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A Note on Lagrangian Loci of Quotients

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Abstract. We study Hamiltonian actions of compact groups in the presence of compatible involutions. We show that the Lagrangian fixed point set on the symplectically reduced space is isomorphic to the disjoint union of the involutively reduced spaces corresponding to involutions on the group strongly inner to the given one. Our techniques imply that the solution to the eigenvalues of a sum problem for a given real form can be reduced to the quasi-split real form in the same inner class. We also consider invariant quotients with respect to the corresponding real form of the complexified group.

1 Introduction

Let a compact group K act in a Hamiltonian way on a manifold X. Let σ be an anti-symplectic involution on X and let τ be an involution on K such that for any $x \in X$ and $k \in K$ we have $\sigma(k.x) = \tau(k).\sigma(x)$. This setup was considered by O'Shea and Sjamaar in [10] in order to establish a real analogue of Kirwan's convexity theorem which lead them to finding inequalities on possible spectrum of sums of two matrices.

The main goal of the present paper is to establish a relationship of the fixed point set on the reduced space $(X /\!/ K)^\sigma$ with the involutively reduced space $X^\sigma /\!/ K^\tau$. The latter is defined as the quotient by K^τ of the zero level set of the momentum map in X^σ. Our main observation is that one needs to consider simultaneously all conjugacy classes of involutions on K which are *strongly inner* to τ. We say that τ_s is strongly inner to τ, if in addition to being inner to τ, *i.e.*, $\tau_s = \mathrm{Ad}_s \circ \tau$, we require $s\tau(s) = 1$. Then we show that when the action of K on the zero level set is free, the space $(X /\!/ K)^\sigma$ is the disjoint union of such $X^{s\sigma} /\!/ K^{\tau_s}$, where s runs through the connected components of the subset Q of K consisting of elements satisfying $\tau(k) = k^{-1}$. The elements of Q correspond to involutions on the group K which are strongly inner to τ. We also discuss the singular case.

We generate many examples by taking a complex semisimple Lie group G and its flag manifold G/P with the property that the real dimension of the closed orbit of a real form G^τ on G/P equals the complex dimension of G/P. Then we show that there exists a symplectic structure on G/P and an anti-symplectic involution σ on G/P which is compatible with τ. To get interesting examples of reduced spaces, one then might take a product of several complex flag manifolds of the above type with the diagonal G-action. A particular case when $G^\tau = \mathrm{SL}(2, \mathbb{H})$ was considered in [3], where the involutively reduced space was identified with the moduli space of polygons in $\mathbb{R}^5$.

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As an application of our techniques we can show that the problem of finding a solution to $A_1 + \cdots + A_k = 0$, where A_i is fixed by τ and has a prescribed spectrum, can be reduced to finding a solution for a quasi-split involution in the same inner class.

We can also consider a similar setup, when a linearly reductive complex Lie group G equipped with an anti-holomorphic involution acts on a projective variety X, which has a compatible anti-holomorphic involution. We employ results of Richardson and Slodowy [11] on minimum vectors to establish an analogue of the aforemention result in this context.

In a subsequent work, we wish to apply our results in order to compute $\mathbb{Z}/2$-cohomology of the real loci of quotients, and extend the Kirwan surjectivity theorem for equivariant cohomology of the real loci in the non-abelian case. For the case of a torus action, such a generalization was studied by Goldin and Holm [6].

2 Involutions and Reduction

Let K be a compact connected Lie group and let $\tau\colon K \to K$ be a group homomorphism satisfying $\tau^2 = \mathrm{Id}$. We will refer to such a τ simply as an *involution* on K. Let (X,ω) be a Hamiltonian K-manifold with an equivariant momentum map $\mu\colon X \to \mathfrak{k}^*$. Besides, let $\sigma\colon X \to X$ be an involution on X, *i.e.*, a diffeomorphism satisfying $\sigma^2 = \mathrm{Id}$. We say that σ is an *anti-symplectic* involution if $\sigma^*\omega = -\omega$. We also shall say that τ and σ are *compatible* if

$$\sigma(k.x) = \tau(k).\sigma(x).$$

We recall that [10, Lemma 2.2] asserts that μ can be shifted by an Ad^*-invariant element of $\mathfrak{k}^*$ to ensure that $\mu(\sigma(x)) = -\tau^*(\mu(x))$. We therefore shall assume that our μ satisfies this property, which in particular implies that $\mu^{-1}(0)$ is σ-stable.

We denote by K^τ the set of points in K fixed by τ and we will denote by X^σ the set of points in X fixed by σ. We notice that X^σ is a smooth submanifold of X and that K^τ has a natural structure of a Lie subgroup of K. Recall that the Marsden–Weinstein symplectic quotient $X /\!/ K$ of X with respect to the K-action is defined as

$$X /\!/ K = \mu^{-1}(0)/K.$$

In the situation when K acts freely on $\mu^{-1}(0)$, the value $0 \in \mathfrak{k}^*$ is regular and the quotient $X /\!/ K$ has a natural structure of a smooth manifold carrying the reduced symplectic form ω_{red}. When 0 is not a regular value of μ, the symplectic quotient $X /\!/ K$ is a *stratified symplectic space* in the terminology of Sjamaar and Lerman [13]. It is easy to check that the involution σ descends to an involution, also called σ, on $X /\!/ K$, where it is anti-symplectic with respect to ω_{red}. Let us denote the fixed point set of the latter involution by $(X /\!/ K)^\sigma$. It is easy to see [10] that the fixed point set of an anti-symplectic involution is either empty or a Lagrangian submanifold. In addition, X^σ is K^τ-stable and its image in $\mathfrak{k}^*$ under the momentum map μ is contained in $(\mathfrak{k}^\tau)^\perp$, which can be identified with $\mathfrak{q}^*$, where $\mathfrak{q} \subset \mathfrak{k}$ is the (-1)-eigenspace of τ.

Remark Our use of ω and μ is dictated by convenience rather than necessity. One can use a more general set-up, where σ is not required to be anti-symplectic, and get the same results.

Let us define the *involutively reduced space*:

$$X^\sigma /\!/ K^\tau = \left(\mu^{-1}(0) \cap X^\sigma\right) / K^\tau.$$

This definition makes sense due to the fact that $\mu^{-1}(0) \cap X^\sigma = (\mu^{-1}(0))^\sigma$ is K^τ-stable.

Given any point in $X^\sigma /\!/ K^\tau$ represented by a K^τ-orbit $K^\tau.x$ in X, we can consider the K-orbit $K.x$, which belongs to $\mu^{-1}(0)$ and thus defines a point in $X /\!/ K$. Moreover, since $\sigma(K.x) = K.x$, the corresponding point in $X /\!/ K$ is fixed by σ. The map thus obtained is denoted by:

$$\psi : X^\sigma /\!/ K^\tau \to (X /\!/ K)^\sigma. \tag{2.1}$$

Remark Our involutively reduced space is related to the *Lagrangian quotients* defined in [10]. Under the assumption that ψ is injective, they are actually the same.

We wish to study properties of the map ψ.

Lemma 2.1 *If K acts freely on $\mu^{-1}(0)$, then ψ is injective.*

Proof Assume that two K^τ-orbits $K^\tau.x_1$ and $K^\tau.x_2$ from $X^\sigma \cap \mu^{-1}(0)$ are mapped to the same point by ψ. This would imply that there exists $k \in K$ such that $x_1 = k.x_2$. Applying σ to both sides:

$$x_1 = \sigma(x_1) = \tau(k).\sigma(x_2) = \tau(k).x_2.$$

The fact that K acts freely on $\mu^{-1}(0)$ implies that $k = \tau(k)$, meaning that x_1 and x_2 are actually in the same K^τ-orbit. ■

Denote by Q the subset of elements in K satisfying $\tau(k) = k^{-1}$. Let us denote by Q_0 the connected component of Q containing the identity element. The group K acts on Q via $k.q = kq\tau(k)^{-1}$ and the connected components of Q are precisely the orbits of this K-action. In particular, Q_0 is the collection of elements in K representable as $k\tau(k)^{-1}$. It is easy to see that Q_0 is diffeomorphic to the symmetric space K/K^τ. Actually, it is true that each connected component of Q is also a symmetric space, for each connected component Q_s of Q and an element $s \in Q_s$ we have:

Lemma 2.2 *For $s \in Q_s$, the map $\tau_s = \mathrm{Ad}_s \circ \tau$ is an involution on K.*

Proof Due to $\tau(s) = s^{-1}$, we have

$$\tau_s \circ \tau_s(k) = \mathrm{Ad}_s \circ \tau \circ \mathrm{Ad}_s \circ \tau(k) = \mathrm{Ad}_s(\tau(s)k\tau(s)^{-1}) = k. \qquad ■$$

Since any other element $s' \in Q_s$ can be presented as $ks\tau(k)^{-1}$, we see that the actions of K on Q_s and on K/K^{τ_s} are compatible. Moreover, the connected component Q_s is isomorphic to the symmetric space K/K^{τ_s}. See [15, Proposition 5.8] for a detailed discussion.

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Definition 2.3 We say that an involution τ' on K is *inner* to τ if there exists an element $s \in K$ such that $\tau' = \mathrm{Ad}_s \circ \tau$. We also say that an involution τ' on K is *conjugate* to τ, if there exists an element $k \in K$ such that $\tau' = \mathrm{Ad}_k \circ \tau \circ \mathrm{Ad}_{k^{-1}}$.

Lemma 2.4 *If τ' is conjugate to τ, then it is also inner to it. If $\tau' = \mathrm{Ad}_s \circ \tau$ is inner to τ, then $s\tau(s)$ belongs to the center of K. The properties of being conjugate or inner are equivalence relations.*

Proof Straightforward. ■

Definition 2.5 We say that an involution τ' on K is *strongly inner* to τ if the element s in the definition can be chosen from Q, *i.e.*, $s\tau(s) = 1$.

For example, if K is of adjoint type, then being inner is the same as being strongly inner.

Lemma 2.6 *Being strongly inner is an equivalence relation.*

Proof Straightforward. ■

Given $s \in Q$ and an involution $\tau_s = \mathrm{Ad}_s \circ \tau$ strongly inner to τ, we can introduce a diffeomorphism σ_s on X:

$$\sigma_s(x) = s.\sigma(x).$$

Lemma 2.7 *If τ_s is strongly inner to τ, then the map σ_s is involutive, anti-symplectic, and compatible with τ_s.*

Proof Involutivity:

$$\sigma_s \circ \sigma_s(x) = s.\sigma(s.\sigma(x)) = s\tau(s).\sigma \circ \sigma(x) = x.$$

Anti-symplecticity readily follows from the facts that σ is anti-symplectic and K acts in a Hamiltonian way.

Compatibility:

$$\sigma_s(k.x) = s.\sigma(k.x) = s\tau(k).\sigma(x) = \mathrm{Ad}_s \circ \tau(k)s.\sigma(x) = \tau_s(k).\sigma_s(x). \qquad ■$$

We can also define the map ψ_s for τ_s and σ_s in the same way the map ψ was defined for σ and τ. However, notice that:

Lemma 2.8 *The right-hand side of equation* (2.1) *only depends on the strongly inner class $[\tau]$ of τ.*

Proof Since for any other real form τ_s strongly inner to τ, the map σ_s is given by $s.\sigma$, an orbit $K.x$ will be mapped to $K.\sigma(x)$ by both σ and σ_s. ■

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Let $\mathcal{I}_\tau$ be a finite set indexing the connected components of Q. It follows from our discussion that each element of this set defines a conjugacy class of involutions strongly inner to τ. It is often convenient to use elements of K to represent elements of $\mathcal{I}_\tau$.

Theorem 2.9 *If K acts freely on $\mu^{-1}(0)$, then $(X \,//\, K)^\sigma$ is diffeomorphic to the disjoint union of all $X^{\sigma_s} \,//\, K^{\tau_s}$ for $s \in \mathcal{I}_\tau$ via the maps ψ_s.*

Proof Consider a point $y \in (X \,//\, K)^\sigma$. First, we need to show that there exists a unique element $s \in \mathcal{I}_\tau$ such that y is in the image of ψ_s. Let y be represented by a σ-stable K-orbit $K.x$ in $\mu^{-1}(0)$. If we let $\sigma(x) = s.x$, then the fact that σ is involutive translates to $\tau(s) = s^{-1}$, so $s \in Q$. If we have a σ-fixed point $k.x$ in the orbit, then we would have:

$$k.x = \sigma(k.x) = \tau(k).\sigma(x) = \tau(k)s.x,$$

which by the freedom of action would imply that $s = \tau(k^{-1})k$ and that s is actually in Q_0. If we do not have a σ-fixed point in the orbit, then s is not in Q_0, but in a different connected component of Q. Then it is easy to check that the involutions τ_s and σ_s defined by s are such that the point $s.x$ is fixed by σ_s:

$$\sigma_s(s.x) = s.\sigma(s.x) = s\tau(s).\sigma(x) = s.x.$$

Therefore, y is in the image of ψ_s. The statement about diffeomorphism follows now from Lemma 2.1 together with a simple computation of the injectivity of ψ_* on the level of tangent spaces. Finally, the fact that image of ψ_s is both open and closed in $(X \,//\, K)^\sigma$ follows from [10, Corollary 7.2]. ■

Corollary 2.10 *If Q is connected, and K acts freely on $\mu^{-1}(0)$, then the map ψ is a diffeomorphism.*

Example Let us consider the case when $K = T$ is a torus. In this case, we can split $T = T_+ \times T_-$ in such a way that τ fixes T_+ pointwise and acts as the inverse map on T_-. If the real dimension of T_+ is n, then Q has exactly 2^n connected components. In this setting, a particular case of the above Corollary, when $T = T_-$, was proven by Goldin and Holm in [6, Proposition 4.3].

Example Let $K = \mathrm{SU}(n)$ and τ act as the complex conjugation. Since every unitary symmetric matrix A can be represented as $\exp(iB)$ for a real symmetric matrix B [5], the set Q is connected, and the map ψ is bijective.

Example Let K be simple of adjoint type (such as $\mathrm{PU}(n)$). Then, as we remarked earlier, our notion of strong inner involution is the same as the standard notion of inner involution. The connected components of Q correspond to equivalence classes of symmetric spaces of K defined by involutions inner to τ. The classification of symmetric spaces is well known [7], as well as the fact that inner classes of involutions are in bijective correspondence with the order two automorphisms of the Dynkin diagram of K [1]. For example, if $K = \mathrm{PU}(n)$, and $\tau = \mathrm{Id}$, there are $[n/2] + 1$ connected components: a point and $[n/2]$ projectivized grassmannians.

It is easy to verify that the connected components of Q are parameterized by the set $H^1(\Gamma, K)$, where $\Gamma \simeq \mathbb{Z}/2$ is generated by τ. Let Z be the center of K, so we have the exact sequence $1 \to Z \to K \to K/Z \to 1$, which induces a long exact sequence in cohomology sets. Since, in principle, we know the answer for the simple adjoint case and the torus case (see examples above), and we can easily compute $H^i(\Gamma, A)$, where A is a compact abelian Lie group acted upon by Γ, we can use this sequence to compute $H^1(\Gamma, K)$.

Example Let τ be inner to the identity automorphism. In this case we can assume that Γ acts trivially on K and therefore we can apply Theorem 6 from [12, Chapter III] to see that $H^1(\Gamma, K)$ can be identified with T_2/W, where T_2 is the set of elements of order 2 in a maximal torus T and W is the Weyl group. For example, when $K = \mathrm{SU}(n)$ and $\tau = \mathrm{Id}$, there are exactly $[n/2] + 1$ elements.

Let us now discuss the situations when the group K does not act freely on $\mu^{-1}(0)$. In general, we cannot expect the map ψ to be injective. An easy example would be when $K = \mathrm{SU}(2)$, τ the complex conjugation, $X = S^2 \times S^2 \times S^2$ with the diagonal action of K and symplectic form (for example) $4\pi_1^*\omega + 3\pi_2^*\omega + 2\pi_3^*\omega$, where π_i is the projection onto the i-th factor and ω the standard invariant symplectic form on S^2. The anti-symplectic involution σ on X is simply the reflection about the equatorial plane on each factor. The involutively reduced space in this case consists of two points and the symplectically reduced space consists of a single point. The reason is that each point on X has a non-trivial stabilizer, the center of K. This example illustrates the fact that there are two equivalence classes of triangles of side lengths $2, 3, 4$ under the action of the motion group in $\mathbb{R}^2$ but only one such in $\mathbb{R}^3$. However, it is important to notice that if we pass to the group $\mathrm{PU}(2)$, then the injectivity will hold, because $\mathrm{PU}(2)^\tau$ has now two connected components.

In general the following is true [10, Proposition 2.3(iii)]:

Lemma 2.11 *For each point $y \in X^\sigma /\!/ K^\tau$ the fiber $\psi^{-1}(\psi(y))$ is finite.*

We see that the question of counting the number of points in the fiber heavily depends on the stabilizer of the point. Let two points $x_1, x_2 \in X^\sigma$ from different K^τ orbits be in the same K-orbit: $x_2 = k.x_1$, $k \in K \setminus K^\tau$. This would imply that $k^{-1}\tau(k) \in K_1$, the stabilizer of x_1 in K. Let us represent $k = k_0 p$ with $p \notin K_1$, $p \in Q_0$, and $k_0 \in K^\tau$. We immediately see that $p^2 \in K_1$. So the problem of counting the number of points in a given fiber amounts to counting the equivalence classes of elements from Q_0 satisfying the above conditions.

Let us now turn to surjectivity questions. Let us assume that the momentum map μ is proper. According to [13], the symplectic quotient $X /\!/ K$ is stratified according to the conjugacy classes of stabilizers of points in $\mu^{-1}(0)$. There is a unique open dense stratum called the *principal* stratum denoted by $(X /\!/ K)_{\mathrm{prin}}$. The involution σ on $X /\!/ K$ clearly maps a stratum to a stratum and thus $(X /\!/ K)_{\mathrm{prin}}$ is σ-stable. Let us denote its fixed point set there by $(X /\!/ K)^\sigma_{\mathrm{prin}}$. If we can show surjectivity for a (connected component) of $(X /\!/ K)^\sigma_{\mathrm{prin}}$, then the surjectivity would also be valid for its closure in the classical topology, because each connected component of $\mu^{-1}(0)^\sigma$

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is compact. We shall assume that there is a point in the principal stratum $\mu^{-1}(0)_{\mathrm{prin}}$, which is σ-stable. This is a rather reasonable assumption, because otherwise there would be no points in $(X^\sigma /\!/ K^\tau)_{\mathrm{prin}}$. The stabilizer of y, which we call H, is therefore a τ-stable subgroup of K. Let N_H stand for the normalizer of H in K and let $L = N_H/H$ be the quotient group. It is easy to check that the N_H is also τ-stable and that the involution τ descends to the quotient group L.

Let us recall another result of Sjammar and Lerman [13, Theorem 3.5]. They have established that the principal stratum $(X /\!/ K)_{\mathrm{prin}}$ can be realized as the smooth Marsden–Weinstein reduced space of the set of points X_H on X with stabilizer H by the action of the group L. Using this result, we can convert the questions of surjectivity for singular reduction to the smooth reduction which we dealt with in Theorem 2.9.

3 Example: Flag Manifolds

Let G be a connected complex semisimple Lie group of adjoint type with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let Δ be the corresponding root system. Let us fix a choice of positive roots Δ^+ and let Σ be the basis of simple roots. For any arbitrary subset $S \subset \Sigma$, we get a parabolic subgroup P_S of G in a standard way as follows. Every root $\beta \in \Delta$ has a unique decomposition

$$\beta = \sum_{\alpha\in\Sigma} n_\alpha(\beta)\alpha,$$

where $n_\alpha(\beta)$ is a collection of either non-positive integers, in which case β is a negative root, or non-negative integers, in which case $\beta \in \Delta^+$. Let $\Delta_S \subset \Delta$ stand for the set of roots which only involve simple roots from S in the above decomposition. Let Δ_S^+ be the subset of Δ_S consisting of positive roots and let $\overline{\Delta_S^+}$ be the complement of Δ_S^+ in Δ^+. Let us further define

$$\mathfrak{p}_S = \mathfrak{h} + \sum_{\alpha\in\Delta_S} \mathfrak{g}_\alpha + \sum_{\alpha\in\overline{\Delta_S^+}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to α. The first two summands in the above formula form the Levi factor of $\mathfrak{p}_S$ and the last one is the nilradical. Let P_S stand for the parabolic subgroup of G corresponding to $\mathfrak{p}_S$. If P is any such subgroup then $X = G/P$ is a complex flag manifold.

Let B be the Killing form on $\mathfrak{g}$ and let $E_\alpha \in \mathfrak{g}_\alpha$ be chosen such that

$$[E_\alpha, E_\beta] = \begin{cases} m_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta,\\ H_\alpha & \text{if } \alpha = -\beta,\\ 0 & \text{otherwise,}\end{cases}$$

where H_α is the unique element of $\mathfrak{h}$ defined by $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. In addition we require that the constants $m_{\alpha,\beta}$ be real and satisfy $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$. We take the compact real form $\mathfrak{k}$ of $\mathfrak{g}$ as the span of iH_α, $E_\alpha - E_{-\alpha}$, and $i(E_\alpha + E_{-\alpha})$. Let θ stand for the corresponding Cartan involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be the Cartan decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces of the involution θ. Let also $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition corresponding to our choices of Δ^+ and $\mathfrak{k}$. Let $G = KAN$ be the corresponding decomposition on the group level.

Each isomorphism class of real forms of G is represented by a Satake diagram D, as explained e.g. in [7]. A Satake diagram is the Dynkin diagram for $\mathfrak{g}$ such that some vertices are painted black and certain white vertices are paired by arrows. As explained in [4], we can construct an involution τ on G such that G^τ is a specific representative of a real form corresponding to D, called an Iwasawa real form, which has the following properties:

(1) τ commutes with θ: $\tau \circ \theta = \theta \circ \tau$. Then $\mathfrak{g}^\tau = \mathfrak{k}^\tau + \mathfrak{q}^\tau$.
(2) $\mathfrak{h}^\tau$ is a maximally non-compact Cartan sub-algebra in $\mathfrak{g}^\tau$.
(3) If we denote by N^τ the subgroup of N consisting of elements fixed by τ, then

$$G^\tau = K^\tau A^\tau N^\tau$$

is an Iwasawa decomposition of G^τ.

Note that the group N is not stabilized by τ in general, but only when D corresponds to the so-called *quasi-split* real form [1].

Since we are working with an adjoint group, the notion of strongly innner involution is the same as inner, and we will use another result of [1] which asserts that there is a unique, up to conjugacy, quasi-split real form represented by a Satake diagram D_{qs} with no black vertices for each inner class of real forms. Let us denote the corresponding Iwasawa involution τ_{qs}. We may assume that τ_{qs} commutes with τ.

Example Let us give examples of Iwasawa real forms for the A_{n-1} case, when $G = \mathrm{PGL}(n, \mathbb{C})$. There are two inner classes of real forms, I and II. Class I contains real forms isomorphic to $\mathrm{PU}(p,q)$'s for $p+q = n$, $p \le q$, and class II contains the split real form $\mathrm{PGL}(n, \mathbb{R})$ and, when $n = 2m$ is even, a real form isomorphic to $\mathrm{PGL}(m, \mathbb{H})$. Let us write down specific Iwasawa involutions. Let

$$Q_p = \begin{pmatrix} 0_{p\times p} & 0_{p\times(q-p)} & Y_p \\ 0_{(q-p)\times p} & 1_{q-p} & 0_{(q-p)\times p} \\ Y_p & 0_{p\times(q-p)} & 0_{p\times p} \end{pmatrix},$$

where Y_p is the $p \times p$ matrix with ones on the anti-diagonal. Then the involution

$$\tau_p(A) = Q_p(\overline{{}^tA^{-1}})Q_p$$

determines a real form isomorphic to $\mathrm{PU}(p,q)$. In this inner class the split real form is given by $\mathrm{PU}\left(\left[\frac{n}{2}\right], \left[\frac{n+1}{2}\right]\right)$.

Class II contains the split real form $\mathrm{PGL}(n, \mathbb{R})$ with the corresponding Iwasawa involution being the complex conjugation and, when $n = 2m$ is even, we have the real form isomorphic to $\mathrm{PGL}(m, \mathbb{H})$ with the corresponding Iwasawa involution given by

$$\tau_{\mathbb{H}}(A) = Q_{\mathbb{H}} \bar{A} Q_{\mathbb{H}}^{-1},$$

where $Q_{\mathbb{H}}$ is $2m \times 2m$ block-diagonal matrix:

$$Q_{\mathbb{H}} = \operatorname{diag}(J, -J, J, \dots, (-1)^m J),\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now let us recall some classical results about the action of the real form G^τ on the complex flag manifold $X = G/P$, which are due to Joseph Wolf [14]. As is standard, we view points in X as G-conjugates of $\mathfrak{p}$ via the correspondence $gP \leftrightarrow \operatorname{Ad}(g)\mathfrak{p}$. We denote by P_x (resp., $\mathfrak{p}_x$) the corresponding parabolic subgroup of G (resp., the parabolic subalgebra of $\mathfrak{g}$). Wolf showed that there exists a τ-stable Cartan subalgebra $\mathfrak{h}_x \subset \mathfrak{p}_x$ of $\mathfrak{g}$, a positive root system Δ_x^+ compatible with $\mathfrak{h}_x$, and a set S_x of simple roots, such that $\mathfrak{p}_x = \mathfrak{p}_{S_x}$ and $P_x = P_{S_x}$. The real co-dimension of the G^τ-orbit $G^\tau(x)$ through x is equal to the cardinality of the intersection of $\overline{\Delta_{S_x}^+}$ with $\tau\overline{\Delta_{S_x}^+}$. The number of G^τ-orbits on X is finite, and there is a unique closed orbit X_0, which is contained in the closure of every G^τ-orbit. If $G^\tau(x)$ is the closed orbit, then there is an Iwasawa decomposition $G^\tau = K^\tau A^\tau N^\tau$ such that $G^\tau \cap P_x$ contains $H^\tau N^\tau$, whenever H^τ is a Cartan subgroup of G^τ containing A^τ. Moreover, if K_0 is any maximal compact subgroup of G^τ, then K_0 is transitive on X_0.

It will be of particular interest to us to investigate those cases when the real dimension of X_0 is half the real dimension of G/P. It was proved in [14] that the closed G^τ-orbit $X_0 = G^\tau(x)$ satisfies $\dim_{\mathbb{R}}(X_0) = \dim_{\mathbb{C}} X$ if and only if the following equivalent conditions hold:

(1) $\tau\overline{\Delta_{S_x}^+} = \overline{\Delta_{S_x}^+}$.
(2) $\tau(\mathfrak{p}_x) = \mathfrak{p}_x$, $\tau(P_x) = P_x$.
(3) X is a projective variety defined over $\mathbb{R}$ and X_0 is the set of real points.

It is therefore appropriate in such situations to refer to X_0 as a *real flag manifold*. In the case when $P = B$, the Borel subgroup, the condition $\dim_{\mathbb{R}}(X_0) = \dim_{\mathbb{C}} X$ holds if and only if the Satake diagram of the symmetric space G^τ/K^τ contains no painted vertices and $\mathfrak{k}^\tau$ does not contain a simple ideal of $\mathfrak{g}^\tau$.

Let X_0 be the closed G^τ-orbit on G/P. We can assume that P is chosen in such a way that X_0 is the orbit through the base point of G/P. If $\dim_{\mathbb{R}} X_0 = \dim_{\mathbb{C}} X$, then the previous discussion implies that $\tau(\mathfrak{p}) = \mathfrak{p}$ and that $\tau\overline{\Delta_S^+} = \overline{\Delta_S^+}$. According to [8], the set of simple roots Σ decomposes into the disjoint union of two subsets Σ_0 and Σ_1 such that for any $\alpha \in \Sigma_0$ we have $\tau(\alpha) = -\alpha$ and for any $\alpha \in \Sigma_1$ we have

$$\tau(\alpha) = \mu(\alpha) + \sum_{\beta\in\Sigma_0} c_{\alpha,\beta}\beta,$$

where μ is an involution on the set Σ_1 and $c_{\alpha,\beta}$ are non-negative integers. Let S, as before, be the subset of simple roots defining the parabolic subgroup P. First of all, the condition $\tau\overline{\Delta_S^+} = \overline{\Delta_S^+}$ implies that $\Sigma_0 \subset S$. Let us denote $S_1 := S \cap \Sigma_1$. For any $\alpha \in S$ let us define by $h_\alpha \in \mathfrak{a}$ the unique element with the property that for any $\beta \in S$ we have $\beta(h_\alpha) = \delta_{\alpha,\beta}$. We define

$$\tilde{\lambda} = \sum_{\alpha\in\Sigma_1\setminus S_1} h_\alpha \ \in \mathfrak{a}.$$

It is straightforward to check that:

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Lemma 3.1

$$\tau(\tilde{\lambda}) = \tilde{\lambda}$$

Let us now identify $i\mathfrak{k}$ with $\mathfrak{k}^*$ using the imaginary part of the Killing form and let us denote by λ the image of $\tilde{\lambda}$ under this identification. Due to the fact that τ is complex anti-linear, and this identification uses i, we have $\tau(\lambda) = -\lambda$.

Let us denote $K_P = K \cap P$, then we have an identification $G/P \simeq K/K_P$. The element $\lambda \in \mathfrak{k}^*$ defines a left-invariant one-form on K, which we also call λ. Let ω be the unique two-form on $X = K/K_P$ such that $p^*\omega = -d\lambda$, where p is the projection $K \to X$. It is a standard fact that ω defines a symplectic structure on X. Since the involution τ preserves the parabolic subgroup P, it descends to an involution denoted by σ on $X = G/P$ and thus we have:

Theorem 3.2 *When* $\dim_{\mathbb{R}}(X_0) = \dim_{\mathbb{C}} X$, *there exists a symplectic structure on* X *and an anti-symplectic involution* σ *of* X *such that its fixed point set* X^σ *is the closed* G^τ*-orbit* X_0.

Proof The fact that $\tau(\lambda) = -\lambda$ immediately implies that the involution σ on X is anti-symplectic and satisfies $\sigma(k \cdot x) = \tau(k).\sigma(x)$, where $k \in K$. The fixed point set of σ is non-empty, because, for example, it contains the closed K^τ-orbit, which is X_0 by [14]. The rest follows from [10, Example 2.9]. ■

Remark O'Shea and Sjamaar studied real flag manifolds as examples of fixed loci of anti-symplectic involutions on what they called *symmetric* co-adjoint orbits. We refer to [10] for more detail.

Therefore a large class of examples for which we can apply our previous results is given by a product of (partial) flag manifolds for the group G such that each term in the product is equipped with an anti-symplectic involution compatible with the real form G^τ.

In fact, for each isomorphism class of real forms represented by a Satake diagram D and an Iwasawa involution τ, we can classify all standard parabolic subgroups, which are stabilized by τ. The condition is that the corresponding subset S of real roots has to contain all black vertices from D and, in addition, if one white vertex from a pair connected by an arrow is in S, then the other one should be in S as well. So the classification boils down to simple combinatorics involving the Satake diagram. Moreover, if we take the quasi-split D_{qs} in the same inner class together with a commuting Iwasawa involution τ_{qs}, then one can easily see that if a standard parabolic subgroup P_S is stabilized by τ, then it will also be stabilized by τ_{qs}.

For each such standard parabolic subroup P_S stabilized by τ, the flag manifold G/P_S, according to our previous discussion, will have an anti-symplectic involution σ compatible with τ such that $(G/P_S)^\sigma$ is a non-empty Lagrangian submanifold of G/P_S. It follows that the involution $\sigma_{qs} = s\sigma$, determined by $\tau_{qs} = \mathrm{Ad}_s \circ \tau$ will also have a non-empty Lagrangian fixed point set on G/P_S. Therefore, after identifying the flag manifolds of the form G/P_S with the co-adjoint orbit carrying the same symplectic form ω, we arrive at the following result:

Proposition 3.3 *Let $\mathcal{O}_i \subset \mathfrak{k}^*$, $1 \le i \le k$ be a collection of co-adjoint orbits such that $\mathcal{O}_i \simeq G/P_i$ and for each i, the standard parabolic subgroup P_i is τ-stable. Then the equation $A_1 + \cdots + A_k = 0$, $A_i \in \mathcal{O}_i^{\sigma_{qs}}$ has a solution if and only if the same equation has a solution with $A_i \in \mathcal{O}_i^{\sigma}$.*

Proof If $A_1 + \cdots + A_k = 0$ has a solution with $A_i \in \mathcal{O}_i^{\sigma}$, then our preceeding discussion implies that $\mathcal{O}_i^{\sigma_{qs}}$ is non-empty. The rest follows from [10, Theorem 3.1(i)]. ■

Thus we can always reduce the problem about the sum of elements with prescribed spectra for a given real form of the group to the same problem for the quasi-split real form in that inner class.

Remark The inequalities for the additive problem are given in [10]. The equivalence to the multiplicative problem for the quasi-split real form with prescribed singular values follows from [2].

Example Let us consider a complex flag manifold $X = \mathrm{Fl}_{\mathbb{C}}(2m_1, 2m_2, \dots, 2m_k)$ which parameterizes complex flags

$$V_{2m_1} \subset V_{2m_2} \subset \cdots \subset V_{2m_k} \simeq \mathbb{C}^{2m},$$

$\dim_{\mathbb{C}} V_{2i} = 2i$ and $0 < m_1 < \cdots < m_k = m$. We will use the natural embedding

$$\iota \colon X \hookrightarrow \prod_{i=1}^{k} \mathrm{Gr}_{\mathbb{C}}(2m_i, 2m)$$

to get a symplectic structure on X, and the symplectic structure on each of the grassmannians is a positive multiple of the one that comes from the standard Plücker embedding and the Fubini–Study form on the projective space of dimension $2m$ choose $2m_i$.

We shall identify $\mathbb{C}^{2m}$ with the right quaternionic space $\mathbb{H}^m$ as follows. The point $(z_1, \dots, z_{2m}) \in \mathbb{C}^{2m}$ corresponds to the point $(q_1, \dots, q_m) \in \mathbb{H}^m$ if $q_i = z_{2i-1} + \mathbf{j} z_{2i}$ for $1 \le i \le m$. Using this identification, let J be the real operator on $\mathbb{C}^{2m}$ which comes from the right multiplication by $\mathbf{j}$ on the space $\mathbb{H}^m$. Since $J^2 = -\mathrm{Id}$, the action of J on $\mathbb{C}^{2m}$ extends to an involution σ on all complex partial flag manifolds. However, this involution is only real and not a complex diffeomorphism. If all the weights of the subspaces are even, then the fixed point set of σ is clearly the quaternionic (partial) flag manifold:

$$(\mathrm{Fl}^{\mathbb{C}}(2m_1, \dots, 2m_k))^{\sigma} = \mathrm{Fl}^{\mathbb{H}}(m_1, \dots, m_k).$$

Moreover, if ω is an invariant Kähler form on the complex flag manifold, then the fixed point set of σ is a Lagrangian submanifold with respect to ω. For example, when $X = Gr_{\mathbb{C}}(2, 4)$, we have $X^{\sigma} = \mathbb{HP}^1 \simeq \mathrm{S}^4$. As usual, we view the group $\mathrm{PGL}(m, \mathbb{H})$ as a real form of the complex semi-simple group $\mathrm{PGL}(2m, \mathbb{C})$, with the corresponding

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Satake diagram [7] having odd numbered vertices painted black and no arrows. The multiplicity of each restricted root is 4. In terms of matrices, we have the embedding

$$\nu\colon \mathrm{PGL}(m,\mathbb{H}) \hookrightarrow \mathrm{PGL}(2m,\mathbb{C}), \quad A + B\mathbf{j} \mapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}. \tag{3.1}$$

If we let J be the $2m \times 2m$ matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then one can define the involution τ on $\mathrm{PGL}(2m,\mathbb{C})$ by $\tau(C) = -J\bar{C}J$, which defines the real form $\mathrm{PGL}(m,\mathbb{H})$. (To simplify the discussion, we use a different τ than earlier.) One can see that σ and τ are compatible since both have the same origin and the action of $\mathrm{PGL}(m,\mathbb{H})$ on the fixed point set of σ on flag manifolds comes from the action of $\mathrm{PGL}(2m,\mathbb{C})$.

Now we will conjugate τ to the Iwasawa involution $\tau_{\mathbb{H}}$ defined previously, and consider together with the involution τ_{qs} which is, simply, the complex conjugation. Any standard parabolic subgroup defined by any set S consisting of all odd-numbered and an arbitrary subset of even-numbered vertices in the Satake diagram will be simultaneously $\tau_{\mathbb{H}}$- and τ_{qs}-stable. Then, as usual, we can take X to be a product of a number of such G/P_S and arrive at a series of examples when $(X /\!/ K)^{\sigma}$ will have two kinds of connected components corresponding to the involutive quotients of the weighted k-fold product of quaternionic flag manifolds by the diagonal action of groups $\mathrm{PSp}(m)$ and weighted k-fold product of real flag manifolds by the diagonal action of $\mathrm{PO}(2m)$.

Example Let us take $m = 2$ and $m_1 = 1$, and X the m-fold product of complex grassmannians of two-planes in $\mathbb{C}^4$:

$$X = \mathrm{Gr}_{\mathbb{C}}(2,4) \times \cdots \times \mathrm{Gr}_{\mathbb{C}}(2,4).$$

Each factor $\mathrm{Gr}_{\mathbb{C}}(2,4)$ carries its own invariant symplectic form, a positive multiple of a standard one. The fixed point set of the involution σ in this case on each factor is the quaternionic projective line $\mathbb{HP}^1 \simeq \mathrm{S}^4$ and the quotient

$$X^{\sigma} /\!/ K^{\tau} = (\mathrm{S}^4 \times \cdots \times \mathrm{S}^4) /\!/ \mathrm{PSp}(2),$$

considered in detail in [3], is identified with the moduli space of polygons in $\mathbb{R}^5$. Indeed, a point in S^4 gives a direction in $\mathbb{R}^5$, the weight on each factor is the corresponding side length, and $\mathrm{PSp}(2) = \mathrm{SO}(5)$. On the other hand, the real locus of

$$\bigl(\mathrm{Gr}_{\mathbb{C}}(2,4) \times \cdots \times \mathrm{Gr}_{\mathbb{C}}(2,4)\bigr) /\!/ \mathrm{SU}(4)$$

defined by σ will have another connected component corresponding to the involutive quotient

$$\bigl(\mathrm{Gr}_{\mathbb{R}}(2,4) \times \cdots \times \mathrm{Gr}_{\mathbb{R}}(2,4)\bigr) /\!/ \mathrm{PO}(4).$$

4 Invariant Theory Quotients and Involutions

Let K be a compact connected Lie group and let G be its complexification. Then G is a connected linearly reductive complex Lie group. Let τ be an antiholomorphic involution on G defining a real form G^τ. Then we have $K^\tau = K \cap G^\tau$. Let $\rho\colon G \to \mathrm{GL}(N, \mathbb{C})$ be a rational representation and let θ be a Cartan involution of $\mathrm{GL}(N, \mathbb{C})$ commuting with τ. This condition is not very restrictive, since any anti-holomorphic involution on G is conjugate to a one commuting with θ. Let V be a complex vector space of dimension N on which $\mathrm{GL}(N, \mathbb{C})$ acts in the usual way. We can also assume that V has a $\mathrm{U}(N)$-invariant hermitian form. Let V be equipped with an anti-holomorphic involution σ compatible with τ. We can always assume that the image of K under ρ is contained in $\mathrm{U}(N) = \mathrm{GL}(N, \mathbb{C})^\theta$ and thus K acts by unitary operators.

Let us recall some of the results from Richardson–Slodowy [11]. Denote by $\mathcal{M}$ the set of minimal vectors in V^σ for the G^τ-action. A vector $v \in V^\sigma$ is a minimal vector for G^τ if $|g(v)| \geq |v|$ for every $g \in G^\tau$. Now consider the map $\pi : V^\sigma \to V^\sigma /\!/ G^\tau$, where $V^\sigma /\!/ G^\tau$ is the Luna quotient of V^σ by G^τ. We recall that the Luna quotient is the space whose points are the closed orbits equipped with the quotient topology. In [11] it was shown that the inclusion $\mathcal{M} \hookrightarrow V^\sigma$ induces a homeomorphism of the orbit space $\mathcal{M}/K^\tau$ with the quotient $V^\sigma /\!/ G^\tau$. The set $\mathcal{M}$ coincides with the σ-invariant minimal vectors for the G-action on V. For any closed G^τ-stable real-algebraic subset $Z \subset V^\sigma$ the induced map $Z/K^\tau \to Z /\!/ G^\tau$ is a homotopy equivalence. Let $\beta : V^\sigma /\!/ G^\tau \to V /\!/ G$ be the natural map. For each $\xi \in V^\sigma /\!/ G^\tau$, the fiber $\beta^{-1}(\beta(\xi))$ is finite.

Now let now ω stand for the Fubini–Study form on $\mathbb{CP}^{N-1}$ for which σ on $\mathbb{CP}^{N-1}$ satisfies $\sigma^*\omega = -\omega$. Let $X \hookrightarrow \mathbb{CP}^{N-1}$ be a smooth σ-stable projective variety equipped with a linear action of G via ρ.

By standard results in invariant theory [9], there is a momentum map $\mu\colon X \to \mathfrak{k}^*$ such that the inclusion of $\mu^{-1}(0)$ into the subset X^{ss} of semi-stable points induces a homeomorphism:

$$\phi\colon \mu^{-1}(0)/K \to X /\!/ G, \tag{4.1}$$

where the right-hand side is the categorical quotient of X by the action of G. According to our previous discussion, the involution σ leaves $\mu^{-1}(0)$ invariant and descends to the left-hand side of (4.1). It is also easy to check that the involution σ descends to the categorical quotient $X /\!/ G$, and the above homeomorphism respects these involutions.

According to [11], we have a homeomorphism

$$\gamma : (X^\sigma \cap \mu^{-1}(0))/K^\tau \to X^\sigma /\!/ G^\tau.$$

Thus we have the following commutative diagram of continuous maps:

$$\begin{array}{ccc} X^\sigma /\!/ K^\tau & \xrightarrow{\psi} & (X /\!/ K)^\sigma \\ \gamma\downarrow & & \downarrow\phi \\ X^\sigma /\!/ G^\tau & \xrightarrow{\eta} & (X /\!/ G)^\sigma \end{array}$$

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where γ and ϕ are homeomorphisms. In particular, each of the four spaces in the diagram is homeomorphic to a closed semi-algebraic set.

Example Let us return to the example of quaternionic flag manifolds from Section 3. Let $m_i < m$ be two positive integers and let $k_i = \binom{2m}{2m_i}$. Consider the Plücker embedding $\mathrm{Gr}_{\mathbb{C}}(2m_i, 2m) \hookrightarrow \mathbb{CP}^{k_i-1}$. It is easy to see that the action defined by J on $\mathbb{C}^{2m}$ lifts to a complex-conjugate involution denoted by σ on $\mathbb{C}^{k_i}$ for the reason that $2m_i$ is an even number. The compatibility of actions is straightforward. In particular, we can think of the moduli space of polygons in $\mathbb{R}^5$ as the real invariant theoretic quotient of $(\mathbb{HP}^1)^n$ by the diagonal action of the group $\mathrm{PGL}(2, \mathbb{H})$.

Returning to the general case, we can translate the results of our previous discussion to the map η and for each involution $\tau_s = \mathrm{Ad}_s \circ \tau$ strongly inner to τ construct the corresponding involution σ_s and map ψ_s as in Section 2. Note that the element s can be chosen from K.

Theorem 4.1 *If $X \,/\!/\, G$ is a smooth variety, then the manifold $(X \,/\!/\, G)^\sigma$ is homeomorphic to the disjoint union of $\psi_s(X^{\sigma_s} \,/\!/\, G^{\tau_s})$, where s runs through the set $\mathcal{I}_\tau$ as in Section 2. If, moreover, G acts freely on stable points, then each ψ_s is injective.*

When $X \,/\!/\, G$ has quotient singularities, then according to our results in Section 2, the maps ψ_s still cover $(X \,/\!/\, G)^\sigma$, although the (finite) fibers over the singular points may differ from fibers over the smooth points.

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