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SOME PROPERTIES ON ISOLOGISM OF GROUPS

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Abstract

In this paper a necessary and sufficient condition will be given for groups to be $\mathcal{V}$-isologic, with respect to a given variety of groups $\mathcal{V}$. It is also shown that every $\mathcal{V}$-isologism family of a group contains a $\mathcal{V}$-Hopfian group. Finally we show that if G is in the variety $\mathcal{V}$, then every $\mathcal{V}$-covering group of G is a Hopfian group.

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1. Introduction and preliminary results

Let F_∞ be the free group freely generated by a countable set $\{x_1, x_2, \dots\}$ and let V be a subset of F_∞. Let the variety of groups $\mathcal{V}$ be defined by the set of laws V. It is assumed that the reader is familiar with the notion of the verbal subgroup, $V(G)$, and the marginal subgroup, $V^*(G)$, associated with the variety $\mathcal{V}$ and a given group G. See Neumann [8] for more information on varieties of groups.

In 1940, Hall [1] introduced the notion of isoclinism and then he extended it to the notion of $\mathcal{V}$-isologism, with respect to a given variety of groups $\mathcal{V}$. If $\mathcal{V}$ is the variety of Abelian or nilpotent groups of class at most n, then $\mathcal{V}$-isologism coincides with isoclinism and n-isoclinism properties, respectively (see [1, 2]).

In the next section we define some closure operation with respect to a variety of groups $\mathcal{V}$, and show that a group G is $\mathcal{V}$-isologic to a group H (written by $G \underset{\mathcal{V}}{\sim} H$) if and only if G and H have the same $\mathcal{V}$-closure (see Theorem 2.5).

Finally, if H_1 and H_2 are two $\mathcal{V}$-covering groups of a given group G and f is an epimorphism of H_1 onto H_2 with some other condition, then f is an isomorphism.

From this result we conclude that all $\mathcal{V}$-covering groups of an arbitrary group in the variety $\mathcal{V}$ are Hopfian.

In the following we recall the definitions of isologism and the Hopf property of groups.

DEFINITION 1.1. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G and H be two groups. Then the pair (α, β) is said to be a *$\mathcal{V}$-isologism* between the groups G and H, if the maps

$$\alpha : G/V^*(G) \longrightarrow H/V^*(H),$$
$$\beta : V(G) \longrightarrow V(H)$$

are isomorphisms such that for all words $\nu(x_1, \ldots, x_r)$ in V and all the elements $g_1, \ldots, g_r$ in G, we have

$$\beta(\nu(g_1, \ldots, g_r)) = \nu(h_1, \ldots, h_r),$$

whenever $h_i \in \alpha(g_i V^*(G))$, for $i = 1, 2, \ldots, r$. In this case we write $G \underset{\mathcal{V}}{\sim} H$ and say that the group G is *$\mathcal{V}$-isologic* to H.

A group G is said to be a *Hopfian group*, if every epimorphism $G \to G$ is an isomorphism, otherwise G is *non-Hopfian*.

Clearly isologism is an equivalence relation, and hence gives rise to a partition on the class of all groups into equivalence classes, the so called *isologism families*.

One notes that if A is any group belonging to the variety $\mathcal{V}$, then $G \times A \underset{\mathcal{V}}{\sim} G$, for all groups G.

The proof of the following lemma is straightforward (see also Hekster [3]).

LEMMA 1.2. *Let $\mathcal{V}$ be a variety of groups and H be a subgroup and N be a normal subgroup of a group G. Then the following statements hold*:

(i) $H \underset{\mathcal{V}}{\sim} HV^*(G)$. *In particular, if $G = HV^*(G)$ then $G \underset{\mathcal{V}}{\sim} H$. Conversely, if the marginal factor group $G/V^*(G)$ satisfies the descending chain condition on subgroups and $G \underset{\mathcal{V}}{\sim} H$, then $G = HV^*(G)$.*

(ii) $G/N \underset{\mathcal{V}}{\sim} G/N \cap V(G)$. *In particular, if $N \cap V(G) = \langle 1 \rangle$, then $G \underset{\mathcal{V}}{\sim} G/N$. Conversely, if $V(G)$ satisfies the ascending chain condition on normal subgroups and $G \underset{\mathcal{V}}{\sim} G/N$, Then $N \cap V(G)$ is trivial.*

Now, in the spirit of the above Lemma 1.2 (ii), we introduce the following

DEFINITION 1.3. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. A group G is said to be *$\mathcal{V}$-Hopfian*, with respect to $\mathcal{V}$-isologism, if G contains no non-trivial normal subgroup N satisfying $N \cap V(G) = \langle 1 \rangle$.

2. $\mathcal{V}$-isologism of groups

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. A group G is called *$\mathcal{V}$-marginal group*, if $G = V^*(G)$.

Now, in the following we define a $\mathcal{V}$-closure operation similar to [9], which is done for the variety of Abelian groups.

DEFINITION 2.1. Let G be a group. Then $\{G\}_{\mathcal{V}}$ denotes the smallest class of groups containing G, closed under the operation of forming direct products with $\mathcal{V}$-marginal groups, and satisfying the following property: if $H \in \{G\}_{\mathcal{V}}$ then every subgroup K of H which satisfies $H = K\,V^*(H)$ is also in $\{G\}_{\mathcal{V}}$, and for every normal subgroup N of H which satisfies $N \cap V(H) = \langle 1 \rangle$ the quotient group H/N is also in $\{G\}_{\mathcal{V}}$. We call the set $\{G\}_{\mathcal{V}}$ the *$\mathcal{V}$-closure* of G.

One should note that we may replace the group G by a set of groups $\{G_i\}$, thus obtaining a $\mathcal{V}$-closure operator for sets of groups.

The following proposition can be proved easily.

PROPOSITION 2.2. *Let $\{G_i\}$ and $\{H_j\}$ be two sets of groups. Then*

(a) $\{G_i\} \subseteq \{G_i\}_{\mathcal{V}}$.
(b) $\{\{G_i\}_{\mathcal{V}}\}_{\mathcal{V}} = \{G_i\}_{\mathcal{V}}$.
(c) *if* $\{G_i\} \subseteq \{H_j\}$, *then* $\{G_i\}_{\mathcal{V}} \subseteq \{H_j\}_{\mathcal{V}}$.

The following result yields the necessary tools for our main result (Theorem 2.6).

THEOREM 2.3. *Let G and H be two groups. Then G and H are $\mathcal{V}$-isologic if and only if a group C and subgroups V_G^*, V_H^* of C exist such that $G \cong C/V_H^*$, $H \cong C/V_G^*$ and the following equivalent statements hold*:

(a) $G \cong C/V_H^* \underset{\mathcal{V}}{\sim} C \underset{\mathcal{V}}{\sim} C/V_G^* \cong H$;
(b) $C/V_H^* \times C/V(C) \underset{\mathcal{V}}{\sim} C_H \cong C \cong C_G \underset{\mathcal{V}}{\sim} C/V_G^* \times C/V(C)$,

for some subgroup C_H of $C/V_H^* \times C/V(C)$ and some subgroup C_G of $C/V_G^* \times C/V(C)$.

PROOF. It is clear that if such groups C, V_G^* and V_H^* exist then $G \underset{\mathcal{V}}{\sim} H$.

Conversely, let $G \underset{\mathcal{V}}{\sim} H$, and (α, β) be a $\mathcal{V}$-isologism between the groups G and H. Assume

$$C = \{(g, h) \in G \times H \mid \alpha(g\,V^*(G)) = h\,V^*(H)\},$$
$$V_G^* = \{(x, 1) \in G \times H \mid x \in V^*(G)\},$$
$$V_H^* = \{(1, y) \in G \times H \mid y \in V^*(H)\}.$$

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Clearly, $V_G^* \cong V^*(G)$ and $V_H^* \cong V^*(H)$. Define the map φ from C into G by $\varphi(g, h) = g$. It is easy to see that φ is an epimorphism with $\ker\varphi = V_H^*$. Hence $C/V_H^* \cong G$. Similarly $C/V_G^* \cong H$.

(a) The verbal subgroup $V(C)$ is generated by

$$\{(\nu(g_1, \dots, g_r), \beta(\nu(g_1, \dots, g_r))) \mid g_1, \dots, g_r \in G, \nu \in V\}.$$

Clearly, $V(C) \cap V_H^* = \langle 1 \rangle$, for if $(g, h) \in V(C) \cap V_H^*$ then $g = 1$ and hence $h = \beta(1) = 1$. Similarly $V(C) \cap V_G^*$ is also trivial. Thus by Lemma 1.2 (ii),

$$C/V_G^* \underset{\mathcal{V}}{\sim} C \underset{\mathcal{V}}{\sim} C/V_H^*,$$

which proves part (a).

(b) We define the subgroup C_G of $C/V_G^* \times C/V(C)$ to be

$$C_G = \{(xV_G^*, xV(C)) \mid x \in C\}.$$

It is clear that the map $\psi : C \to C_G$, given by $\psi(x) = (xV_G^*, xV(C))$, defines an isomorphism and hence $C \cong C_G$. Now, in view of Lemma 1.2 (i), to show

$$C/V_G^* \times C/V(C) \underset{\mathcal{V}}{\sim} C_G$$

it is enough to prove that $C/V_G^* \times C/V(C) = C_G V^*(C/V_G^* \times C/V(C))$. Let $a = (xV_G^*, yV(C))$ be an arbitrary element of $C/V_G^* \times C/V(C)$. Clearly $a = bc$, where $b = (xV_G^*, xV(C)) \in C_G$ and $c = (V_G^*, x^{-1}yV(C))$. It is easily seen that

$$c \in V^*(C/V_G^* \times C/V(C)).$$

This implies that

$$C/V_G^* \times C/V(C) \subseteq C_G V^*(C/V_G^* \times C/V(C)).$$

The reverse containment follows immediately. Hence

$$C \cong C_G \underset{\mathcal{V}}{\sim} C/V_G^* \times C/V(C).$$

By a similar argument it follows that

$$C \cong C_H \underset{\mathcal{V}}{\sim} C/V_H^* \times C/V(C),$$

in which $C_H = \{(yV_H^*, yV(C) \mid y \in C\}$. □

The following corollary generalizes a result of Weichsel [9] to an arbitrary variety of groups.

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COROLLARY 2.4. *Let G and H be two groups and $\mathcal{V}$ be a variety of groups. Then $G \underset{\mathcal{V}}{\sim} H$ if and only if there exists a $\mathcal{V}$-marginal group K, a subgroup L of $G \times K$ with $LV^*(G \times K) = G \times K$ and a normal subgroup N of L such that $N \cap V(L) = 1$ and $H \cong L/N$.*

PROOF. Assume that $G \underset{\mathcal{V}}{\sim} H$, then the result follows from the above theorem by taking $K = C/V(C)$, $L = C_H$, and $N = V_G^*$.

Conversely, suppose the required groups exist, then it follows immediately that $H \underset{\mathcal{V}}{\sim} L \underset{\mathcal{V}}{\sim} G \times K \underset{\mathcal{V}}{\sim} G$. □

Using the notation as in Definition 2.1 we obtain the following.

THEOREM 2.5. *$\{G\}_{\mathcal{V}}$ is the $\mathcal{V}$-isologism family of the group G, and hence $G \underset{\mathcal{V}}{\sim} H$ if and only if $\{G\}_{\mathcal{V}} = \{H\}_{\mathcal{V}}$.*

PROOF. Clearly the $\mathcal{V}$-isologism family of the group G contains G and it is closed under the operations given in Definition 2.1, and hence it contains $\{G\}_{\mathcal{V}}$. But by Corollary 2.4, any group isologic to G can be constructed from G using the allowable operations of $\{G\}_{\mathcal{V}}$, and so is contained in $\{G\}_{\mathcal{V}}$. □

Finally, in this section we show that for any group G, the set $\{G\}_{\mathcal{V}}$ contains a group, H say, which is $\mathcal{V}$-Hopfian with respect to $\mathcal{V}$-isologism.

THEOREM 2.6. *Let G be a group. Then there exists a normal subgroup N of G such that $G \underset{\mathcal{V}}{\sim} G/N$ and G/N is $\mathcal{V}$-Hopfian.*

PROOF. Let $\mathcal{N} = \{N \trianglelefteq G \mid N \cap V(G) = \langle 1 \rangle\}$. Clearly the set $\mathcal{N}$ is non-void, as it contains the trivial subgroup. We define a partial ordering on $\mathcal{N}$ by inclusion and clearly by Zorn's Lemma we can find a maximal normal subgroup N in $\mathcal{N}$. Since $N \cap V(G) = \langle 1 \rangle$, it follows, by Lemma 1.2, that $G \underset{\mathcal{V}}{\sim} G/N$. Now, suppose there exists $M/N \trianglelefteq G/N$ such that $M/N \cap V(G/N) = \langle 1 \rangle$. By [3, Proposition 2.3] and Dedekind's modular law, we have $M \cap V(G) \subseteq N$. Since $N \cap V(G) = \langle 1 \rangle$, it follows that $M \in \mathcal{N}$. On the other hand, we have $N \subseteq M$, so by the maximality of N, it follows that $M = N$. Therefore M/N is trivial, and hence G/N is $\mathcal{V}$-Hopfian with respect to $\mathcal{V}$-isologism. □

3. Hopfian property

Let H_1 and H_2 be two $\mathcal{V}$-covering groups of a given group G. In this final section we give a sufficient condition for an epimorphism of H_1 onto H_2 to be an isomorphism.

Then we conclude that every $\mathcal{V}$-covering group of a group in the variety $\mathcal{V}$ has the Hopf property.

Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of a group G, where F is a free group and $R = \ker \pi$. Then the *Baer-invariant* of G with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be $R \cap V(F)/[RV^*F]$, where $V(F)$ is the verbal subgroup of F and $[RV^*F]$ is the least normal subgroup T of F contained in R such that $R/T \subseteq V^*(F/T)$. One may check that the Baer-invariant of a group G is always Abelian and independent of the choice of the free presentation of G. In particular, if $\mathcal{V}$ is the variety of Abelian or nilpotent groups of class at most c ($c \geq 1$), then the Baer-invariant of the group G will be $(R \cap F')/[R, F]$, which is the Schur-multiplicator of G, or $(R \cap \gamma_{c+1}(F))/[R, {}_cF]$ (where F repeated c times), respectively (see [4]).

We recall that an exact sequence $1 \to A \to G^* \to G \to 1$ is called a *$\mathcal{V}$-stem extension* with respect to the variety of groups $\mathcal{V}$, when $A \subseteq V(G^*) \cap V^*(G^*)$. If in addition $A \cong \mathcal{V}M(G)$, then the above extension is called a *$\mathcal{V}$-stem cover*. In this case G^* is said to be a $\mathcal{V}$-covering group of G. It is of interest to know the class of groups that do not have $\mathcal{V}$-covering groups (see [7]). In [6] we have also shown that a given group G has always a $\mathcal{V}$-covering group with respect to some specific variety $\mathcal{V}$. So whenever we talk about a $\mathcal{V}$-covering of a group, it is assumed that $\mathcal{V}$ is a suitable variety.

The following results of [5] are needed to prove the main result of this section.

THEOREM 3.1 (Moghaddam and Salemkar [5]). *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let $1 \to R \to F \to G \to 1$ be a free presentation of a group G. Then*

(i) *If S is a normal subgroup of F such that*

$$\frac{R}{[RV^*F]} = \frac{R \cap V(F)}{[RV^*F]} \times \frac{S}{[RV^*F]},$$

then $G^* = F/S$ is a $\mathcal{V}$-covering group of G.

(ii) *Every $\mathcal{V}$-covering group of G is a homomorphic image of $F/[RV^*F]$.*

(iii) *For any $\mathcal{V}$-covering group G^* of G with an exact sequence $1 \to A \to G^* \to G \to 1$, such that $A \subseteq V^*(G^*) \cap V(G^*)$ and $A \cong \mathcal{V}M(G)$, then there exists a normal subgroup S of F, as in* (i), *such that $F/S \cong G^*$ and $R/S \cong A$.*

COROLLARY 3.2. *With the above assumption, for any $\mathcal{V}$-covering group G^* of a given group G, there exists an epimorphism $\bar{\psi}$ from $F/[RV^*F]$ onto G^* such that*

$$\frac{R}{[RV^*F]} = \frac{R \cap V(F)}{[RV^*F]} \times \ker \bar{\psi},$$

where the image under $\bar{\psi}$ of the first factor is equal to A.

The following lemma is needed for the proof of Theorem 3.4 below, which is the main result of this section.

LEMMA 3.3. *Let G be a group, and*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \longrightarrow & H_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \\ 1 & \longrightarrow & A_2 & \longrightarrow & H_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

a commutative diagram of groups such that the first row is exact and the second one is a $\mathcal{V}$-stem extension of G. If the homomorphism φ is onto, then so is ψ.

PROOF. It is easily shown that $H_2 = (\operatorname{Im}\psi)A_2$. Hence by [3, Theorem 2.4],

$$V(H_2) = V(\operatorname{Im}\psi)[A_2 V^* H_2].$$

But $A_2 \subseteq V^*(H_2)$, by the assumption. Thus $V(H_2) = V(\operatorname{Im}\psi)$. We also have $A_2 \subseteq V(H_2)$, which implies that $A_2 \subseteq V(\operatorname{Im}\psi) \subseteq \operatorname{Im}\psi$, and hence $H_2 = \operatorname{Im}\psi$. □

THEOREM 3.4. *Let G be a group and let*

$$1 \longrightarrow A_i \longrightarrow H_i \longrightarrow G \longrightarrow 1, \qquad i = 1, 2$$

be two $\mathcal{V}$-stem covers of G with respect to the variety $\mathcal{V}$. If $\psi : H_1 \to H_2$ is an epimorphism such that $\psi(A_1) = A_2$, then ψ is an isomorphism.

PROOF. Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G. By Theorem 3.1 (iii), there exist normal subgroups S_i of F, $i = 1, 2$, such that $H_i \cong F/S_i$ and $A_i \cong R/S_i$, and

$$\frac{R}{[R V^* F]} = \frac{R \cap V(F)}{[R V^* F]} \times \frac{S_i}{[R V^* F]}.$$

So we may regard ψ as an epimorphism from F/S_1 onto F/S_2 such that $\psi(R/S_1) = R/S_2$. Therefore, by Corollary 3.2, there exists an epimorphism $\varphi\colon F/[R V^* F] \to F/S_2$ such that $\ker\varphi = S_2/[R V^* F]$ and the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R/[R V^* F] & \longrightarrow & F/[R V^* F] & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \\ 1 & \longrightarrow & R/S_2 & \longrightarrow & F/S_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where φ_1 and φ' are the restriction and the induced homomorphisms of φ, respectively. One can easily check that φ' is an isomorphism. We claim that there exists a homomorphism $f : F/[R V^* F] \to F/S_1$ such that the following diagrams are commutative.

$$
\begin{array}{ccc}
 & & F/[R V^* F] \\
 & \swarrow^{f} & \uparrow \varphi \\
F/S_1 & \xrightarrow[\psi]{} & F/S_2
\end{array}
$$

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R/[R V^* F] & \longrightarrow & F/[R V^* F] & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow f_1 & & \downarrow f & & \downarrow \varphi' \circ \psi'^{-1} & & \\
1 & \longrightarrow & R/S_1 & \longrightarrow & F/S_1 & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

where $\psi' : G \to G$ is induced by ψ, and $\varphi' \circ \psi'^{-1}$ is an isomorphism. The homomorphism f is obtained as follows. Since ψ is surjective there is a homomorphism $\tilde{f} : F \to F/S_1$ such that $\psi(\tilde{f}(x)) = \varphi(x[R V^* F])$ for all $x \in F$. We see that $\psi(\tilde{f}(R)) = R/S_2$, and so $\tilde{f}(R) \subseteq \psi^{-1}(R/S_2) = R/S_1$. Since $R/S_1 \subseteq V^*(F/S_1)$ it follows $\tilde{f}([R V^* F])$ is trivial; thus $\tilde{f}$ induces a map $f : F/[R V^* F] \to F/S_1$, as required.

Lemma 3.3 implies that f is onto. Put $\ker f = T/[R V^* F]$. Then $T(R \cap V(F)) = R$. But $\ker f \subseteq \ker \varphi$, and hence $T \subseteq S_2$ and so $T = S_2$. Therefore $\ker f = \ker \varphi$, which implies that ψ is an isomorphism. □

The following corollary shows that all $\mathcal{V}$-covering groups of any group in the variety $\mathcal{V}$ are Hopfian.

COROLLARY 3.5. *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and G be an arbitrary group of $\mathcal{V}$. Then every $\mathcal{V}$-covering group of G is Hopfian.*

PROOF. Let G^* be a $\mathcal{V}$-covering group of G. Then there exists a normal subgroup A of G^* such that $A \subseteq V(G^*) \cap V^*(G^*)$, $A \cong \mathcal{V}M(G)$, and $G^*/A \cong G$. Since G is in the variety, it follows that $V(F) \subseteq R$, and hence $\mathcal{V}M(G) = V(F)/[R V^* F]$. Thus if $f : G^* \to G^*$ is an epimorphism, then $f(A) = A$; and hence by the above theorem G^* is a Hopfian group. □

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