k G, G/Z_{k-1}G)/\alpha\big(\ker(\beta)\big)$. Note that $\otimes^{c+1}(\gamma_k G, G/Z_{k-1}G)$ is just an iteration of the usual tensor product of abelian groups.

PROPOSITION 5. *Let G be a nilpotent group of class $k \geq 2$. Then*

(i) $|\gamma_k(G)|\,|M^{(c)}(G)|$ *divides* $|M^{(c)}(G/\gamma_k G)|\,|\wedge^{c+1}(\gamma_k G, G/Z_{k-1}G)|$.

(ii) $e\big(M^{(c)}(G)\big)$ *divides* $e\big(M^{(c)}(G/\gamma_k G)\big) \times e\big(\wedge^{c+1}(\gamma_k G, G/Z_{k-1}G)\big)$.

(iii) $d\big(M^{(c)}(G)\big) \leq d\big(M^{(c)}(G/\gamma_k G)\big) + d\big(\wedge^{c+1}(\gamma_k G, G/Z_{k-1}G)\big)$.

PROOF. The sequence $(*)$ with $N = \gamma_k(G)$ yields an exact sequence

$$\wedge^{c+1}(\gamma_k G, G/Z_{k-1}G) \longrightarrow M^{(c)}(G) \longrightarrow M^{(c)}(G/\gamma_k G) \longrightarrow \gamma_k G \longrightarrow 1$$

from which (i), (ii) and (iii) follow. ∎

When $c = 1$, Proposition 5 reduces to [11, Proposition 2.4].

The following theorem is a particularly useful "starting key" for the computer program [3] mentioned above.

THEOREM 6. *Let G be a group of prime-power exponent p^e and nilpotency class $k \geq 2$. Then $e\big(M^{(c)}(G)\big)$ divides $p^{e(k-1)}$.*

PROOF. The result follows from Proposition 5(ii) and induction on k, once we have proved the case $k = 2$. So suppose $k = 2$. The sequence $(*)$ with $N = G$ yields $M^{(c)}(G)$ as a quotient of $\otimes^{c+1}(G, G)$. We shall show that the exponent of $\otimes^{c+1}(G, G)$ divides p^e. Since G is of class 2, for any integer m and elements x, y in G the identity

$$x \otimes y^m = (x \otimes y)^m \Big(y \otimes [x, y]^{\binom{m}{2}}\Big)$$

holds in $\otimes^2(G, G) = G \otimes G$ thanks to [1, Lemma 3.4]. In particular, for $m = p^e$ the integer $\binom{m}{2}$ is divisible by m when $p \geq 3$, and divisible by $m/2$ when $p = 2$. But when $p = 2$ and $m = p^e$ the identity

$$1 = (xy)^m = x^m y^m [x, y]^{m(m-1)/2} = [x, y]^{(m/2)(m-1)}$$

holds for all x, y in G, and implies $[x, y]^{m/2} = 1$. Thus the exponent of $\otimes^2(G, G)$ divides p^e. Since $\otimes^2(G, G)$ acts trivially on G, the exponent of $\otimes^{c+1}(G, G) = \big(\otimes^2(G, G) \otimes G\big) \otimes \cdots \otimes G$ divides the exponent of $\otimes^2(G, G)$. ∎

When $c = 1$, Theorem 6 reduces to [11, Corollaries 2.6 and 2.7].

There is a string of further interesting results on $M^{(c)}(G)$ that can be deduced from the sequence $(*)$. For instance, generalisations of Theorem 3.1 in [11], and its corollaries, are consequences of the sequence. Details are left to the reader.

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