A NUMERICAL SCHEME TO INTEGRATE THE ROTATIONAL MOTION OF A RIGID BODY

TOSHIO FUKUSHIMA
National Astronomical Observatory
2-21-1, Ohsawa, Mitaka, Tokyo 181, Japan
(Internet) toshio@spacetime.mtk.nao.ac.jp

1. Introduction

Once, we numerically integrated the precession and nutation of a spheroidal rigid Earth (Kubo and Fukushima 1987). As a natural extension, we tried to integrate the rotation of a triaxial rigid Earth numerically and faced a problem: a loss of precision in long-term integration. This is due to the smallness of the characteristic period of the problem: 1 day. Of course, one can integrate the rotational motion in higher precision arithmetics with a smaller stepsize. However, the quadruple precision integration is roughly 30 times more time-consuming than the double precision integration. See Table 1. Therefore, it is desirable if there is a formulation 1) reducing the overall integration error, 2) being independent on the choice of the integrator and 3) requiring no extra computations. The key points to achieve this goal will be to find a set of variables which 1) are efficiently convertible to the physical quantities required finally, say, the orientation matrix in the case of the rotational dynamics, *and* 2) vary with time as smoothly as possible. In this note, we report a discovery of such an example.

2. Scheme

As the basic variables describing the rotational motion of a rigid body, we adopt $(L, L_X, L_Y, L_A, L_B, f)$; the magnitude and X-, Y-, A-, B- components of the rotational angular momentum vector $\vec{L}$, and the angle f measured from the plane containing $\vec{L}$ and the X-axis to the plane containing $\vec{L}$ and the A-axis. The orientation matrix $\mathcal{E}$ is expressed in terms

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TABLE 1. Comparison of CPU time

Scheme	Precision	CPU time	
		Evaluation of $\mathcal{E}$	Eq. of Motion
New	Double	117	195
Euler		117	206
Serret-Andoyer		147	250
Euler	Quadruple	3378	7195

Note: The unit of CPU time is μs in HP/9000 715/50MHz.

of these variables as

$$\mathcal{E} \equiv (\vec{e}_{\mathrm{A}}, \vec{e}_{\mathrm{B}}, \vec{e}_{\mathrm{C}}) = \mathcal{Q}_{12}\left(\frac{L_{\mathrm{X}}}{L}, \frac{L_{\mathrm{Y}}}{L}\right)\mathcal{R}_3(-f)\mathcal{Q}_{12}^T\left(\frac{L_{\mathrm{A}}}{L}, \frac{L_{\mathrm{B}}}{L}\right). \tag{1}$$

Here, $\vec{e}_j$ is the unit column vector defining the j-th axis, and

$$\mathcal{Q}_{12}(x,y) \equiv \mathcal{R}_1\left(+\tan^{-1}\frac{y}{z}\right)\mathcal{R}_2\left(-\sin^{-1}x\right) = \begin{pmatrix} w & 0 & x \\ -xy/w & z/w & y \\ -xz/w & -y/w & z \end{pmatrix},$$

where $w = \sqrt{1-x^2}$ and $z = \sqrt{1-(x^2+y^2)}$. Thus, $\mathcal{E}$ is generated by the five successive rotations in the sense of 1-2-3-2-1. In the actual integration, we integrate not these basic variables but their departures from nominal constants, their initial values for the first five and a linear function of time for f. In the language of the Earth rotation, these departures correspond to the variation of LOD, the nutation in obliquity and in declination, the polar motion and the variation of UT1. The equation of motion is simple. Those for the first five are just the translation of the conservation law of $\vec{L}$ in the inertial and body-fixed coordinate systems. That for f is

$$\begin{aligned} \frac{\mathrm{d}\Delta f}{\mathrm{d}t} &= \frac{\Delta L}{C} + \left(\frac{1}{B} - \frac{1}{C}\right)\frac{LL_{\mathrm{B}}^2}{L^2 - L_{\mathrm{A}}^2} \\ &\quad + \frac{L_{\mathrm{A}}(L_{\mathrm{C}}N_{\mathrm{B}} - L_{\mathrm{B}}N_{\mathrm{C}})}{L(L^2 - L_{\mathrm{A}}^2)} - \frac{L_{\mathrm{X}}(L_{\mathrm{Z}}N_{\mathrm{Y}} - L_{\mathrm{Y}}N_{\mathrm{Z}})}{L(L^2 - L_{\mathrm{X}}^2)} \end{aligned} \tag{2}$$

where $L_{\mathrm{Z}} = \sqrt{L^2 - (L_{\mathrm{X}}^2 + L_{\mathrm{Y}}^2)}$, $L_{\mathrm{C}} = \sqrt{L^2 - (L_{\mathrm{A}}^2 + L_{\mathrm{B}}^2)}$, and $N_j = \vec{N}\cdot\vec{e}_j$ is the j-th component of the torque $\vec{N}$.

3. Comparisons with Other Schemes

Figure 1 illustrates the growth of integration errors in $\approx$ 90 years of the rotational motion of a rigid Earth perturbed by Moon and Sun in model

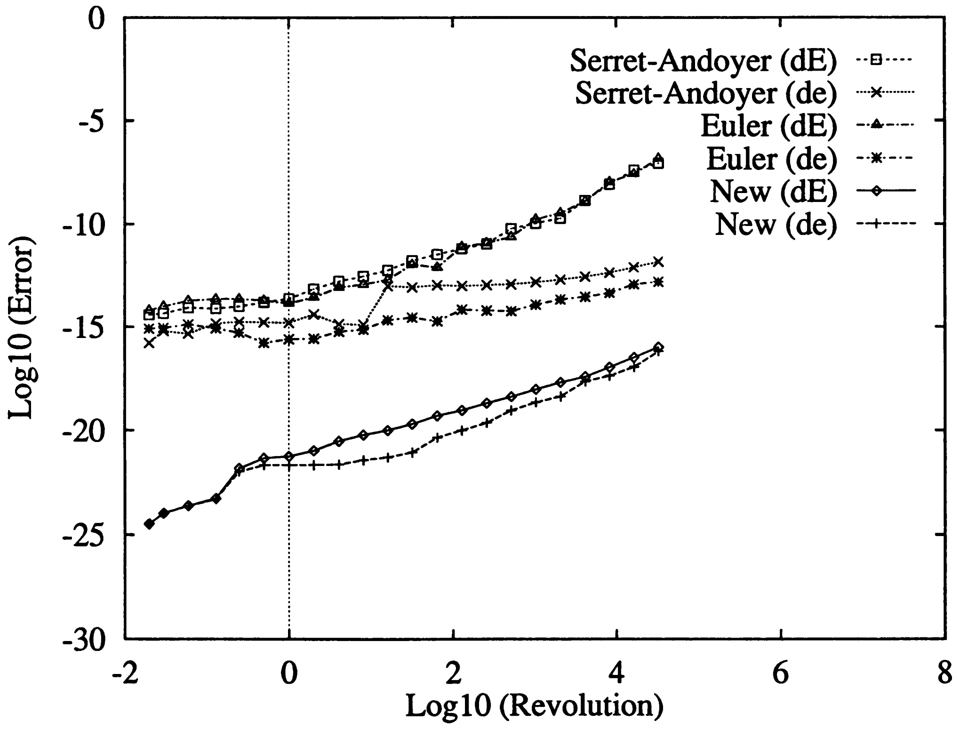

Figure 1. Growth of Integration Error of Earth Rotation

orbits for the new and two other schemes: the well-known Eulerian and the Serret-Andoyer canonical ones. The lines (dE) and (de) denote the errors in $\mathcal{E}$, the total orientation, and in $\vec{e}_C$, the figure axis, respectively. We obtained similar results for the Moon's rotation also. The integrations were done by the 12th order Adams method in 53-bits mantissa arithmetics with the stepsize of 1/128 nominal rotational period. It is clear that the integration error of the new scheme is drastically smaller, say, by 7-9 digits for the overall orientation, and 3-10 digits for the figure axis. The observed large differences in the errors are due to the large differences in the magnitude of the integrated variables, especially those related to the rotation angle. All the integrated variables remain very small in the new scheme, namely less than 10", which is the magnitude of nutation. Also, the growth of error seems smaller, with an almost linear growth for the first 10^4 revolutions. The reason is not clarified yet. On the other hand, Table 1 shows the averaged CPU times for the three formulations. As is seen, there is no actual difference between the Euler and new schemes.

4. Conclusion

A new numerical scheme to integrate the orientation of a rigid body was presented. The adopted basic variables are the magnitude, the X-, Y-, A- and B-components of the angular momentum vector, and the longitude of the A-axis measured from the X-axis along the great circle perpendicular to the angular momentum. Not these basic variables, but the correction to their nominal constants and/or linear motion are integrated. Numerical simulations showed that the new scheme integrates the orientation matrices of the Earth and the Moon 5-9 digits more precisely than the ordinary Eulerian approach or the alternative Serret-Andoyer one does while the required computational time does not change significantly.

We will make two comments on the new variables themselves. First, they are translated to the set of canonical variables $(L, L_{\rm X}, L_{\rm A}; f, -\sigma, \xi)$ where the auxiliary angles σ and ξ are defined in Fukushima (1994). This indicates the possibility to develop another symplectic integrator for the rotational motion as was done for the Serret-Andoyer set by Touma and Wisdom (1994). Next, the new variables are closely connected to the concept of the Non Rotating Origin (Guinot 1981). For example, the angle f looks similar to the sidereal angle based on NROs, although the rigorous relation between them remains an open problem. These two facts may imply the possibility to develop an analytical theory of the Earth rotation based on NROs.

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