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ON THE OPTIMALITY OF CERTAIN ESTIMATES FOR ALGEBRAIC VALUES OF ANALYTIC FUNCTIONS

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Abstract

We prove, by constructing a function with given parameters, that the estimate by G. V. Chudnovsky of the number of points at which a meromorphic function has algebraic Taylor coefficients is optimal. The construction is carried out by the use of interpolation series.

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Introduction

Let $f_1, \ldots, f_m$ be meromorphic functions on the complex plane $\mathbf{C}$. We discuss the optimality of certain estimates of the number of points at which $f_1, \ldots, f_m$ have algebraic Taylor coefficients. The classical Schneider–Lang theorem, improved by M. Waldschmidt [8, Theorem 3.3.1], asserts the following. Suppose $f_1, \ldots, f_m$ satisfy suitable differential equations with coefficients in a number field K. Suppose also f_1, f_2 are algebraically independent over the rational number field $\mathbf{Q}$, and of order at most ρ_1, ρ_2 respectively. Then the number of points at which all the derivatives of $f_1, \ldots, f_m$ take values in K is at most $[K:\mathbf{Q}](\rho_1 + \rho_2)$. Let us note that E. Bombieri conjectures that the number of such points will be at most $\rho_1 + \rho_2$. On the other hand, D. Bertrand [1] extended the above theorem to general meromorphic functions; that is, in place of considering functions satisfying differential equations, he considered functions such that the size of their Taylor coefficients satisfies certain conditions (we find analogous conditions also in Waldschmidt [9].) Then he obtained a similar but more general estimate

(Theorem 1) (see also Bertrand–Waldschmidt [2] for the detailed proof). But in the special case of Bertrand's theorem where the size of the Taylor coefficients of the functions behave as that of functions satisfying differential equations, the number of exceptional points is at most $[K:\mathbf{Q}](\rho_1 + \rho_2)$. So also for general meromorphic functions the estimate is the same as that of Schneider, Lang and Waldschmidt. G. V. Chudnovsky [4, 5] further extended these results: he succeeded in removing a certain condition imposed on the variable z, and obtained, under weaker assumptions, an estimate of the same form as Bertrand (Theorem 2).

The purpose of the present paper is to show that Chudnovsky's estimate is optimal (Theorem 3). Our method also fits the situation of Bertrand's theorem, so the estimate of Bertrand is also optimal, though it is for a slightly restricted case. From this result, we observe especially that in Bombieri's conjecture, the assumption that the functions satisfy suitable differential equations is certainly necessary. In other words, we can not replace this assumption by the assumption that they are general meromorphic functions with moderate Taylor coefficients.

The proof will be achieved by constructing a function with given parameters satisfying Chudnovsky's estimate, and the construction will be carried out by the use of interpolation series.

1. Statement of result

We shall state here Bertrand's theorem, Chudnovsky's theorem and our result. To this aim we recall the definition of well-behaved points, which is found in Bertrand [1] (Bertrand informed us that the terminology "well-behaved points" was suggested to him by D. Masser.) We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$, and for $\alpha \in \overline{\mathbf{Q}}$ we denote by $\overline{|\alpha|}$ the maximum of the absolute values of its conjugates, and for a number field K we denote by I_K the ring of algebraic integers in K.

DEFINITION 1. Let $f_1,\ldots,f_m$ be functions holomorphic at a point $w \in \mathbf{C}$. We say that w is a *well-behaved point* of $\{f_1,\ldots,f_m\}$, if there exist a number field K_w (of degree d_w), a positive number μ_w, a natural number δ_w and nonnegative integers δ'_w, δ''_w such that the following conditions are satisfied:

(i) all the numbers $f_i^{(k)}(w)$ for $k \geqslant 0$, $i = 1,\ldots,m$ lie in K_w;

(ii)

$$\limsup_{k\to\infty} \frac{\log \overline{|f_i^{(k)}(w)|}}{k \log k} \leqslant \mu_w$$

for $i = 1,\ldots,m$;

(iii) $\delta_w^{k+1}((\delta'_w k)!)^{\delta''_w} f_i^{(k)}(w) \in I_{K_w}$ for $k \geqslant 0$, $i = 1,\ldots,m$.

DEFINITION 2. An entire function f is *of order* ρ if

$$\limsup_{R\to\infty} \frac{\log\log|f|_R}{\log R} = \rho,$$

where

$$|f|_R = \max_{|z|=R} |f(z)|.$$

A meromorphic function is *of order* at most ρ if it is the quotient of two entire functions of order at most ρ.

DEFINITION 3. An entire function f is of *strict order* at most ρ if there exists a constant $c > 0$ such that $|f|_R < e^{cR^\rho}$ for $R \gg 1$. A meromorphic function is of *strict order* at most ρ if it is the quotient of two entire functions of strict order at most ρ.

REMARK 1. The order of an entire function f is equal to $\inf\{\rho | f$ is of strict order at most $\rho\}$.

Extending the Schneider–Lang theorem on meromorphic functions satisfying suitable differential equations, Bertrand obtained the following theorem on general meromorphic functions (see also [2]).

THEOREM 1 (Bertrand [1]). *Let f_1 and f_2 be two algebraically independent meromorphic functions of orders at most ρ_1, ρ_2 respectively. Then*

$$\sum_w \frac{1}{d_w\delta'_w\delta''_w + 1 + (d_w - 1)\mu'_w} \leqslant \rho_1 + \rho_2,$$

where w ranges over all well-behaved points of $\{f_1, f_2\}$, and $\mu'_w = \max(1, \mu_w)$.

Let $f(z)$ be a transcendental meromorphic function of order at most ρ. Then for the pair of functions $\{z, f(z)\}$, he obtained by the same method

$$\sum_w \frac{1}{d_w\delta'_w\delta''_w + 1 + (d_w - 1)\mu_w} \leqslant \rho,$$

where w ranges over all well-behaved points of $\{z, f\}$,

Let us note that by the definition of well-behaved points, here the value w of the variable z and the values $f^{(k)}(w)$ must belong to the same number field K_w.

However Chudnovsky succeeded in removing this condition on values of the variable, and proved that the same inequality holds provided only that w are algebraic numbers.

THEOREM 2 (Chudnovsky [4, 5]). *Let f be a transcendental meromorphic function of order at most ρ. Then*

$$\sum_w \frac{1}{d_w\delta'_w\delta''_w + 1 + (d_w - 1)\mu_w} \leqslant \rho,$$

where w ranges over all well-behaved points of $\{f\}$ such that $w \in \overline{\mathbf{Q}}$.

We shall show that the estimate of Theorem 2 is optimal. To this aim, we shall show that, for arbitrary parameters satisfying the estimate of Theorem 2, there exists a function f whose well-behaved points correspond to these parameters. It will turn out that these well-behaved points are all rational (and even integral): hence they are well-behaved points of $\{z, f\}$. So our result also shows that Theorem 1 is optimal, though it is for a slightly restricted case, that is, for the case where one of two functions is z.

THEOREM 3. *Let n range over the natural numbers from 1 to N or from 1 to ∞. Suppose we are given $\rho > 0$, $d_n \in \mathbf{N}$, $\mu_n > 0$, $\delta'_n, \delta''_n \in \mathbf{N} \cup \{0\}$ such that*

$$\sum_n \frac{1}{d_n\delta'_n\delta''_n + 1 + (d_n - 1)\mu_n} = \rho, \tag{1}$$

and

$$\mu_n \geqslant 1 - 1/\rho \tag{2}$$

for all n. Let K_n be real number fields such that $d_n = [K_n : \mathbf{Q}]$.

Then there exists a transcendental entire function f of strict order at most ρ such that the following conditions are satisfied for all n:

(i′) *all the numbers $f^{(k)}(n)$ for $k \geqslant 0$ lie in K_n;*

(ii′)
$$\limsup_{k\to\infty} \frac{\log \overline{|f^{(k)}(n)|}}{k \log k} \leqslant \mu_n;$$

(iii′) *there exists $\delta_n \in \mathbf{N}$ such that for $k \geqslant 0$*

$$\delta_n^{k+1}\left((\delta'_n k)!\right)^{\delta''_n} f^{(k)}(n) \in I_{K_n}.$$

REMARK 2. The condition (2) is a necessary condition for the existence of f. In fact (ii′) implies $|f^{(k)}(n)| \leqslant k^{(\mu_n+\varepsilon)k}$ for $\varepsilon > 0$ and $k \gg 1$. We may assume $\mu_n < 1$ and $\mu_n + \varepsilon < 1$. Then by applying Lemma 1 below to the Taylor expansion of f at $z = n$, we see that f is of strict order at most $1/(1 - \mu_n - \varepsilon)$. So by Remark 1, f is of order at most $1/(1 - \mu_n)$. Then Theorem 2 and (1) imply $\mu_n \geqslant 1 - 1/\rho$.

2. Preliminary lemmas

Concerning the strict order of a function, we have the following lemma.

LEMMA 1. *Let P_m $(m = 0, 1, \ldots)$ be polynomials such that the degree of P_m is at most m. Suppose $\rho > 0$ and there exists a constant c such that $|P_m|_R \leqslant c^m m^{-m/\rho} R^m$, for all $R \gg 1$ and $m \gg 1$. Then the series $\sum_{m=0}^{\infty} P_m$ converges uniformly on any compact set and defines an entire function of strict order at most ρ.*

PROOF. We may assume $c \geqslant 1$. Let $R \gg 1$ and $l_0 \gg 1$. Then using the fact that the number of integers m with $\rho(l-1) \leqslant m < \rho l$ is at most $[\rho] + 1$, we have

$$\begin{aligned}
\sum_{m \geqslant \rho(l_0 - 1)} |P_m|_R &\leqslant \sum_{m \geqslant \rho(l_0-1)} c^m m^{-m/\rho} R^m \\
&= \sum_{l=l_0}^{\infty} \sum_{\rho(l-1) \leqslant m < \rho l} c^m m^{-m/\rho} R^m \\
&\leqslant ([\rho] + 1) \sum_{l=l_0}^{\infty} c^{\rho l} (\rho(l-1))^{-(l-1)} R^{\rho l} \\
&\leqslant \sum_{l=l_0}^{\infty} c'^l l^{-l} R^{\rho l} \leqslant \sum_{l=l_0}^{\infty} c'^l R^{\rho l}/l! \\
&< e^{c' R^\rho},
\end{aligned}$$

with a constant c' independent of l, l_0 and R. This proves that $\sum_{m=0}^{\infty} P_m(z)$ converges uniformly on $|z| \leqslant R$, so on any compact set, and defines an entire function of strict order at most ρ, as desired.

The following lemma on simultaneous approximation is due to P. G. L. Dirichlet, and is well known (cf. W. M. Schmidt [6, page 34]).

LEMMA 2. *Let $\theta_1, \ldots, \theta_m$ be real numbers and suppose $Q > 1$. Then there exist integers $p, q_1, \ldots, q_m$ with*

$$1 \leqslant \max(|q_1|, \ldots, |q_m|) < Q$$

and

$$|p + \theta_1 q_1 + \cdots + \theta_m q_m| \leqslant Q^{-m}.$$

Among the following lemmas we shall need later Lemma 5 which is derived from Lemmas 3 and 4. Lemma 3 is on bad approximation, and is easily derived

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from the fact that the product of an algebraic integer with its all other conjugates is at least 1 in absolute value (cf. J. W. S. Cassels [3, page 79]).

LEMMA 3. *Let $\theta_1,\dots,\theta_m$ be m numbers in a real number field of degree $m+1$ such that $1, \theta_1,\dots,\theta_m$ are linearly independent over $\mathbf{Q}$. Then there is a constant $c > 0$ (depending only on $\theta_1,\dots,\theta_m$) such that*

$$|p + \theta_1 q_1 + \cdots + \theta_m q_m| \geqslant c\big(\max(|q_1|,\dots,|q_m|)\big)^{-m},$$

for any integers $p, q_1,\dots,q_m$ with $(q_1,\dots,q_m) \neq (0,\dots,0)$.

The next lemma is called a *transference theorem* with respect to simultaneous approximation (cf. Cassels [3, page 82]).

LEMMA 4. *Let $L_j(x)$, $x = (x_1,\dots,x_m)$ be l linear forms in m variables with real coefficients. Suppose that the simultaneous homogeneous inequalities*

$$|y_j + L_j(x)| < C, \qquad |x_i| < X,$$

have no integer solution $x, y_1,\dots,y_l$ with $x \neq 0$. Then for any real numbers $\alpha_1,\dots,\alpha_l$ the simultaneous inhomogeneous inequalities

$$|y_j + L_j(x) - \alpha_j| \leqslant C', \qquad |x_i| \leqslant X',$$

have no integer solution $x, y_1,\dots,y_l$, where

$$C' = \frac{(h+1)C}{2}, \qquad X' = \frac{(h+1)X}{2}, \qquad h = [X^{-m}C^{-l}].$$

By applying Lemmas 3 and 4 to $L_1(x) = \theta_1 x_1 + \cdots + \theta_m x_m$, we obtain the following lemma on inhomogeneous approximation.

LEMMA 5. *Let $\theta_1,\dots,\theta_m$ be m numbers in a real number field of degree $m+1$ such that $1, \theta_1,\dots,\theta_m$ are linearly independent over $\mathbf{Q}$. Then there is a constant $c > 0$ (depending only on $\theta_1,\dots,\theta_m$) such that for any real number α and $Q > 0$, there exist integers $p, q_1,\dots,q_m$ with*

$$\max(|q_1|,\dots,|q_m|) \leqslant Q$$

and

$$|p + \theta_1 q_1 + \cdots + \theta_m q_m - \alpha| \leqslant cQ^{-m}.$$

3. Proof of Theorem 3

We now turn to the proof of Theorem 3. So we suppose that n ranges over the natural numbers from 1 to N or from 1 to ∞ and that ρ, d_n, μ_n, δ_n', δ_n'' and K_n satisfy the assumptions of Theorem 3. Let

$$\zeta_n = \left(d_n\delta_n'\delta_n'' + 1 + (d_n - 1)\mu_n\right)\rho. \tag{3}$$

Then (1) is equivalent to

$$\sum_n \frac{1}{\zeta_n} = 1. \tag{1'}$$

We divide the proof into steps.

Step 1: *interpolation series*. We shall construct the desired function f by the use of interpolation series (cf. Th. Schneider [7, Chapter 2]). For interpolation points, we take N points $1, 2, \ldots, N$ or a countable number of points $1, 2, \ldots$ according as n ranges from 1 to N or from 1 to ∞. Furthermore, we take each point n infinitely often, but we take n in such a way that the frequency of taking each n has weight $1/\zeta_n$. We state this precisely below. To this aim, we introduce some notation.

First for any integer $m \geqslant 0$, let

$$\Lambda_m = \left\{n \mid [m/\zeta_n] < [(m+1)/\zeta_n]\right\}, \tag{4}$$

[ ] denoting the Gauss symbol, and let

$$s(m) = \text{the number of the elements of } \Lambda_m.$$

We see that $s(m)$ is finite. More precisely, we have the following estimate. By (1′) we have $\zeta_n \geqslant 1$ for any n, and so $[(m+1)/\zeta_n] - [m/\zeta_n] = 0$ or 1. Therefore we have

$$\sum_n [(m+1)/\zeta_n] = \sum_n [m/\zeta_n] + s(m),$$

and by (1′) again we have

$$s(0) + \cdots + s(m) \leqslant m + 1. \tag{5}$$

Let us arrange the elements of Λ_m in the order of size, and denote them by $n_{m,1}, \ldots, n_{m,s(m)}$ so that we have

$$n_{m,1} < n_{m,2} < \cdots < n_{m,s(m)}.$$

In the following we shall consider only the pairs of integers (m, s) such that $\Lambda_m \neq \varnothing$ and $1 \leqslant s \leqslant s(m)$. We introduce a linear order among such pairs by defining lexicographically

$$(l, \sigma) < (m, s) \qquad \text{if } l < m \text{ or } l = m, \text{ and } \sigma < s. \tag{6}$$

Now for every pair (m, s) we define a function $\phi_{m,s}(z)$ by

$$(7) \qquad \phi_{m,s}(z) = \prod_{(l,\sigma)<(m,s)} \frac{z - n_{l,\sigma}}{n_{l,\sigma}}.$$

With this notation, we shall construct the desired function f in the form

$$(8) \qquad f(z) = \sum_{(m,s)} a_{m,s}\phi_{m,s},$$

where (m, s) ranges over all pairs such that $\Lambda_m \neq \varnothing$ and $1 \leqslant s \leqslant s(m)$.

Note that $\phi_{m,s}(z)$ can be written

$$(9) \qquad \phi_{m,s}(z) = \prod_n \left(\frac{z-n}{n}\right)^{[m/\zeta_n]} \times \frac{z - n_{m,1}}{n_{m,1}} \cdot \ \cdots \ \cdot \frac{z - n_{m,s-1}}{n_{m,s-1}},$$

because for some $l < m$, n is written $n = n_{l,\sigma}$ if n belongs to Λ_l, equivalently if $[l/\zeta_n] < [(l+1)/\zeta_n]$ and because there are $[m/\zeta_n]$ number of such l. In view of this expression we see that for m fixed, $(z - n)$ appears in $\phi_{m,s}(z)$ almost $[m/\zeta_n]$ times for each n. This is the meaning of our former phrase "the frequency of taking each n has weight $1/\zeta_n$."

Step 2: *properties of* $\phi_{m,s}$. We show that $\phi_{m,s}$ has the following properties.

ASSERTION 1.

(a) $\phi_{m,s}$ *has at most m factors.*

(b) $|\phi_{m,s}^{(k)}(n)| \leqslant n^m k!$ *for all* $k \geqslant 0$ *and* n.

(c) $|\phi_{m,s}|_R \leqslant (2R)^m$ *for* $R \geqslant 1$.

(d) $\phi_{m,s}^{(k)} \neq 0$ *if* $k > m$.

(e) *Suppose* $k = [m/\zeta_{n_{m,s}}]$. *Then* $\phi_{l,\sigma}^{(k)}(n_{m,s}) = 0$ *if* $(l, \sigma) > (m, s)$.

(f) *Suppose* $k = [m/\zeta_{n_{m,s}}]$. *Then* $|\phi_{m,s}^{(k)}(n_{m,s})| \geqslant k!/(2n_{m,s})^m$.

PROOF.

(a) By the definition of linear order (6) and the definition (7), $\phi_{m,s}$ has $s(0) + \cdots + s(m-1) + s - 1$ factors, and by (5) this number is at most m.

(b) Let Γ_n be the circle of center n with radius 1. Then Cauchy's integral formula implies

$$\left|\phi_{m,s}^{(k)}(n)\right| \leqslant k!|\phi_{m,s}|_{\Gamma_n},$$

where $|\phi_{m,s}|_{\Gamma_n} = \max_{z\in\Gamma_n}|\phi_{m,s}(z)|$. Note that for any positive integer n' we have

$$\left|\frac{z-n'}{n'}\right|_{\Gamma_n} = \frac{|n'-n|+1}{n'}$$

$$= \begin{cases} \dfrac{n'-n+1}{n'} \leqslant 1 \leqslant n & \text{if } n' > n, \\[2ex] \dfrac{n-n'+1}{n'} \leqslant \dfrac{n}{n'} \leqslant n & \text{if } n' \leqslant n. \end{cases}$$

Then since $\phi_{m,s}$ has at most m factors by (a), we obtain $|\phi_{m,s}|_{\Gamma_n} \leqslant n^m$, which implies (b).

(c) For any n, $|z-n|_R = R+n$. Also $R+n \leqslant 2Rn$, since $R, n \geqslant 1$. So $|(z-n)/n|_R \leqslant 2R$. Then by (a) we obtain $|\phi_{m,s}|_R \leqslant (2R)^m$.

(d) By (a) the degree of $\phi_{m,s}$ is at most m. Hence clearly (d) holds.

(e) Suppose $k = [m/\zeta_{n_{m,s}}]$. If $(l,\sigma) > (m,s)$, then by the expression (9) we see that $\phi_{l,\sigma}$ contains the factor $(z - n_{m,s})$ at least $[m/\zeta_{n_{m,s}}]+1$ times. Therefore $\phi_{l,\sigma}^{(k)}(n_{m,s}) = 0$.

(f) Suppose $k = [m/\zeta_{n_{m,s}}]$. By the expression (9), $\phi_{m,s}$ has the factor $(z-n_{m,s})$ exactly $[m/\zeta_{n_{m,s}}]$ times. So we have

$$|\phi_{m,s}^{(k)}(n_{m,s})| = \frac{k!}{(n_{m,s})^{[m/\zeta_{n_{m,s}}]}} \prod_{n \neq n_{m,s}} \left| \frac{n_{m,s}-n}{n} \right|^{[m/\zeta_n]}$$

$$\times \left| \frac{n_{m,s}-n_{m,1}}{n_{m,1}} \right| \cdots \left| \frac{n_{m,s}-n_{m,s-1}}{n_{m,s-1}} \right|.$$

On the other hand, we have $|(n_{m,s}-n)/n| \geqslant 1/(2n_{m,s})$ for $n \neq n_{m,s}$. In fact, if $n \leqslant 2n_{m,s}$, then this is clear, and if $n > 2n_{m,s}$, then $|(n_{m,s}-n)/n| > 1/2 \geqslant 1/(2n_{m,s})$. Therefore (a) implies that the right-hand side of the above equality is greater than or equal to $k!/(2n_{m,s})^m$, as desired.

Step 3: *induction for* $a_{m,s}$. Our objective is to choose $a_{m,s}$ well so that the function f defined by (8) has the desired properties. We wish to choose $a_{m,s}$ by induction on (m,s) with respect to our lexicographic linear order. In order to formulate the induction, we make some preliminary observation.

Let us suppose that $a_{m,s}$ have been chosen and the termwise differentiation of (8) is allowed. Then by (e) we find that for any $k \geqslant 0$ and any n, $f^{(k)}(n)$ is expressed as a finite sum

$$f^{(k)}(n) = \sum_{(l,\sigma) \leqslant (m,s)} a_{l,\sigma} \phi_{l,\sigma}^{(k)}(n).$$

Indeed, the smallest (m,s) admitting such an expression is given as follows. Let m be the integer with $k = [m/\zeta_n] < [(m+1)/\zeta_n]$. Then n is an element of Λ_m. Hence we can find s with $1 \leqslant s \leqslant s(m)$ such that $n = n_{m,s}$. Clearly (m,s) is determined in a unique way by (k,n). Let us denote this correspondence by G: $(k,n) \to (m,s)$. Then (e) tells us that if $(l,\sigma) > (m,s)$ then $\phi_{l,\sigma}^{(k)}(n_{m,s}) = 0$; that is, $\phi_{l,\sigma}^{(k)}(n) = 0$. Moreover (f) tells us that $\phi_{m,s}^{(k)}(n) \neq 0$. Therefore this (m,s) is just the smallest one having the above mentioned property. Note that G is a one to one mapping of the set of all pairs (k,n) with $k \geqslant 0$ onto the set of all pairs (m,s) such that $\Lambda_m \neq \varnothing$ and $1 \leqslant s \leqslant s(m)$.

Now with this observation we introduce the following notation. For any $k \geqslant 0$ and n, we set $(m, s) = G(k, n)$, and write

(10) $$\beta_{k,n} = \sum_{(l,\sigma)<(m,s)} a_{l,\sigma}\phi_{l,\sigma}^{(k)}(n)$$

and

$$\gamma_{k,n} = \sum_{(l,\sigma)\leqslant(m,s)} a_{l,\sigma}\phi_{l,\sigma}^{(k)}(n).$$

Note that

(11) $$k = [m/\zeta_n] < [(m+1)/\zeta_n] \quad \text{and} \quad n = n_{m,s}.$$

Also we have

(12) $$\gamma_{k,n} = \beta_{k,n} + a_{m,s}\phi_{m,s}^{(k)}(n).$$

Furthermore, $f^{(k)}(n) = \gamma_{k,n}$ if the termwise differentiation of (8) is allowed.

With this notation our objective is stated as follows.

ASSERTION 2. *There exist positive constants C_n, positive integers δ_n (both depending only on the n-th date in Theorem 3) and real algebraic numbers $a_{m,s}$ such that the following conditions are satisfied for all (m, s) and (k, n) with $\Lambda_m \neq \varnothing$, $1 \leqslant s \leqslant s(m)$ and $k \geqslant 0$:*

(i″) $|a_{m,s}| \leqslant m^{-m/\rho}$;

(ii″) $\gamma_{k,n} \neq 0$ *and* $\overline{|\gamma_{k,n}|} \leqslant C_n^{k+1}k^{\mu_n k}$;

(iii″) $\delta_n^{k+1}((\delta'_n k)!)\delta''_n\gamma_{k,n} \in I_{K_n}$.

For the proof we use the following induction on (m, s).

For a pair (m, s) with $\Lambda_m \neq \varnothing$ and $1 \leqslant s \leqslant s(m)$, we assume the following condition (A) holds.

(A) For all $(l, \sigma) < (m, s)$, real algebraic numbers $a_{l,\sigma}$ have been chosen, and they satisfy (i″).

Under this assumption, we show that the following condition (B) holds.

(B) There exists a real algebraic number $a_{m,s}$ such that $a_{m,s}$ satisfies (i″), and such that for $(k, n) = G^{-1}(m, s)$, $\gamma_{k,n}$ satisfies (ii″) and (iii″).

Note that for the smallest pair, say $(m_0, 1)$, we may consider that (A) holds trivially. Also recall that G is a one to one mapping. Then it is easy to see that by induction this implies Assertion 2.

In this paper we set $0^0 = 1$.

In the following steps we shall prove that (B) holds under the assumption (A). To this aim, we choose a basis of I_{K_n}, $\{\omega_1 = 1, \omega_2, \ldots, \omega_{d_n}\}$ for every n. For simplicity we use the same letters for all n. Then we fix a pair (m, s) with $\Lambda_m \neq \varnothing$ and $1 \leqslant s \leqslant s(m)$, and assume (A) holds for this (m, s). Let $(k, n) = G^{-1}(m, s)$; that is, let k, n, m, s satisfy (11). We fix this notation: so in the following (up to Step 8), by (k, n) we shall always mean this (k, n) defined here.

Step 4: $|\beta_{k,n}| \leqslant (2n)^m k^{(1-1/\rho)k}$. Indeed, by (10) and Assertion 1(d) we have

$$\beta_{k,n} = \sum_{(l,\sigma)<(m,s),\, l \geqslant k} a_{l,\sigma}\phi_{l,\sigma}^{(k)}(n).$$

Then by (b), (A) and (5), we obtain

$$\begin{aligned} |\beta_{k,n}| &\leqslant \sum_{(l,\sigma)<(m,s),\, l\geqslant k} l^{-l/\rho} n^l k! \\ &\leqslant k! n^m k^{-k/\rho} \sum_{l=k}^{m} s(l) \\ &\leqslant k! n^m k^{-k/\rho}(m+1) \leqslant (2n)^m k^{(1-1/\rho)k}, \end{aligned}$$

as desired.

Step 5: *determination of* $a_{m,s}$ *in the case* $d_n = 1$. Here we suppose $d_n = 1$; that is, $K_n = \mathbf{Q}$. Let δ_n be a positive integer such that

$$\delta_n \geqslant (2n)^{\zeta_n}(2e\zeta_n)^{\zeta_n/\rho}. \tag{13}$$

For simplicity, let us write

$$\delta_{k,n} = \delta_n^{k+1}\left((\delta_n' k)!\right)^{\delta_n''}. \tag{14}$$

If $[\delta_{k,n}\beta_{k,n}] = 0$, then we set $q = 1$, and otherwise we set $q = [\delta_{k,n}\beta_{k,n}]$. We define $a_{m,s}$ by

$$a_{m,s} = \frac{q - \delta_{k,n}\beta_{k,n}}{\delta_{k,n}\phi_{m,s}^{(k)}(n)}. \tag{15}$$

Then $a_{m,s} \in \overline{\mathbf{Q}}$. Further together with (12), we have

$$\gamma_{k,n} = q/\delta_{k,n}. \tag{16}$$

Also we have $\gamma_{k,n} \neq 0$, as $q \neq 0$.

Step 6: *verification of* (i″), (ii″) *and* (iii″) *in the case* $d_n = 1$. First let us verify (i″) for the above $a_{m,s}$. Since the absolute value of the numerator of (15) is at most 1, (11) and (f) yield $|a_{m,s}| \leqslant (2n)^m/(\delta_{k,n} k!)$. Note that $(\delta_n' k)! \geqslant (k!)^{\delta_n'}$, and thus $\delta_{k,n} k! \geqslant \delta_n^{k+1}(k!)^{1+\delta_n'\delta_n''}$ by (14). Recall that $k = [m/\zeta_n]$ by (11). So $k + 1 > m/\zeta_n$. Then evaluating $k!$ by the use of the inequality $[x]! \geqslant x^x/(2e)^x$ $(x \geqslant 0)$, we obtain

$$|a_{m,s}| \leqslant \frac{(2n)^m (2e)^{(1+\delta_n'\delta_n'')m/\zeta_n}}{\delta_n^{m/\zeta_n}(m/\zeta_n)^{(1+\delta_n'\delta_n'')m/\zeta_n}}.$$

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On the other hand, since $d_n = 1$, we have $(1 + \delta_n'\delta_n'')/\zeta_n = 1/\rho$ by (3). Therefore together with (13) we obtain

$$|a_{m,s}| \leqslant \left(\frac{2n(2e\zeta_n)^{1/\rho}}{\delta_n^{1/\zeta_n}}\right)^m \frac{1}{m^{m/\rho}} \leqslant m^{-m/\rho},$$

as desired.

Next let us verify (ii″) and (iii″) for the above $\gamma_{k,n}$. It was already verified that $\gamma_{k,n} \neq 0$. By (16) and the definition of q, we have $\gamma_{k,n} \in \mathbf{Q}$ and $|\gamma_{k,n}| \leqslant |\beta_{k,n}| + 1$. Then in view of (2), the inequality $m/\zeta_n < k + 1$ and the estimate of $|\beta_{k,n}|$, we obtain

$$\begin{aligned}|\gamma_{k,n}| &\leqslant (2n)^m k^{(1-1/\rho)k} + 1 < (2n)^{\zeta_n(k+1)} k^{\mu_n k} + 1\\ &< (4n)^{\zeta_n(k+1)} k^{\mu_n k} \leqslant C_n^{k+1} k^{\mu_n k},\end{aligned}$$

with a constant C_n greater than or equal to $(4n)^{\zeta_n}$. Thus (ii″) holds. By (14) and (16), (iii″) holds clearly.

Step 7: determination of $a_{m,s}$ in the case $d_n > 1$. Next we suppose $d_n > 1$. In this case we use Lemma 5 to determine $a_{m,s}$. For simplicity, here we write

$$\delta_{k,n} = \left((\delta_n' k)!\right)^{\delta_n''}, \tag{17}$$

instead of (14). Recall that $\{\omega_1 = 1, \omega_2, \ldots, \omega_{d_n}\}$ is a basis of I_{K_n}. Let us apply Lemma 5 to $\omega_2, \ldots, \omega_{d_n}$ in place of $\theta_1, \ldots, \theta_m$ and to the number field K_n. Then there exists a constant C_n' depending only on $\omega_2, \ldots, \omega_{d_n}$, and playing the same role as the constant c in Lemma 5. Therefore for $\delta_{k,n}\beta_{k,n}$ and

$$Q = m^{(\mu_n + \delta_n'\delta_n'')m/\zeta_n} C_n''^{m} \left(C_n' + 1\right)^{1/(d_n - 1)} \tag{18}$$

with a constant C_n'' such that

$$C_n''^{d_n - 1} \geqslant 2n(2e\zeta_n)^{(1+\delta_n'\delta_n'')/\zeta_n}, \tag{19}$$

there exist integers $q_1', \ldots, q_{d_n}'$ such that $\max(|q_2'|, \ldots, |q_{d_n}'|) \leqslant Q$ and

$$\left|\sum_{i=1}^{d_n} \omega_i q_i' - \delta_{k,n}\beta_{k,n}\right| \leqslant C_n' Q^{-(d_n - 1)}.$$

If $\sum_{i=1}^{d_n} \omega_i q_i' \neq 0$, then we set $q_i = q_i'$. If it is equal to 0, then $(q_1', \ldots, q_{d_n}') = (0, \ldots, 0)$ as $\omega_1 = 1$, $\omega_2, \ldots, \omega_{d_n}$ are linearly independent over $\mathbf{Q}$, and in this case we set $q_i = q_i''$, where $q_i'', \ldots, q_{d_n}''$ are integers such that $1 \leqslant \max(|q_2''|, \ldots, |q_{d_n}''|) < Q$ and $|\sum_{i=1}^{d_n} \omega_i q_i''| \leqslant Q^{-(d_n - 1)}$. The existence of these q_i'' follows from Lemma 2, as $Q > 1$ by (18). Note that $\sum_{i=1}^{d_n} \omega_i q_i'' \neq 0$. Therefore in both the cases we have

$$\max\left(|q_2|, \ldots, |q_{d_n}|\right) \leqslant Q \tag{20}$$

and

$$\left| \sum_{i=1}^{d_n} \omega_i q_i - \delta_{k,n}\beta_{k,n} \right| \leqslant \left(C_n' + 1\right) Q^{-(d_n - 1)}. \tag{21}$$

Now using these $q_1, \dots, q_{d_n}$, we define $a_{m,s}$ by

$$a_{m,s} = \frac{\omega_1 q_1 + \cdots + \omega_{d_n} q_{d_n} - \delta_{k,n}\beta_{k,n}}{\delta_{k,n}\phi_{m,s}^{(k)}(n)}. \tag{22}$$

Then $a_{m,s} \in \overline{\mathbf{Q}}$. Further by (12) we have

$$\gamma_{k,n} = \frac{\omega_1 q_1 + \cdots + \omega_{d_n} q_{d_n}}{\delta_{k,n}}. \tag{23}$$

Also $\gamma_{k,n} \neq 0$.

Step 8: *verification of* (i″), (ii″) *and* (iii″) *in the case* $d_n > 1$. First let us verify (i″) for $a_{m,s}$ defined in Step 7. By (18), (21), (22) and (f), we have

$$|a_{m,s}| \leqslant \frac{(2n)^m \left(C_n' + 1\right)}{\delta_{k,n} k! Q^{d_n - 1}} = \frac{(2n)^m}{\delta_{k,n} k! m^{(d_n-1)(\mu_n + \delta_n'\delta_n'')m/\zeta_n} C_n''^{(d_n-1)m}}.$$

Then since $k = [m/\zeta_n]$ and we have (17), the same argument as Step 6 implies

$$|a_{m,s}| \leqslant \frac{(2n)^m (2e)^{(1+\delta_n'\delta_n'')m/\zeta_n}}{(m/\zeta_n)^{(1+\delta_n'\delta_n'')m/\zeta_n} m^{(d_n-1)(\mu_n+\delta_n'\delta_n'')m/\zeta_n} C_n''^{(d_n-1)m}}.$$

On the other hand, by the definition (3) we have

$$\begin{aligned} &\left((1 + \delta_n'\delta_n'') + (d_n - 1)\left(\mu_n + \delta_n'\delta_n''\right)\right)/\zeta_n \\ &\qquad = \left(d_n\delta_n'\delta_n'' + 1 + (d_n - 1)\mu_n\right)/\zeta_n = 1/\rho. \end{aligned}$$

Thus by (19) we have

$$|a_{m,s}| \leqslant \left(\frac{2n(2e\zeta_n)^{(1+\delta_n'\delta_n'')/\zeta_n}}{C_n''^{d_n - 1}} \right)^m \frac{1}{m^{m/\rho}} \leqslant m^{-m/\rho},$$

as desired.

Next let us verify (ii″) for $\gamma_{k,n}$. It was already verified that $\gamma_{k,n} \neq 0$. By (23) we have

$$\left| \overline{|\gamma_{k,n}|} - \gamma_{k,n} \right| \leqslant \frac{1}{\delta_{k,n}} \sum_{i=2}^{d_n} 2 \, \overline{|\omega_i|} \, |q_i|.$$

Also by (18), (21), (23) and Step 4, we have

$$|\gamma_{k,n}| \leqslant \frac{\left(C_n' + 1\right) Q^{-(d_n - 1)}}{\delta_{k,n}} + |\beta_{k,n}| \leqslant 1 + (2n)^m k^{(1-1/\rho)k}.$$

Hence by (17), (18) and (20) we have

$$\overline{|\gamma_{k,n}|} \leqslant 1 + (2n)^m k^{(1-1/\rho)k} + \frac{2(d_n - 1)\max_{i \geqslant 2}\left(\overline{|\omega_i|}\right) m^{(\mu_n + \delta'_n\delta''_n)m/\zeta_n} C''^m_n \left(C'_n + 1\right)^{1/(d_n - 1)}}{\left((\delta'_n k)!\right)^{\delta''_n}}.$$

Note that we have $m/\zeta_n < k + 1$, and $1 - 1/\rho \leqslant \mu_n$ by (2). Then using again the inequality $(\delta'_n k)! \geqslant (k!)^{\delta'_n} \geqslant k^{\delta'_n k}(2e)^{-\delta'_n k}$, we obtain

$$\overline{|\gamma_{k,n}|} < C_n^{k+1} k^{\mu_n k},$$

with a sufficiently large constant C_n independent of k.

Finally, by (17) and (23), (iii″) holds clearly with $\delta_n = 1$.

Thus the results of Step 4 to Step 8 together imply that (B) holds under the assumption (A). Therefore by induction, Assertion 2 in Step 3 is proved.

Step 9: *verification of properties of* f. Let us define f by (8), using $a_{m,s}$ of Assertion 2. We wish to verify that f has the desired properties of Theorem 3.

First let us verify that f is a transcendental entire function of strict order at most ρ. We apply Lemma 1 with $P_m = \sum_{1 \leqslant s \leqslant s(m)} a_{m,s}\phi_{m,s}$. By (5), (c) and (i″), we have for any m and $R \geqslant 1$.

$$|P_m|_R \leqslant \sum_{1 \leqslant s \leqslant s(m)} |a_{m,s}||\phi_{m,s}|_R \leqslant (m+1)m^{-m/\rho}(2R)^m.$$

Hence f is an entire function of strict order at most ρ. Further, since the termwise differentiation is allowed, we have $f^{(k)}(n) = \gamma_{k,n}$ by the preliminary observation in Step 3. So by (ii″), f is transcendental.

Next let us verify (i′), (ii′) and (iii′). Since $f^{(k)}(n) = \gamma_{k,n}$, clearly (iii″) implies (i′) and (iii′), and also (ii″) implies (ii′). Thus the proof of Theorem 3 is completely achieved.

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