

The equivalence between rational G -sheaves and rational G -Mackey functors for profinite G

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Abstract

For G a profinite group, we construct an equivalence between rational G -Mackey functors and a certain full subcategory of G -sheaves over the space of closed subgroups of G called Weyl- G -sheaves. This subcategory consists of those sheaves whose stalk over a subgroup K is K -fixed.

This extends the classification of rational G -Mackey functors for finite G of Thévenaz and Webb, and Greenlees and May to a new class of examples. Moreover, this equivalence is instrumental in the classification of rational G -spectra for profinite G , as given in the second author’s thesis.

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1. Introduction

The classification of rational Mackey functors for finite groups is well known and highly useful. An algebraic version of this result is given by Thévenaz and Webb [TW95]. A version more suited to algebraic topology can be found in work of Greenlees and May [GM95, appendix A]. In this case the classification is given as an equivalence of categories

$$\text{Mackey}_{\mathbb{Q}}(G) \cong \prod_{(H) \leq G} \mathbb{Q}[W_G H] - \text{mod}, \tag{1.1}$$

where the product runs over G -conjugacy classes of subgroups of G . Greenlees and May proved this result in order to classify rational G -spectra (for finite G) in terms of an algebraic model. A gentle introduction to classifying rational G -spectra (and similar results) can be found in work of the first author and Kędziołek [BK20].

Profinite groups (compact Hausdorff totally-disconnected topological groups) are an important class of topological groups that contain all finite groups. Examples of where profinite groups occur include the automorphism groups of Galois extensions of fields, the étale fundamental groups of algebraic geometry and the Morava stabiliser group \mathbb{S}_n from chromatic homotopy theory. The standard example is the p -adic integers \mathbb{Z}_p^\wedge for p a prime.

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The main result of this paper is an extension of the classification of rational Mackey functors to profinite groups.

THEOREM A. *For G a profinite group, there is an exact equivalence*

$$\text{Mackey} : \text{Weyl-}G\text{-sheaf}_{\mathbb{Q}}(\mathcal{S}G) \xrightleftharpoons{\quad} \text{Mackey}_{\mathbb{Q}}(G) : \text{Sheaf} .$$

Here $\mathcal{S}G$ denotes the space of closed subgroups of G , see Definition 2.8 for the topology. A Weyl- G -sheaf over $\mathcal{S}G$ is a G -equivariant sheaf over $\mathcal{S}G$ such that the stalk at a closed subgroup $K \leq G$ is K -fixed. Hence, the stalk at K has an action of the Weyl group $W_G K$. This result appears as Theorem 4.8 in the main body.

The reason why Mackey functors for finite groups have a simpler description is that $\mathcal{S}G$ is a discrete space when G is finite. In which case, a Weyl- G -sheaf over $\mathcal{S}G$ is a product of \mathbb{Q} -modules with actions of the Weyl groups. Choosing representatives for conjugacy classes gives the description (1.1) above. Furthermore, the rational Burnside ring for a profinite group is isomorphic to the ring of continuous functions from $\mathcal{S}G/G$ to \mathbb{Q} . Hence, when G is finite, the rational Burnside ring is simply a product of copies of \mathbb{Q} , indexed over the set of conjugacy classes of subgroups of G .

Our next result, see Lemma 2.17, extends a vital identity on rational G -Mackey functors from the finite case to the profinite case. In the finite setting, the restriction map induces an isomorphism

$$e_K^H M(H) \cong e_K^K M(K)^{N_H(K)}, \tag{1.2}$$

where e_K^H is the idempotent in the rational Burnside ring of H corresponding to the conjugacy class of $K \leq H$. To extend this to the profinite case, we use the idempotents $e_{N,K}^H$ and $e_{N,K}^K$ of Definition 2.13, which are analogous to e_K^H and e_K^K .

LEMMA B. *Let $K \leq H$ be open subgroups of G and N an open normal subgroup of K . Restriction from H to K induces an isomorphism*

$$e_{N,K}^H M(H) \cong e_{N,K}^K M(K)^{N_H(K)} .$$

The following is Proposition 2.21 and relates the equivariance of the stalks to that of representing sections. This proposition and the preceding lemma are key to proving the equivalence of categories of Theorem A.

PROPOSITION C. *Let E be a Weyl- G -sheaf over $\mathcal{S}G$ and K a closed subgroup of G . A germ over K can be represented by an NK -equivariant section over a neighbourhood of K , for some open normal subgroup N of G .*

This work is part of the second author’s PhD thesis, [Sug19], completed under the supervision of the first author. That thesis applies the main result of this paper to obtain a classification of rational G -spectra for profinite G in terms of chain complexes of Weyl- G -sheaves over $\mathcal{S}G$, see [Sug19, chapters 3 and 6]. This extends the work of the first author on the case where G is the p -adic integers, [Bar11]. A further application of the classification of rational G -Mackey functors in terms of sheaves is a calculation of the injective dimension of these categories in terms of the Cantor–Bendixson rank of the space $\mathcal{S}G$, see [Sug19, chapter 8].

1.1. Organisation

We define the categories involved in Section 2 and construct the two functors of the classification in Section 3. We prove directly that the functors give an equivalence in Section 4. We end the paper with some examples and the relation between G -sheaves over SG and Weyl- G -sheaves over SG , see Section 5.

2. Mackey functors and sheaves

We introduce the definitions and categories that are used in the paper.

2.1. Basic facts on profinite groups

We give a few reminders of useful facts on profinite groups. More details can be found in Wilson [Wil98] or Ribes and Zalesskii [RZ00].

A **profinite group** is a compact, Hausdorff, totally disconnected topological group. A profinite group G is homeomorphic to the inverse limit of its finite quotients:

$$G \cong \varprojlim_{\substack{N \trianglelefteq G \\ \text{open}}} G/N \subseteq \prod_{\substack{N \trianglelefteq G \\ \text{open}}} G/N.$$

The limit has the canonical topology which can either be described as the subspace topology on the product or as the topology generated by the pre-images of the open sets in G/N under the projection map $G \rightarrow G/N$, as N runs over the open normal subgroups of G .

Closed subgroups and quotients by closed subgroups of profinite groups are also profinite. A subgroup of a profinite group is open if and only if it is finite index and closed. The trivial subgroup $\{e\}$ is open if and only if the group is finite. The intersection of all open normal subgroups is $\{e\}$. Any open subgroup H contains an open normal subgroup, the **core** of H in G , defined as the following intersection. Note that as H has finite index in G , this intersection has only finitely many distinct terms.

$$\text{Core}_G(H) = \bigcap_{g \in G} gHg^{-1}.$$

We can also define a profinite topological space to be a Hausdorff, compact and totally disconnected topological space. As with profinite groups, such a space is homeomorphic to the inverse limit of a filtered diagram of finite spaces. Moreover, a profinite topological space has a closed-open basis.

2.2. Mackey functors for profinite groups

There are many equivalent definitions of Mackey functors for a finite group G . Our preferred version is a collection of abelian groups indexed over the subgroups of G with induction (transfer), restriction and conjugation maps satisfying a list of axioms. In turn, there are several (inequivalent) extensions to profinite groups. Since we have in mind applications to algebraic topology and the work of Fausk [Fau08], we choose a generalisation that emphasises the role of the open subgroups.

The following definition can be found in Bley and Boltje [BB04, definition 2.1 and examples 2.2] as the “finite natural Mackey system” and in Thiel [Thi11, definition 2.2.12]. In the case of a finite group, it restricts to the usual definition.

Definition 2.1. Let G be a profinite group, then a rational **G -Mackey functor** M is:

- (i) a collection of \mathbb{Q} -modules $M(H)$ for each open subgroup $H \leq G$;
- (ii) for open subgroups $K, H \leq G$ with $K \leq H$ and any $g \in G$ we have a **restriction** map, an **induction** map and a **conjugation** map

$$R_K^H : M(H) \implies M(K), \quad I_K^H : M(K) \implies M(H) \quad \text{and} \quad C_g : M(H) \implies M(gHg^{-1}).$$

These maps satisfy the following conditions.

- (i) For all open subgroups H of G and all $h \in H$ the structure maps are **unital**

$$R_H^H = \text{Id}_{M(H)} = I_H^H \quad \text{and} \quad C_h = \text{Id}_{M(H)}.$$

- (ii) For $L \leq K \leq H$ open subgroups of G and $g, h \in G$, there are composition rules

$$I_L^H = I_K^H \circ I_L^K, \quad R_L^H = R_L^K \circ R_K^H, \quad \text{and} \quad C_{gh} = C_g \circ C_h.$$

The first two are **transitivity** of induction and restriction. The last is **associativity** of conjugation.

- (iii) For $g \in G$ and $K \leq H$ open subgroups of G , there are composition rules

$$R_{gKg^{-1}}^{gHg^{-1}} \circ C_g = C_g \circ R_K^H \quad \text{and} \quad I_{gKg^{-1}}^{gHg^{-1}} \circ C_g = C_g \circ I_K^H.$$

This is the **equivariance** of restriction and induction.

- (iv) For open subgroups $K, L \leq H$ of G

$$R_K^H \circ I_L^H = \sum_{x \in [K \backslash H / L]} I_{K \cap xLx^{-1}}^K \circ C_x \circ R_{L \cap x^{-1}Kx}^L.$$

This condition is known as the **Mackey axiom**.

A map of Mackey functors $f : M \rightarrow N$ is a collection of \mathbb{Q} -module maps

$$f(H) : M(H) \longrightarrow N(H)$$

for each open subgroup $H \leq G$, which commute with the induction, restriction and conjugation maps. We use $\text{Mackey}_{\mathbb{Q}}(G)$ to denote this category.

See Thévenaz and Webb [TW95] for an introduction into Mackey functors for finite groups. The case of rational Mackey functors is also discussed in work of the first author with Kędziorek [BK20]. Alternative definitions of Mackey functors in a general setting are discussed in Lindner [Lin76, definition 1].

We have some standard examples similar to those for finite groups.

Example 2.2. Consider the profinite group given by the p -adic integers, \mathbb{Z}_p^\wedge . The open subgroups in this case are of the form $p^k \mathbb{Z}_p^\wedge$ for $k \geq 0$. We define a constant Mackey functor by sending each open subgroup to \mathbb{Q} with all restriction and conjugation maps taken to be the identity. The subgroup $p^l \mathbb{Z}_p^\wedge$ is contained inside $p^k \mathbb{Z}_p^\wedge$ if and only if $l \geq k$. The Mackey axiom forces the induction maps to be

$$I_{p^l \mathbb{Z}_p^\wedge}^{p^k \mathbb{Z}_p^\wedge} = \text{multiplication by } p^{l-k}.$$

In general, there is a constant Mackey functor for any profinite group G , which takes value \mathbb{Q} at every open subgroup H of G . The conjugation and restriction maps are identity maps and induction from K to H is multiplication by the index of K in H .

Example 2.3. Let $R(G)$ denote the ring of finite dimensional complex representations of the profinite group G . We define a rational Mackey functor M_R by $M_R(H) = R(H) \otimes \mathbb{Q}$, with induction and restriction induced by induction and restriction of representations. Conjugation is induced from precomposition with the conjugation homomorphism of groups.

In the case of finite groups, another common example is the fixed point Mackey functor. A profinite analogue exists, but one needs to be more careful over the action of G on a \mathbb{Q} -module.

Definition 2.4. A rational **discrete** G -module M is a \mathbb{Q} -module such that

$$\operatorname{colim}_{\substack{H \leq G \\ \text{open}}} M^H \cong M.$$

The definition is equivalent to the statement that any $m \in M$ is stabilised by an open subgroup of G . If we give a rational G -module M the discrete topology, then the action of G on M is continuous if and only if M is discrete in the preceding sense.

Example 2.5. Given M a discrete G -module, there is a Mackey functor which at an open subgroup $H \leq G$ takes value M^H . The restriction maps are inclusion of fixed points and the conjugation maps are given by left multiplication. The induction map from K to H is given by summing over the action of a left transversal of K in H (which is finite as the subgroups are open).

Example 2.6. The rational Burnside ring $\mathbf{A}_{\mathbb{Q}}(G)$ of a profinite group G defines a rational Mackey functor. For H an open subgroup of G , the ring $\mathbf{A}_{\mathbb{Q}}(H)$ is the rational Grothendieck ring of finite H -sets (that is, finite discrete topological spaces with a continuous action of H). The multiplication is given by the product of H -sets.

Restriction and induction between $\mathbf{A}_{\mathbb{Q}}(H)$ and $\mathbf{A}_{\mathbb{Q}}(K)$ are given by the usual restriction and induction of sets with group actions. Moreover, the restriction maps are maps of rings. Conjugation is induced from precomposition with the conjugation homomorphism of groups.

Given a G -Mackey functor M , we can define an action of the Burnside ring $\mathbf{A}_{\mathbb{Q}}(H)$ on the abelian group $M(H)$ by

$$[H/K] := I_K^H R_K^H : M(H) \longrightarrow M(K)$$

and extending linearly from the additive basis for $\mathbf{A}_{\mathbb{Q}}(H)$ given by H/K for open subgroups K of H . The Mackey axiom implies that this action is compatible with the multiplication of $\mathbf{A}_{\mathbb{Q}}(H)$, so that $M(H)$ is a module over $\mathbf{A}_{\mathbb{Q}}(H)$. Moreover, the following square commutes

$$\begin{array}{ccc} \mathbf{A}_{\mathbb{Q}}(H) \otimes M(H) & \longrightarrow & M(H) \\ R_K^H \otimes R_K^H \downarrow & & R_K^H \downarrow \\ \mathbf{A}_{\mathbb{Q}}(K) \otimes M(K) & \longrightarrow & M(K) \end{array}$$

We also have the relation, known as **Frobenius reciprocity**, between induction and the action of Burnside rings. Let $\alpha \in \mathbf{A}_{\mathbb{Q}}(H)$, $\beta \in \mathbf{A}_{\mathbb{Q}}(K)$, $m \in M(K)$ and $n \in M(H)$, then

$$\alpha \cdot I_K^H(m) = I_K^H(R_K^H(\alpha) \cdot m) \quad I_K^H(\beta) \cdot n = I_K^H(\beta \cdot R_K^H(n)).$$

See Yoshida [**Yos80**, definition 2.3 and example 2.11]. We summarise this discussion in the following proposition.

PROPOSITION 2.7. *If M is a Mackey functor for a profinite group G and H is an open subgroup of G , then each $M(H)$ has an $\mathbf{A}_{\mathbb{Q}}(H)$ -module structure. Moreover, the restriction map $M(H) \rightarrow M(K)$ is a map of modules over the restriction map $\mathbf{A}_{\mathbb{Q}}(H) \rightarrow \mathbf{A}_{\mathbb{Q}}(K)$, which is compatible with the conjugation maps. The Burnside ring actions satisfy Frobenius reciprocity with respect to the induction maps.*

2.3. Burnside ring idempotents

As with finite groups, there is a description of the rational Burnside ring in terms of continuous maps out of the space of subgroups of G . We will use this structure repeatedly, so we examine the result in detail and use it to describe idempotents of the Burnside ring.

Definition 2.8. For G a profinite group, the space of closed subgroups of G , SG , is the topological space

$$SG := \lim_{\substack{N \triangleleft G \\ \text{open}}} \mathcal{S}(G/N).$$

where $\mathcal{S}(G/N)$ is a finite space with the discrete topology. For $N \leq N'$, the corresponding map of the limit diagram is

$$p_{N,N'}: \mathcal{S}(G/N) \longrightarrow \mathcal{S}(G/N') \quad H/N \longmapsto HN'/N'.$$

The limit is topologised as either the subspace of the product or as the coarsest topology so that the projection maps

$$p_N: SG \longrightarrow \mathcal{S}(G/N) \quad H \longmapsto HN/N$$

are continuous. The group G acts continuously on SG by conjugation, $g \in G$ sends a closed subgroup K to $gKg^{-1} = {}^gK$.

Following Gartside and Smith [**GS10**, lemma 2.14], we can give a compact-open basis for this topology.

LEMMA 2.9. *An explicit basis for SG is the collection of closed and open sets*

$$O_G(N, J) = \{K \in SG \mid NK = J\}$$

such that N is an open normal subgroup of G , and J is an open subgroup of G (that contains N).

Proof. The set $O_G(N, J)$ is the pre-image of the set $\{J/N\} \in \mathcal{S}(G/N)$ under p_N , hence it is open and closed. To see this is a basis, let $A \in O_G(N, J) \cap O_G(N', J')$ and set $N'' = N \cap N'$, then

$$A \in O_G(N'', N''A) \subseteq O_G(N, J) \cap O_G(N', J').$$

These sets are compatible with the G -action:

$$gO_G(N, J) = \{ {}^gK \mid NK = J \} = O_G(N, {}^gJ).$$

Since any open subgroup of G has form NK for N some open normal subgroup of G and K a closed subgroup of G , we can rewrite the basis to consist of the sets

$$\left\{ O_G(N, NK) \mid \begin{array}{l} N \trianglelefteq_{\text{open}} G, \\ K \leq_{\text{closed}} G \end{array} \right\}.$$

If we fix K and allow N to vary, these sets give a neighbourhood basis for $K \in SG$. We further define

$$\overline{O}_G(N, J) = O_G(N, J)/G \subseteq SG/G,$$

the quotient under the conjugation action. This set is open in SG/G as the quotient map is open. It is closed as $O_G(N, K)$ is compact. We will also make use of the equality

$$O_G(N, NK) = O_{NK}(N, NK).$$

THEOREM 2.10. *For G a profinite group, there is an isomorphism of commutative rings*

$$\mathbf{A}_{\mathbb{Q}}(G) \cong C(SG/G, \mathbb{Q}).$$

Proof. This is Dress [Dre71, theorem B.2.3(a)]. As with finite groups, the isomorphism maps $[G/H]$ to the function which sends a conjugacy class of closed subgroup K to $|(G/H)^K|$.

LEMMA 2.11. *For G a profinite group, there is a natural isomorphism of commutative rings*

$$\operatorname{colim}_{\substack{N \trianglelefteq_{\text{open}} G}} \mathbf{A}_{\mathbb{Q}}(G/N) \cong \mathbf{A}_{\mathbb{Q}}(G).$$

Proof. Since any G/N -set is a G -set by precomposing with the quotient $G \rightarrow G/N$, we have the given map of rings, which is injective. Surjectivity follows as a continuous action of G on a finite set factors through some finite quotient of G .

Remark 2.12. Similar to Pontryagin duality, the previous lemma and theorem imply that the canonical map

$$\operatorname{colim}_{\substack{N \trianglelefteq_{\text{open}} G}} C(\mathcal{S}(G/N)/(G/N), \mathbb{Q}) \cong C\left(\lim_{\substack{N \trianglelefteq_{\text{open}} G}} \mathcal{S}(G/N)/(G/N), \mathbb{Q}\right) = C(SG/G, \mathbb{Q})$$

induced by the surjections $SG/G \rightarrow \mathcal{S}(G/N)/(G/N)$ is an isomorphism.

This type of isomorphism holds for any profinite space X . The difficult step is proving surjectivity. This comes from the fact that any map $X \rightarrow \mathbb{Q}$ must have finite image and hence factor through some finite quotient of X . See Ribes and Zalesskii [RZ00, proposition 1.1.16 (a) and lemma 5.6.3].

An idempotent of $\mathbf{A}_{\mathbb{Q}}(H) = C(SH/H, \mathbb{Q})$ corresponds to an open and closed subset of SH/H . The projection map $p_H: SH \rightarrow SH/H$ is open and both domain and codomain are compact, hence an open and closed subset of SH defines an idempotent of $\mathbf{A}_{\mathbb{Q}}(H)$. Indeed,

the set of idempotents of $\mathbf{A}_{\mathbb{Q}}(H)$ is in bijection with the set of open, closed and H -invariant subsets of SH .

Definition 2.13. For U an open and closed subset of SH/H , there is an idempotent e_U^H , the characteristic function of U .

An open and closed subset V of SH (that is not necessarily H -invariant) defines an idempotent $e_V^H := e_{p_H(V)}^H$ of $\mathbf{A}_{\mathbb{Q}}(H)$, the characteristic function of V/H . In turn, we define

$$e_{N,J}^H = e_{O_H(N,J)}^H = e_{\overline{O_H(N,J)}}^H$$

the characteristic function for $\overline{O_H(N, J)} = O_H(N, J)/H$, for J an open subgroup of H .

We relate our closed-open basis for SG to idempotents of $C(SG/G, \mathbb{Q})$ and the additive basis of that ring. For the formula we will use the Moebius function μ for pairs of subgroups $D \leq K$

$$\mu(D, K) = \sum_{i \geq 0} (-1)^i c_i,$$

where c_i is the number of strictly increasing chains from D to K of length i . Note that a chain of the form $D < K$ has length 1 and, by convention, $\mu(K, K) = 1$.

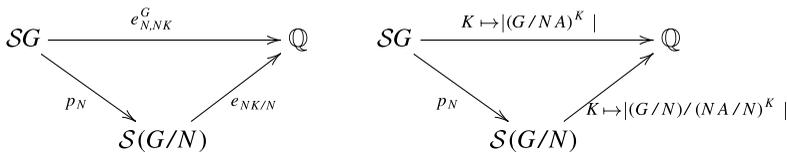
PROPOSITION 2.14. *For G a profinite group, with open normal subgroup N and closed subgroup K*

$$e_{N,NK}^G = \sum_{N \leq NA \leq NK} \frac{|NA|}{|N_G(NK)|} \mu(NA, NK)[G/NA].$$

Proof. The sum is finite as we range over open subgroups of G that contain N .

The method is to reduce the calculation to the case of finite groups, and use the formula for idempotents of the rational Burnside ring of the finite group G/N from Gluck [Glu81, section 2]. The reduction comes from the following commuting diagrams, where $p_N : SG \rightarrow S(G/N)$ sends H to HN/N . The second diagram uses the homeomorphism of G -spaces:

$$G/NA \cong (G/N)/(NA/N).$$



If we replace p_N by $\bar{p}_N : SG/G \rightarrow S(G/N)/(G/N)$ the two diagrams above remain commutative.

The first diagram and the formula for idempotents gives

$$e_{N,NK}^G = \sum_{N \leq NA \leq NK} \frac{|NA|}{|N_G(NK)|} \mu(NA, NK) [(G/N) / (NA/N)] \circ p_N.$$

The second diagram completes the proof.

The coefficients are constant under certain conjugations. More precisely, for $K \leq J \leq G$ open subgroups of G and $a \in N_G(J)$, if we define

$$\alpha_{K,J} = \frac{|K|}{|N_G(J)|} \mu(K, J), \quad \text{then} \quad \alpha_{K,J} = \alpha_{aKa^{-1}, J}.$$

A simple observation shows that for a rational G -Mackey functor M , each component $M(H)$ is a sheaf over the profinite space SH/H . This holds as $M(H)$ is a module over $\mathbf{A}_{\mathbb{Q}}(H) = C(SH/H, \mathbb{Q})$.

PROPOSITION 2.15. *Suppose X is a profinite space and M is a $C(X, \mathbb{Q})$ -module, then M determines a sheaf of \mathbb{Q} -modules over X .*

Proof. Let \mathfrak{B} be an open-closed basis for X . Given a module M , we define a sheaf F by giving its values on \mathfrak{B} . Let $U \in \mathfrak{B}$ and define the characteristic function of U to be the map which sends elements of U to 1 and the rest of X to 0. This map, which is denoted $e_U \in C(X, \mathbb{Q})$, is continuous as U is both open and closed. We define $F(U) = e_U M$. For $V \subseteq U$ in \mathfrak{B} , we define the restriction map to be the projection (where V^c is the complement of V in X)

$$e_U M \cong e_V M \oplus e_{V^c \cap U} M \longrightarrow e_V M$$

which is equivalent to multiplying by e_V .

The sheaf condition follows as we have a closed-open basis of a compact space and hence can write any open cover of a basis element as a finite partition of basis elements.

COROLLARY 2.16. *If M is a Mackey functor over a profinite group G , then for each open subgroup H of G , the \mathbb{Q} -module $M(H)$ defines a sheaf over SH/H , the space of H -conjugacy classes of closed subgroups of H .*

Paraphrasing the corollary, we see that a G -Mackey functor is a collection of sheaves, one for each open subgroup of G . The rough idea of the equivalence is to patch parts of these sheaves together to construct a G -equivariant sheaf. It will take some time to make this rigorous, but we can give a useful relation between the various sheaves $M(H)$. This relation is an extension of a result for finite groups, as referenced in the proof, see also Equation (1.2) of the introduction.

LEMMA 2.17. *Let $K \leq H$ be open subgroups of G and N an open normal subgroup of K . Restriction from H to K gives an isomorphism*

$$e_{N,K}^H M(H) \cong e_{N,K}^K M(K)^{N_H(K)}.$$

Proof. The method is to reduce the problem to the case of finite groups. Similarly to Greenlees and May [GM92, definition 3] or Thévenaz and Webb [TW95], we may define a G/N -Mackey functor \overline{M} by

$$\overline{M}(J/N) = M(J).$$

Proposition 2.14 implies that when $J \geq N$,

$$e_{N,K}^J M(J) \cong e_{K/N}^{J/N} \overline{M}(J/N).$$

Now that we have Mackey functors for finite groups, we may use [Gre98, example 5C(i) and corollary 5.3] to see that restriction induces an isomorphism

$$e_{K/N}^{K/N} \overline{M}(K/N)^{N_{H/N}(K/N)} \cong e_{K/N}^{H/N} \overline{M}(H/N).$$

Alternative proofs of that result occur in [Sug19, lemma 6.1.9] and [BK20]. Combining these isomorphisms and the fact $N_{H/N}(K/N) \cong N_H(K)/N$, we obtain

$$\begin{aligned} e_{N,K}^K M(K)^{N_H(K)} &= e_{N,K}^K M(K)^{N_H(K)/N} \\ &= e_{K/N}^{K/N} \overline{M}(K/N)^{N_{H/N}(K/N)} \\ &= e_{K/N}^{H/N} \overline{M}(H/N) \\ &= e_{N,K}^H M(H). \end{aligned}$$

2.4. *Equivariant sheaves*

We begin with the definition of a G -equivariant sheaf over a profinite G -space X where G is a profinite group. The second author gives two equivalent definitions in [Sug19], see also [BS20]. We work with just one for brevity.

Definition 2.18. A **G -equivariant sheaf** of \mathbb{Q} -modules over X is a map of topological spaces $p: E \rightarrow X$ such that:

- (i) p is a G -equivariant map $p: E \rightarrow X$ of spaces with continuous G -actions,
- (ii) (E, p) is a sheaf space (étale space) of \mathbb{Q} -modules,
- (iii) each map $g: p^{-1}(x) \rightarrow p^{-1}(gx)$ is a map of \mathbb{Q} -modules for every $x \in X, g \in G$.

We will write this as either the pair (E, p) or simply as E . We call E the **total space**, X the **base space** and p the structure map.

We call $p^{-1}(x)$ the **stalk** of E at x and denote it E_x . An element of a stalk is called a **germ**.

Note that points (i) and (ii) give a map of sets for point (iii), but they do not imply that it is a map of \mathbb{Q} -modules. Given a G -equivariant sheaf (E, p) over X and $x \in X$, the stalk E_x (equipped with the discrete topology) has a continuous action of $\text{stab}_G(x)$.

As in the non-equivariant case, given an open subset $U \subseteq X$ the space of (continuous) sections

$$E(U) = \Gamma(U, E) = \{s: U \rightarrow E \mid p \circ s = \text{Id}_U\}$$

has an addition operation (defined stalk-wise). The sections are not required to be G -equivariant. Allowing U to vary defines a functor from the set of open subsets of X to \mathbb{Q} -modules. Moreover, one can show that

$$\text{colim}_{U \ni x} \Gamma(U, E) = E_x = p^{-1}(x)$$

as in the non-equivariant setting.

Given a section $s: U \rightarrow E$ and $g \in G$, we can define

$$g * s = g \circ s \circ g^{-1}: gU \rightarrow E$$

which sends $v = gu$ to $gs(u) = gs(g^{-1}v)$. Hence, if U is invariant under the action of a subgroup H (that is, $hU = U$ for all $h \in H$), the space of sections $\Gamma(U, E)$ has an H -action.

The fixed points of this space are those sections which commute with the action of H , which we call **H -equivariant** sections.

Definition 2.19. A **Weyl- G -sheaf** of \mathbb{Q} -modules over SG is a G -sheaf of \mathbb{Q} -modules over SG such that action of H on E_H is trivial. Hence E_H has an action of the Weyl group $W_G H$ of H in G . We use $\text{Weyl-}G\text{-sheaf}_{\mathbb{Q}}(SG)$ to denote this category.

The idea of the classification result is to construct a Mackey functor M from a Weyl- G -sheaf E by setting $M(H) = E(SH)^H$. For this we need SH to be an open subset of SG when H is open in G .

LEMMA 2.20. *For H a closed subgroup of G , SH is closed in SG . This subspace is also open when H is open.*

Proof. The open statement follows from noting that

$$SH = \bigcup_{K \in SH} O_H(\text{Core}(H), \text{Core}(H)K).$$

For the closed statement we see that H is the limit of finite groups of the form $H/(H \cap N) = HN/N$ for N an open normal subgroup of G . Since $\mathcal{S}(HN/N)$ is a closed subset of $\mathcal{S}(G/N)$, the result follows by taking limits.

We also need a result which can be described as saying that a section of a G -equivariant sheaf over a profinite G -space is “locally sub-equivariant”. See [Sug19, section 4.3] for related results.

PROPOSITION 2.21. *If E is a Weyl- G -sheaf over SG and K a closed subgroup of G , then any $s_K \in E_K$ can be represented by an NK -equivariant section*

$$s: O_{NK}(N, NK) \longrightarrow E$$

for N some open normal subgroup of G .

Proof. Let $s: O_{MK}(M, MK) = U \rightarrow E$ be a section representing s_K . The set $s(U)$ is not necessarily closed under the action of G , but if $gu \in U$ and $gs(u) \in s(U)$ for some $g \in G$ and $u \in U$, then

$$ps(gu) = gu = gp(s(u)) = p(gs(u))$$

as p is G -equivariant. Since p is injective when restricted to $s(U)$, we see that $s(gu) = g(s(u))$. The method of the proof is to restrict the domain and codomain so that they are closed under the action of NK , for N some open normal subgroup of G .

The set $s(U)$ is open (by definition of the topology on a sheaf space) and is the image of compact set. Hence, there is an open normal subgroup M' of G such that $M's(U) = s(U)$ by [BS20, lemma A.1]. Let

$$V = \bigcap_{k \in K} ks(U)$$

which consists of only finitely many distinct terms as $M' \cap K$ has finite index in K . Moreover, $s_K = s(K) \in V$ as $s(K)$ is K -fixed, so V is a non-empty compact open subset of E which is invariant under $M''K$ for some open normal subgroup M'' of G .

The set $p(V)$ is open and contains K , hence we can find a basic open set containing K of the form

$$W = O_{NK}(N, NK) \subseteq O_{MK}(M, MK) \cap p(V)$$

for some $N \leq M \cap M''$. The section $s|_W$ is NK -equivariant by our earlier argument as $gw \in W$ and $gs(w) \in V \subseteq s(U)$ for all $g \in NK$ and $w \in W$.

3. The functors

In this chapter we will construct a correspondence between rational G -Mackey functors and Weyl- G -sheaves over $\mathcal{S}G$. We shall explicitly construct functors between the categories, see Theorems 3.5 and 3.13. In Theorem 4.8 we will see that these functors are equivalences of categories.

3.1. Weyl- G -sheaves determine Mackey functors

We define a functor:

$$\text{Mackey} : \text{Weyl-}G\text{-sheaf}_{\mathbb{Q}}(\mathcal{S}G) \longrightarrow \text{Mackey}_{\mathbb{Q}}(G).$$

We will not need the input sheaf to be a Weyl- G -sheaf, a detail we return to in Subsection 5.2.

Construction 3.1. Let (E, p) be a G -sheaf over $\mathcal{S}G$, we define a Mackey functor $\text{Mackey}(E)$ as follows.

For $H \leq G$ an open subgroup define

$$\text{Mackey}(E)(H) = E(\mathcal{S}H)^H = \{s : \mathcal{S}H \longrightarrow E \mid p \circ s = \text{Id}_{\mathcal{S}H}\}^H$$

the set of H -equivariant sections on $\mathcal{S}H$. The conjugation maps C_g are given by the G -action on the sheaf E .

For $K \leq H$ another open subgroup, the restriction map

$$R_K^H : E(\mathcal{S}H)^H \implies E(\mathcal{S}K)^K$$

is given by restriction of a section to the subspace $\mathcal{S}K$.

Let $T \subset H$ be a left transversal of K in H . For β a section of $\mathcal{S}K$, let $\bar{\beta}$ be the extension by zero to $\mathcal{S}H$. We define the induction map by

$$I_K^H : E(\mathcal{S}K)^K \implies E(\mathcal{S}H)^H, \quad \beta \mapsto \sum_{h \in T} h * \bar{\beta}.$$

We prove that the induction functor is well defined in Lemma 3.2. The structure maps compose and interact appropriately by Lemma 3.3. Lemma 3.4 proves that the Mackey axiom holds.

LEMMA 3.2. *The induction map given in Construction 3.1 is well defined.*

Proof. Let $hK = h'K$ be two representatives for the same coset. Since the section β is K -fixed,

$$h * \bar{\beta} = h' * \bar{\beta}$$

as h and h' differ by an element of K . It follows that I_K^H is independent of the choice of T .

The sum of $h * \bar{\beta}$ over $h \in T$ is H -equivariant as T is a transversal.

LEMMA 3.3. *The structure maps of Mackey(E) are unital, associative, transitive and equivariant.*

Proof. The part of most interest is that the induction maps are transitive and equivariant. The first follows from the fact that given open subgroups $J \leq K \leq H$, one can combine a transversal of J in K with a transversal of K in H to get a transversal of J in H . This gives the transitivity. For equivariance, the result follows as one can conjugate a transversal to get a transversal of the conjugate.

We note that $\mathcal{S}H$ is invariant under the action of $N_G H \leq G$, so $E(\mathcal{S}H)$ has a continuous action of $N_G H$.

It remains to show the Mackey axiom, which is a direct calculation.

LEMMA 3.4. *The construction Mackey(E) satisfies the Mackey axiom.*

Proof. We start with $J, L \leq H$ where $H, J, L \leq G$ are open. We can decompose H into double cosets.

$$H = \coprod_{hL \in H/L} hL = \coprod_{x \in [J \backslash H / L]} JxL = \coprod_{x \in [J \backslash H / L]} \bigcup_{j \in J} jxL = \coprod_{\substack{x \in [J \backslash H / L] \\ j_x \in J / J \cap xLx^{-1}}} j_x xL.$$

Given a transversal T for L in H we have

$$R_J^H I_L^H(\beta) = \left(\sum_{h \in T} h * \bar{\beta} \right) \Big|_{SJ} = \left(\sum_{\substack{x \in [J \backslash H / L] \\ j_x \in J / J \cap xLx^{-1}}} j_x x * \bar{\beta} \right) \Big|_{SJ} = \sum_{\substack{x \in [J \backslash H / L] \\ j_x \in J / J \cap xLx^{-1}}} \left(\overline{j_x x \beta \Big|_{\mathcal{S}(x^{-1} J x \cap L)}} \right),$$

where the extension by zero in the third term is with respect to H and in the in the fourth term it is with respect to J . If we start from the other direction we have:

$$\begin{aligned} \sum_{x \in [J \backslash H / L]} I_{J \cap xLx^{-1}}^J \circ C_x \circ R_{x^{-1} J x \cap L}^L(\beta) &= \sum_{\substack{x \in [J \backslash H / L] \\ j_x \in J / J \cap xLx^{-1}}} j_x x * \left(\overline{\beta \Big|_{\mathcal{S}(L \cap x^{-1} J x)}} \right) \\ &= \sum_{\substack{x \in [J \backslash H / L] \\ j_x \in J / J \cap xLx^{-1}}} \left(\overline{j_x x \beta \Big|_{\mathcal{S}(L \cap x^{-1} J x)}} \right) \end{aligned}$$

proving that the two sides coincide.

Similar arguments to the above show that a map of sheaves $E \rightarrow E'$ induces a map of G -Mackey functors $\text{Mackey}(E) \rightarrow \text{Mackey}(E')$. We summarise this section in the following theorem.

THEOREM 3.5. *Let G be a profinite group. If E is a G -sheaf of \mathbb{Q} -modules over $\mathcal{S}G$, then $\text{Mackey}(E)$ from Construction 3.1 is a Mackey functor and the assignment is functorial. Hence, there is a functor*

$$\text{Mackey} : \text{Weyl-}G\text{-sheaf}_{\mathbb{Q}}(\mathcal{S}G) \longrightarrow \text{Mackey}_{\mathbb{Q}}(G).$$

3.2 Mackey functors determine Weyl- G -sheaves

We construct a functor in the opposite direction, from rational G -Mackey functors to Weyl- G -sheaves of \mathbb{Q} -modules over $\mathcal{S}G$. We start by constructing the stalks of the Weyl- G -sheaf.

Definition 3.6. For M a rational G -Mackey functor and K a closed subgroup of G , we define a \mathbb{Q} -module

$$\text{Sheaf}(M)_K = \text{colim}_{J \supseteq K} M(J)_{(K)} = \text{colim}_{\substack{N \leq G \\ \text{open}}} M(NK)_{(K)} = \text{colim}_{\substack{N \leq G \\ \text{open}}} e_{N,NK}^{NK} M(NK).$$

The maps in the colimits are induced by the restriction maps of M and applying idempotents. The notation $M(J)_{(K)}$ refers to the stalk of the SJ/J -sheaf $M(J)$ at the J -conjugacy class of K , for J an open subgroup of G containing K .

The restriction maps are compatible with taking stalks since $e_{N,NK}^H \in \mathbf{A}_{\mathbb{Q}}(H)$ restricts to an idempotent $f \in \mathbf{A}_{\mathbb{Q}}(J)$ for $K \leq J \leq H$, where the support of f is those subgroups of J (up to J -conjugacy) which are H -conjugate to an element of $O_H(N, NK)$. This support contains

$$K \in O_J(N, NK) = O_H(N, NK) \cap SJ.$$

We begin our construction of a rational G -equivariant sheaf (E, p) over $\mathcal{S}G$ from a Mackey functor M . We first define E and p .

Construction 3.7. If M is a rational G -Mackey functor, we define the underlying set of the sheaf space

$$\text{Sheaf}(M) = \coprod_{K \in \mathcal{S}G} \text{Sheaf}(M)_K.$$

The projection map $p : \text{Sheaf}(M) \rightarrow \mathcal{S}G$ sends all of $\text{Sheaf}(M)_K$ to K . The conjugation maps of M induces maps

$$\text{Sheaf}(M)_K \longrightarrow \text{Sheaf}(M)_{gKg^{-1}}$$

for each $g \in G$. Thus $\text{Sheaf}(M)$ has a G -action and p is G -equivariant.

We construct a set of sections that will give a basis for a topology on $\text{Sheaf}(M)$. For H an open subgroup of G , define a map

$$\theta_H : M(H) \longrightarrow \left\{ s : SH \longrightarrow \coprod_{K \in SH} \text{Sheaf}(M)_K \right\}^H,$$

where $\theta_H(m)$ sends $K \leq H$ to m_K , the image of $m \in M(H)$ in $\text{Sheaf}(M)_K$. We must check that $\theta_H(m)$ defines an H -equivariant map. As restriction and applying idempotents are equivariant, the square below commutes:

$$\begin{array}{ccc} M(H) & \longrightarrow & \text{Sheaf}(M)_K \\ \downarrow C_h & & \downarrow C_h \\ M(H) & \longrightarrow & \text{Sheaf}(M)_{hKh^{-1}}. \end{array}$$

Since $C_h = \text{Id}_{M(H)}$, we have the desired equivariance. We also see that m_K is fixed by $N_H(K)$.

We can restrict a section $s = \theta_{NK}(m)$ coming from $m \in e_{N,NK}^{NK}M(NK)$ to a section over

$$O_{NK}(N, NK) = O_G(N, NK).$$

The sets $s(O_G(N, NK))$ for varying open and normal N , closed K , and $s = \theta_{NK}(m)$ for m in $e_{N,NK}^{NK}M(NK)$, define a topology on $\text{Sheaf}(M)$ by Lemma 3.8.

The G -action is continuous by Lemma 3.9 and the projection map p is a G -equivariant local homeomorphism by Proposition 3.10.

Lemma 3.11 shows that $(\text{Sheaf}(M), p)$ is a sheaf of \mathbb{Q} -modules. Lemma 3.12 completes the construction by showing that we have a Weyl- G -sheaf.

LEMMA 3.8. *The sets $s(O_G(N, NL))$ constructed above form a basis for a topology on*

$$\text{Sheaf}(M) = \coprod_{K \in SG} \text{Sheaf}(M)_K = \coprod_{K \in SG} \text{colim}_{\substack{N \leq G \\ \text{open}}} e_{N,NK}^{NK}M(NK).$$

Proof. Given $a \in \text{Sheaf}(M)_L$, we can find a representative $m \in e_{N,NL}^{NL}M(NL)$. We then take the composite

$$e_{N,NL}^{NL}M(NL) \longrightarrow M(NL) \xrightarrow{\theta_{NL}} \left\{ s: \mathcal{S}NL \longrightarrow \coprod_{K \in \mathcal{S}NL} \text{Sheaf}(M)_K \right\}^{NL}.$$

We see that $m_L = \theta_{NL}(m)(L) = a$ and so each germ is in the image of one of our chosen sections. It follows that the open sets $s(O_G(N, NK))$ cover $\text{Sheaf}(M)$.

Now we show that the intersection of two basis sets is a union of basis elements. Take sections $t_1 = \theta_{N_1K_1}(m_1)$ and $t_2 = \theta_{N_2K_2}(m_2)$ and let

$$x \in t_1(O_G(N_1, N_1K_1)) \cap t_2(O_G(N_2, N_2K_2)).$$

We construct a $t(O_G(N, NK))$ satisfying

$$x \in t(O_G(N, NK)) \subseteq t_1(O_G(N_1, N_1K_1)) \cap t_2(O_G(N_2, N_2K_2)).$$

Given such an x , we let $L = p(x)$, which is a closed subgroup of G . We see that

$$L \in O_G(N_1, N_1K_1) \cap O_G(N_2, N_2K_2)$$

and $t_1(L) = t_2(L)$ in $\text{Sheaf}(M)_L$. Hence, there is an open normal subgroup $N \leq N_1 \cap N_2$ of G such that m_1 and m_2 restricted to $e_{N,NL}^{NL}M(NL)$ agree. It follows that t_1 and t_2 agree when restricted to $O_{NL}(N, NL) = O_G(N, NL)$. We define t to be this common refinement and see

$$t(O_G(N, NL)) \subseteq t_1(O_G(N_1, N_1K_1)) \cap t_2(O_G(N_2, N_2K_2)).$$

LEMMA 3.9. *The G -action on the space $\text{Sheaf}(M)$ from Construction 3.7 is continuous.*

Proof. Take a basic open set $s(O_G(N, NK))$ for $s = \theta_{NK}(m)$ with $m \in M(NK)$. Take any point (g, t_L) in the pre-image of $s(O_G(N, NK))$ under the group action map. Then

$$(gt)(gLg^{-1}) = s(gLg^{-1}) \in \text{Sheaf}(M)_{gLg^{-1}}.$$

We can therefore find an open normal subgroup N and representatives such that

$$(gt)|_{O_G(N,NgLg^{-1})} = s|_{O_G(N,NgLg^{-1})}, \quad \text{hence} \quad t|_{O_G(N,NL)} = (g^{-1}s)|_{O_G(N,NL)}.$$

These sections have N -invariant domains and are obtained from an N -fixed module $M(NL)$. We can conclude that the open set:

$$W = gN \times t(O_G(N, NL))$$

is contained in pre-image of $s(O_G(N, NK))$ under the group action map.

PROPOSITION 3.10. *The projection map in Construction 3.7 is a continuous G -map and a local homeomorphism.*

Proof. We first prove that p is continuous. Given a basic open subset of SG of the form $O_G(N, NK)$, we have that

$$p^{-1}(O_G(N, NK)) = \coprod_{L \in O_G(N, NK)} \text{Sheaf}(M)_L.$$

Take $s \in \text{Sheaf}(M)_L$ and a section

$$t: O_G(N', N'L) \longrightarrow \coprod_{L \in O_G(N', N'L)} \text{Sheaf}(M)_L,$$

with $t(L) = s$ and $N' \leq N$. Then $t(O_G(N', N'L)) \subseteq p^{-1}(O_G(N, NK))$ and is open.

We now show that p is a local homeomorphism. A point s in $\text{Sheaf}(M)_L$ has a neighbourhood of the form $t(O_G(N, NK))$ as seen previously. Let $f = p|_{t(O_G(N, NK))}$, we claim that

$$f: t(O_G(N, NK)) \longrightarrow O_G(N, NK)$$

is a homeomorphism. Since f is bijective and continuous, we need only show that it is open.

A basic open set of $t(O_G(N, NK))$ is a set of the form $t'(O_G(N', N'K')) \leq t(O_G(N, NK))$ for $O_G(N', N'K') \subseteq O_G(N, NK)$. The map f sends $t'(O_G(N', N'K'))$ to $O_G(N', N'K')$, which is open in $O_G(N, NK)$.

LEMMA 3.11. *If U is any open subset of SG , then the set of sections of p over U from Construction 3.7 is a \mathbb{Q} -module. Hence $(\text{Sheaf}(M), p)$ is a sheaf of \mathbb{Q} -modules.*

Proof. Given two sections s and t of p we can restrict ourselves to neighbourhoods of SG and E such that p is a local homeomorphism. From the construction of the topology on E it follows that on this region, s and t come from elements of some $M(NK)$, where N is an open normal subgroup and K is a closed subgroup of G . The addition of $M(NK)$ defines a section $s + t$. Similarly, the \mathbb{Q} -action on $M(NK)$ defines $q \cdot t$ for $q \in \mathbb{Q}$.

LEMMA 3.12. *The G -sheaf of \mathbb{Q} -modules $(\text{Sheaf}(M), p)$ defined in Construction 3.7 is a Weyl- G -sheaf. That is, $g \in G$ gives maps of \mathbb{Q} -modules*

$$p^{-1}(K) = \text{Sheaf}(M)_K \longrightarrow \text{Sheaf}(M)_{gKg^{-1}}$$

and $\text{Sheaf}(M)_K$ is K -fixed for each $K \in SG$.

Proof. If $K \in SG$, then

$$p^{-1}(K) = \text{Sheaf}(M)_K = \text{colim}_J M(J)_{(K)}$$

with J running over all of the open subgroups containing K . An element $g \in G$ induces maps

$$\text{Sheaf}(M)_K \longrightarrow \text{Sheaf}(M)_{gKg^{-1}}$$

by acting as C_g on the terms $M(J)_{(K)}$. This action is a map of \mathbb{Q} -modules and hence gives a \mathbb{Q} -module map on the stalks.

The \mathbb{Q} -module $M(J)$ is J -fixed and the idempotents defining the stalk $M(J)_{(K)}$ are J -fixed and hence K -fixed. Thus $(\text{Sheaf}(M), p)$ is a Weyl- G -sheaf.

We summarise this work in the following theorem. The additional statement here is that the construction is functorial. This follows from the fact that maps of Mackey functors commute with actions of Burnside rings.

THEOREM 3.13. *For G a profinite group, there is a functor*

$$\text{Sheaf}: \text{Mackey}_{\mathbb{Q}}(G) \longrightarrow \text{Weyl-}G\text{-sheaf}_{\mathbb{Q}}(SG)$$

which sends a Mackey functor M over G to a Weyl- G -sheaf denoted $\text{Sheaf}(M)$, as defined in Construction 3.7.

We end this subsection by noting that the maps θ_H from Construction 3.7 are injective.

LEMMA 3.14. *Let $m \in M(H)$. If $\theta_H(m)(L) = m_L = 0$ for all $L \leq H$, then $m = 0$ in $M(H)$.*

Proof. For $L \leq H$, there is an open normal subgroup N such that the image of m in $e_{N,NL}^{NL}M(NL)$ is zero. As m is H -fixed, the image of m is $N_H(NL)$ -fixed. By Lemma 2.17, restriction and applying idempotents gives an isomorphism

$$e_{N,NL}^H M(H) \cong e_{N,NL}^{NL} M(NL)^{N_H(NL)}.$$

Hence, we see that $e_{N,NL}^H m$ is zero in $e_{N,NL}^H M(H)$. As the sets $\overline{O}_H(N, NL)$ are an open cover of the compact space SH/H , we see that m must be zero in $M(H)$.

4. The equivalence

In this section we prove that the two functors Mackey and Sheaf are inverse equivalences, see Theorem 4.8.

4.1. The equivalence on sheaves

We prove that for a Weyl- G -sheaf of \mathbb{Q} -modules F , we have an isomorphism of equivariant sheaves of \mathbb{Q} -modules

$$\text{Sheaf} \circ \text{Mackey}(F) \cong F.$$

The first part is to show that we have an isomorphism on each stalk. For that, we need to know how the action of the Burnside ring on Mackey functors translates to sheaves.

LEMMA 4.1. *Let F be G -sheaf of \mathbb{Q} -modules on $\mathcal{S}G$ and $K \leq H$ open subgroups of G . The Mackey functor $M = \text{Mackey } F$ satisfies*

$$([H/K](s))(L) = |(H/K)^L|s(L)$$

for all $L \in \mathcal{S}H$ and $s \in M(H) = F(\mathcal{S}H)^H$.

Proof. The element $[H/K] \in \mathbf{A}_{\mathbb{Q}}(H)$ acts on $M(H)$ by the formula

$$[H/K](s) = \sum_{hK \in H/K} C_h \overline{s|_{\mathcal{S}K}}$$

As $s \in M(H)$ is H -fixed, the section $t = C_h \overline{s|_{\mathcal{S}K}}$ satisfies $t(hJh^{-1}) = s(hJh^{-1})$ for $J \in \mathcal{S}K$. It is zero outside of $\mathcal{S}(hKh^{-1})$. Hence, for $L \in \mathcal{S}H$, $t(L)$ is non-zero exactly when $L \in \mathcal{S}(hKh^{-1})$. This is equivalent to the condition $hNA \in (H/K)^L$. We see that

$$([H/K](s))(L) = \sum_{hK \in G/K} \overline{s|_{\mathcal{S}(hKh^{-1})}}(L) = \sum_{hK \in (H/K)^L} s(L).$$

We can now see that a idempotent of $\mathbf{A}_{\mathbb{Q}}(H)$ acts by restricting a section and then extending the result by zero.

PROPOSITION 4.2. *For H an open subgroup of G , let U be an H -invariant open and closed subset of $\mathcal{S}H$. If F is a G -sheaf then $e_U^H \text{Mackey}(F)(H)$ is equal to the set of H -equivariant sections of $\mathcal{S}H$ which are zero outside U . Hence, $e_U^H \text{Mackey}(F)(H) \cong F(U)^H$.*

Proof. The definition of $\text{Mackey}(F)$ gives:

$$\text{Mackey}(F)(H) = F(\mathcal{S}H)^H.$$

Since the sets of the form $O_H(N, NK)$ form a basis for $\mathcal{S}H$ where N is open and normal in H , we can assume that $U = O_H(N, NK)$. By Proposition 2.14 (and using the notation $\alpha_{NA, NK}$ from the proof of that proposition), we have the first equality below for $s \in F(\mathcal{S}H)^H$.

$$\begin{aligned} e_{N, NK}^H s(L) &= \sum_{N \leq NA \leq NK} (\alpha_{NA, NK} [G/NA](s))(L) \\ &= \left(\sum_{N \leq NA \leq NK} \alpha_{NA, NK} |(G/NA)^L| \right) s(L) \\ &= e_{N, NK}^H(L) s(L). \end{aligned}$$

The second is Lemma 4.1. The last is an instance of how $[G/NA] \in \mathbf{A}_{\mathbb{Q}}(H)$ defines a function from $\mathcal{S}H/H$ to \mathbb{Q} , see Theorem 2.10.

PROPOSITION 4.3. *If F is a Weyl- G -Sheaf over $\mathcal{S}G$, then for each $K \in \mathcal{S}G$ we have an isomorphism of \mathbb{Q} -modules*

$$\psi_K : \text{Sheaf} \circ \text{Mackey}(F)_K \longrightarrow F_K.$$

Proof. The left-hand side can be expanded to

$$\begin{aligned} \operatorname{colim}_{\substack{N \leq G \\ \text{open}}} e_{N,NK}^{NK} \operatorname{Mackey}(F)(NK) &= \operatorname{colim}_{\substack{N \leq G \\ \text{open}}} e_{N,NK}^{NK} F(\mathcal{S}(NK))^{NK} \\ &\cong \operatorname{colim}_{\substack{N \leq G \\ \text{open}}} F(O_{NK}(N, NK))^{NK} \end{aligned}$$

using Proposition 4.2 for the last term. The right-hand side is

$$\operatorname{colim}_{\substack{N \leq G \\ \text{open}}} F(O_{NK}(N, NK)).$$

The inclusions $F(O_{NK}(N, NK))^{NK} \rightarrow F(O_{NK}(N, NK))$ induce the desired map ψ_K . It is an isomorphism as any germ can be represented by a section that is locally sub-equivariant by Proposition 2.21

The proof requires F to be a Weyl- G -sheaf as we need the local sub-equivariance property. We now check that the preceding isomorphism is compatible with the group actions.

LEMMA 4.4. *Let F be a Weyl- G -sheaf of \mathbb{Q} -modules. If $K \in \mathcal{S}G$ and $g \in G$, then the following square commutes*

$$\begin{array}{ccc} \operatorname{Sheaf} \circ \operatorname{Mackey}(F)_K & \xrightarrow{\psi_K} & F_K \\ \downarrow C_g & & \downarrow C_g \\ \operatorname{Sheaf} \circ \operatorname{Mackey}(F)_{gKg^{-1}} & \xrightarrow{\psi_{gKg^{-1}}} & F_{gKg^{-1}} \end{array}$$

where ψ_K is the map defined in the proof of Proposition 4.3.

Proof. A germ $s_K \in \operatorname{Sheaf} \circ \operatorname{Mackey}(F)_K$ can be represented by an NK -equivariant section

$$s: O_{NK}(N, NK) \rightarrow F.$$

This section is also a representative for $\psi_K(s_K)$, from which the commutativity of the square follows.

THEOREM 4.5. *If F is any Weyl- G -sheaf of \mathbb{Q} -modules, then the maps ψ_K induce an isomorphism*

$$\psi: \operatorname{Sheaf} \circ \operatorname{Mackey}(F) \longrightarrow F$$

in the category of Weyl- G -Sheaves.

Proof. By Proposition 4.3 and Lemma 4.4, we see that they are isomorphic as G -sets. To prove that they are topologically equivalent, we need to show that they have the same sections. As both objects are sheaves, it suffices to do so locally, which amounts to considering sections that represent stalks. By Proposition 4.3, a germ over $K \in \mathcal{S}G$ of $\operatorname{Sheaf} \circ \operatorname{Mackey}(F)$ is represented by some element of $F(O_{NK}(N, NK))^{NK}$. By the local sub-equivariance property of Proposition 2.21 the same is true for a stalk of F .

4.2. The equivalence on Mackey functors

The starting point is to give a levelwise isomorphism of \mathbb{Q} -modules.

PROPOSITION 4.6. *If H is an open subgroup of a profinite group G and M a Mackey functor for G , then there is an isomorphism of \mathbb{Q} -modules (constructed in the proof)*

$$\theta_H : M(H) \longrightarrow \text{Mackey} \circ \text{Sheaf}(M)(H).$$

Proof. Recall from Constructions 3.1 and 3.7 that the stalks of $\text{Sheaf}(M)$ are given by

$$\text{Sheaf}(M)_K = \underset{\substack{J \supseteq K \\ \text{closed}}}{\text{colim}} M(J)_{(K)} = \underset{\substack{N \leq G \\ \text{open}}}{\text{colim}} M(NK)_{(K)} = \underset{\substack{N \leq G \\ \text{open}}}{\text{colim}} e_{N, NK}^{NK} M(NK).$$

Recall the map θ_H from Construction 3.7

$$\theta_H : M(H) \longrightarrow \left\{ s : SH \longrightarrow \coprod_{K \in SH} \text{Sheaf}(M)_K \right\}^H,$$

where $\theta_H(m)(K) = m_K$, the image of m in $\text{Sheaf}(M)_K$. The topology on $\text{Sheaf}(M)$ is defined in terms of $\theta_H(m)$ for varying H and m .

The composite of the two functors at H is

$$\begin{aligned} \text{Mackey} \circ \text{Sheaf}(M)(H) &= \text{Sheaf}(M)(SH)^H \\ &= \left\{ s : SH \longrightarrow \coprod_{K \in SH} \text{Sheaf}(M)_K \mid s \text{ continuous, } p \circ s = \text{Id} \right\}^H. \end{aligned}$$

Since the topology was defined using the images of θ_H , it follows that $\theta_H(m)$ is a continuous section. Thus θ_H defines a map as in the statement. Moreover, it is additive and surjective, injectivity follows from Lemma 3.14.

THEOREM 4.7. *The maps θ_H induce a natural isomorphism of Mackey functors*

$$\theta : M \longrightarrow \text{Mackey} \circ \text{Sheaf}(M).$$

Proof. We need to show that the correspondence between M and $\text{Mackey} \circ \text{Sheaf}(M)$ commutes with the three maps; restriction, induction and conjugation.

For conjugation and restriction one can calculate the effect directly on a stalk of some $m \in M(H)$, for H an open subgroup of G . One will see that the map θ is compatible with conjugation as the restriction maps of a Mackey functor are equivariant and the action of the Burnside ring is compatible with conjugation. Similarly, since θ is defined via the restriction maps of Mackey functors and the restriction of $\text{Mackey} \circ \text{Sheaf}(M)$ is defined by restricting a section to a smaller domain, θ is compatible with restrictions.

Induction requires more work. Suppose $K \leq H$ are open and $s \in M(K)$. We must show that the square

$$\begin{array}{ccc} M(H) & \xrightarrow{\theta_H} & \text{Mackey} \circ \text{Sheaf}(M)(H) \\ I_K^H \uparrow & & \uparrow \overline{I}_K^H \\ M(K) & \xrightarrow{\theta_K} & \text{Mackey} \circ \text{Sheaf}(M)(K) \end{array}$$

commutes, where \overline{I}_K^H is the induction map for $\text{Mackey} \circ \text{Sheaf}(M)$.

Chasing through the definitions gives

$$\left(\overline{I_K^H} \theta_K(m) \right)_L = \left(\sum_{h \in H/K} h * \overline{\theta_K(m)} \right)_L = \sum_{\substack{h \in H/K \\ h^{-1}Lh \leq K}} (h * \theta_K(m))_L.$$

In the final equality we replace the extension by zero by an explicit condition on the subgroups.

Conversely, we use the Mackey axiom to rewrite $(\theta_H I_K^H(m))_L \in \text{Sheaf}(M)_L$. We may choose a representative $a \in e_{N,NL}^{NL} M(NL)$ for this germ, with $N \leq K$ an open normal subgroup of G . Then

$$\begin{aligned} a &= e_{N,NL}^{NL} R_{NL}^H I_K^H(m) \\ &= e_{N,NL}^{NL} \sum_{[NL \setminus H/K]} I_{NL \cap xKx^{-1}}^{NL} \circ C_x \circ R_{K \cap x^{-1}NLx}^K(m) \\ &= e_{N,NL}^{NL} \sum_{[NL \setminus H/K]} I_{NL \cap xKx^{-1}}^{NL} \circ R_{NL \cap xKx^{-1}}^{xKx^{-1}} \circ C_x(m). \\ &= \sum_{[NL \setminus H/K]} I_{NL \cap xKx^{-1}}^{NL} \left(R_{NL \cap xKx^{-1}}^{NL} (e_{N,NL}^{NL}) \cdot R_{NL \cap xKx^{-1}}^{xKx^{-1}} C_x(m) \right). \end{aligned}$$

Consider the term $R_{NL \cap xKx^{-1}}^{NL} e_{N,NL}^{NL}$. If this is non-zero, there is a subgroup J of $NL \cap xKx^{-1}$ in the support of the idempotent. Hence,

$$L \leq NL = NJ \leq xKx^{-1}.$$

If $x \in [NL \setminus H/K] = [L \setminus H/K]$ is such that $L \leq xKx^{-1}$, then $NL \cap xKx^{-1} = NL$. The corresponding summand is

$$e_{N,NL}^{NL} \cdot R_{NL}^{xKx^{-1}} C_x(m)$$

which is sent to $C_x(m)_L$ in the L -stalk.

The collection of $x \in [L \setminus H/K]$ such that $L \leq xKx^{-1}$, is exactly the set $hK \in H/K$ such that $h^{-1}Lh \leq K$ (the canonical isomorphism sends LxK to xK). Hence,

$$(\theta_H I_K^H(m))_L = \sum_{\substack{h \in H/K \\ h^{-1}Lh \leq K}} (\theta_{hKh^{-1}} C_h(m))_L = \sum_{\substack{h \in H/K \\ h^{-1}Lh \leq K}} (h * \theta_K(m))_L.$$

We combine Theorems 4.5 and 4.7 and note that our constructions are compatible with maps in the two categories to obtain the main result.

THEOREM 4.8. *If G is a profinite group then the category of rational G -Mackey functors is equivalent to the category of Weyl- G -sheaves over SG . Furthermore, this is an exact equivalence.*

5. Consequences

5.1. Examples

Just as non-equivariantly, one can make constant sheaves and skyscraper sheaves. We leave the details to the second author’s thesis [Sug19] and the forthcoming paper by the authors [BS20]. The stalks of the constant sheaf at a \mathbb{Q} -module B are all isomorphic to B . The sections are as for the non-equivariant constant sheaf.

Example 5.1. The Burnside ring Mackey functor corresponds to the constant Weyl- G -Sheaf at \mathbb{Q} . The constant sheaf at a \mathbb{Q} -module B corresponds to the Mackey functor $\mathbf{A}_{\mathbb{Q}} \otimes B$.

For an equivariant skyscraper sheaf, one picks a closed subgroup K of G and a discrete $W_G K$ -module C . The stalk at K is C , for $g \in G$, the stalk at gKg^{-1} is gC (with the expected gKg^{-1} -action), all other stalks are zero. The sections of this sheaf over U are given by a direct sum of copies of gB , with one copy for each $gKg^{-1} \in U$, where the sums runs over G/K .

Example 5.2. For K a closed subgroup of G , we can describe the Mackey functor M that corresponds to the skyscraper sheaf $\text{Sky}_K(C)$ built from a discrete $W_G K$ -module C . At an open subgroup H , $M(H)$ is the H -fixed points of $\text{Sky}_K(C)(SH)$. When K is open, there are only finitely many distinct conjugates of K , hence we can express $M(H)$ as a finite direct sum indexed by the distinct conjugates gKg^{-1} of K that are subgroups of H

$$M(H) = \text{Sky}_K(C)(SH)^H = \left(\bigoplus_{\{gKg^{-1} \leq H\}} gC \right)^H.$$

If K is the trivial group, then the Mackey functor M_C corresponding to $\text{Sky}_{\{e\}}(C)$ is the fixed point Mackey functor, see Example 2.5, of the discrete G -module C .

It can be illuminating to perform the reverse calculation. The fixed point Mackey functor M_C is cohomological, that is, restriction followed by induction is multiplication by the subgroup index

$$I_K^H R_K^H = [H, K].$$

See Thévenaz and Webb [TW95, section 16] for further details of cohomological Mackey functors.

This equation simplifies the behaviour of idempotents, particularly when we calculate the stalk of $\text{Sheaf}(M_C)$ at some closed subgroup K . As a self-map of $M_C(NK)$,

$$\begin{aligned} e_{N, NK}^{NK} &= \sum_{N \leq NA \leq NK} \frac{|NA|}{|N_{NK}(NK)|} \mu(NA, NK)[NK : NA] \\ &= \sum_{N \leq NA \leq NK} \frac{|NA|}{|NK|} \mu(NA, NK)[NK : NA] \\ &= \sum_{N \leq NA \leq NK} \mu(NA, NK) \end{aligned}$$

which is zero for non-trivial K and the identity map for K the trivial group. Hence, the stalks at non-trivial subgroups are zero. We then use the fact that C is discrete to complete the

calculation

$$C = \operatorname{colim}_{\substack{N \leq G \\ \text{open}}} C^N = \operatorname{colim}_{\substack{N \leq G \\ \text{open}}} M_C(N).$$

5.2. Weyl Sheaves from equivariant sheaves

As mentioned, Construction 3.1 does not require the input G -sheaf to be a Weyl- G -sheaf. Hence, given a sheaf F , $\text{Sheaf} \circ \text{Mackey}(F)$ is a Weyl- G -sheaf. We can construct this operation explicitly.

PROPOSITION 5.3. *If F is a G -sheaf then the underlying set of $\text{Sheaf} \circ \text{Mackey}(F)$ is*

$$\coprod_{K \in SG} F_K^K.$$

Furthermore, the map $F \rightarrow \text{Sheaf} \circ \text{Mackey}(F)$ is induced from the stalkwise inclusions of fixed points.

Proof. The stalks are given by

$$\operatorname{colim}_{N \leq G} e_{O(N, NK)}^{NK} F(SNK)^{NK} \cong F_K^K.$$

One can prove directly that the stalks F_K^K form a Weyl- G -sheaf, as Proposition 2.21 implies that a K -fixed germ can be represented by a section that is NK -equivariant for some open normal subgroup N . One can also prove that a map from a Weyl- G -sheaf E to F will factor through the composite $\text{Sheaf} \circ \text{Mackey}(F)$. See [BS20, section 10] for further such results.

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