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A NOTE ON SUBDIRECTLY IRREDUCIBLE DISTRIBUTIVE DOUBLE p-ALGEBRAS

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Abstract

A regular double p-algebra L satisfying (i) $\wedge(x^{n(+*)}; n<\omega)$ for every $1 \neq x \in L$ and (ii) L is not subdirectly irreducible, is constructed. The construction is purely topological and the desired result is obtained via the known Priestley duality. The notion of an auxiliary regular double p-algebra is introduced and the algebras having this property are characterized.

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1. Introduction

By H. Priestley [9] every distributive (double) p-algebra can be associated with a special totally order disconnected compact topological space, and vice versa. By means of this topological duality theory of distributive double p-algebras it is possible to dualize every fact and every concept, converting algebraic facts and concepts into topological ones. So, in [4], B. Davey presents a topological characterization of subdirectly irreducible double p-algebras. In [7], the same algebras are characterized by algebraic means as follows:

THEOREM A ([7, Theorem 4]). *Let L be a distributive double p-algebra and let $|L| \geqslant 3$. Then L is subdirectly irreducible if and only if*

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(i) *L is nearly regular; that is to say, for every $a \in L$,*

$$|G[a]| = |\{x \in L\colon x^* = a^* \text{ and } x^+ = a^+\}| \leqslant 2;$$

(ii) $C(L) = \{0, 1\}$;

(iii) *If L is regular then there exists $1 \neq d \in D(L)$ such that*

$$x^{n(+*)} \leqslant d$$

for all $1 \neq x \in D(L)$ and some $n < \omega$;

(iv) *If L is not regular then for all $1 \neq x \in D(L)$ with $|G[x]| = 1$ there exists $d \in D(L)$ satisfying $|G[d]| \neq 1$ such that*

$$x^{n(+*)} \leqslant d$$

for some $n < \omega$.

The conditions (i)–(iv) *are independent.*

Furthermore, in [7], the following is also shown.

THEOREM B ([7, Lemma 7]). *Let L be a distributive double p-algebra and let $|L| \geqslant 3$. Then L is subdirectly irreducible if and only if L satisfies the conditions* (i), (iii), (iv) *from Theorem* A *and the condition*

(ii′) $\bigwedge(x^{n(+*)}; n < \omega) = 0$ *for all $1 \neq x \in L$.*

The aim of the present note is twofold. First, it is shown that the conditions of Theorem B are independent. The crucial example of a regular double p-algebra satisfying (i), (ii′) and (iv) but not (iii) is constructed by topological means (see Lemmas 1–5). This example also demonstrates the importance of the Priestley's duality. Furthermore, the connections between (ii) of Theorem A and (ii′) of Theorem B are investigated. Independently, R. Beazer (see [2], [3]) has considered the interrelationship of conditions similar to those given here. In particular, see [3], he characterized the finitely subdirectly irreducible distributive double p-algebras. He then raises the question of whether every finitely subdirectly irreducible regular double p-algebra is subdirectly irreducible; we show that this is not the case. Second, the notion of an auxiliary regular double p-algebra is introduced. The auxiliary algebras are characterized and, as will be seen, provide another instance of the dual rôle occupied by the conditions (ii) and (ii′) of Theorems A and B.

2. Preliminaries

A *distributive double p-algebra* $\langle L; \wedge, \vee, ^*, ^+, 0, 1\rangle$ is a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ with two unary operations. For $a \in L$, $a \wedge x = 0$ if and

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only if $x \leqslant a^*$ (that is, a^* is the *pseudocomplement* of a) and, dually, $a \vee x = 1$ if and only if $x \geqslant a^+$ (that is, a^+ is the *dual pseudocomplement* of a). Further, $D(L) = \{x \in L\colon x^* = 0\}$ and $C(L)$ is the set of all elements of L that have a complement. In a pseudocomplemented distributive lattice, the relation $x \equiv y(\gamma)$ if and only if $x^* = y^*$ is a congruence called the *Glivenko congruence*. Similarly, in a dually pseudocomplemented distributive lattice the dual relation $x \equiv y(\bar{\gamma})$ if and only if $x^+ = y^+$ is a congruence. For any distributive double p-algebra $x \equiv y(\gamma \wedge \bar{\gamma})$ if and only if $x^* = y^*$ and $x^+ = y^*$ is a congruence called the *determination congruence*. Moreover, for $a \in L$, $a^{0(+*)} = a$, $a^{+*} = (a^+)^*$, and for $n \geqslant 0$, $a^{(n+1)(+*)} = a^{n(+*)+*}$.

An algebra L is *finitely subdirectly irreducible* if and only if the equality relation is meet irreducible in the congruence lattice $\mathrm{Con}(L)$ of L. Observe that, since an algebra is subdirectly irreducible if and only if the equality relation is completely meet irreducible in $\mathrm{Con}(L)$, every subdirectly irreducible algebra is finitely subdirectly irreducible.

For further information on distributive double p-algebras, see R. Beazer [1], B. A. Davey [4] or [7].

Since one of the constructions presented here is better described by an ordered topological space, we briefly describe the topological duality for distributive double p-algebras as given in H. A. Priestley [9]. A topological space T with a partial ordering is *totally order disconnected* if and only if, for $x, y \in T$, whenever $x \not\geqslant y$ there exists a clopen order-ideal $X \subseteq T$ such that $x \in X$ and $y \notin X$. H. A. Priestley established a duality between the category of bounded distributive lattices and the category of compact totally order disconnected topological spaces. Under this duality, the elements of a bounded distributive lattice correspond to the clopen order-ideals of a compact totally order disconnected space. Such a space is the dual space of a distributive double p-algebra if and only if $[X)$ is clopen for every clopen order-ideal X and $(X]$ is clopen for every clopen order-filter. For further information, see H. A. Priestley [9] or B. A. Davey [4].

3. The conditions (ii) and (ii′)

We begin with the construction of a compact totally order disconnected space that will be needed in the proof of Theorem 1.

Let C denote the Cantor discontinuum set; that is to say, the set of all real numbers $0 \leqslant t \leqslant 1$ which can be written in the form

$$(1) \qquad t = t_1/3 + t_2/3^2 + \cdots + t_n/3^n + \cdots \quad \text{where } t_i \in \{0, 2\}.$$

It is known (see, for example, J. L. Kelley [8]) that C is a compact and totally disconnected metric space. More precisely, the sets

$$(2)\qquad \begin{aligned} P_1 &= [0, 1/3] \cap C, & P_2 &= [2/3, 1] \cap C,\\ P_3 &= [0, 1/9] \cap C, & P_4 &= [2/9, 1/3] \cap C, \ldots \end{aligned}$$

are clopen and they form a base of the topology on C. It is easy to see that $P_m = [a, b] \cap C$ if and only if (i) $a_n, b_n \in \{0, 2\}$ for every $n < \omega$, (ii) there exists an integer $k \geqslant 1$ such that $a_n = b_n$ for all $n \leqslant k$, and (iii) $a_n = 0$ and $b_n = 2$ for every $n > k$. If there is no danger of confusion we shall write plainly $[a, b]$ for $[a, b] \cap C$, whenever $a, b \in C$.

For every $n < \omega$ there is a mapping $\tau_n: C \to C$ defined in the following way:

(i) for $x_n = 2$, $\tau_n(x) = x$,

(ii) for $x_n = 0$, $(\tau_n(x))_i = x_i$ for every $i < n$, $(\tau_n(x))_n = 2$,

and $(\tau_n(x))_i = x_{i-1}$ for every $i > n$. It can be readily proved that τ_n is a continuous mapping for every $n < \omega$.

For two elements $x, y \in C$ we have $x = y$ if and only if $x_n = y_n$ for every $n < \omega$. Moreover,

$$(3)\qquad x < y \quad \text{if and only if } x_i < y_i \text{ for the first } i < \omega \text{ with } x_i \neq y_i.$$

For more details on the set C, see J. L. Kelley [8].

Denote by $\bar{C}$ a homeomorphic image of C satisfying $C \cap \bar{C} = \varnothing$. We shall construct a new partially ordered topological space $S = C \cup \bar{C}$. A set $X \subseteq S$ is said to be open in S if $X = X_1 \cup X_2$ such that X_1 and X_2 are open in C and $\bar{C}$, respectively. Evidently, S is again compact and totally disconnected. We shall define a partial order on S as follows: Let

$$\bar{t} = \bar{t}_1/3 + \bar{t}_2/3^2 + \cdots + \bar{t}_n/3^n + \cdots \quad \text{with } \bar{t}_i = t_i \quad \text{for all } i < \omega$$

denote the corresponding element from $\bar{C}$ associated with $t \in C$ (see (1)). The mapping $t \to \bar{t}$ is a homeomorphism between C and $\bar{C}$. The partial order $\subseteq *$ on S can be introduced in the following way:

(i) $\bar{t} \subseteq * \, t$ for every $t \in C$,

(ii) having $p, t \in C$ we say that $\bar{p} \subseteq * \, t$ if $p = \tau_r(t)$ for some r,

(iii) for $t, z \in S$ and $t \neq z$, $t \subseteq * \, z$ if and only if it can occur by (i) or (ii).

LEMMA 1. *Let U be an open (clopen) set of the partially ordered topological space $S = (S; \subseteq *)$. Then $(U]$ and $[U)$ are open (clopen).*

PROOF. First we shall assume that U is a base set of C, that is $U = [a, b]$ with $a_n = b_n$ for $n \leqslant k$ and $a_n = 0$, $b_n = 2$ for $n > k$. It is easy to check that

$$(U] = U \cup \bar{U} \cup \bar{U}_{i_1} \cup \cdots \cup \bar{U}_{i_r},$$

where $1 \leqslant i_1 < \cdots < i_r \leqslant k$ is the set of all $1 \leqslant j \leqslant k$ with $a_j = 0$ and $U_{i_j} = [\tau_{i_j}(a), \tau_{i_j}(b)]$ for every $j = 1, \ldots, r$. (Note that $\bar{U}_j = [\bar{c}, \bar{d}]$ is the clopen base set corresponding to $U_j = [c, d]$.) Therefore $(U]$ is clopen in S. Since $P_1, \ldots, P_m, \ldots; \bar{P}_1, \ldots, \bar{P}_m, \ldots$ form a base of S (see (2)), $(U]$ is an open set whenever U is open. For U clopen, there is a finite family of $P_{i_1}, \ldots, P_{i_r}, \bar{P}_{j_1}, \ldots, \bar{P}_{j_s}$ such that

$$U = P_{i_1} \cup \cdots \cup P_{i_r} \cup \bar{P}_{j_1} \cup \cdots \cup \bar{P}_{j_s},$$

because U is compact. Hence $(U]$ is also clopen.

For the second part of the proof take first $U = \bar{P}$ for a base set $P = [a, b]$ with a and b as above (see also (2)). Evidently, for every $c \in C$ with $c_r = 2$ and $c_{r+1} = 0$ there is a unique $c \neq d \in C$ with $\tau_r(d) = c$, that is, $d = \tau_r^{-1}(c)$. Now, two cases can arise:

(i) $a_k = b_k = 0$ or

(ii) $a_k = b_k = 2$.

In the first event we obtain

$$[U) = P \cup \bar{P} \cup P_{j_1} \cup \cdots \cup P_{j_s}$$

where $1 \leqslant j_1 < \cdots < j_s < k$ is the set of all $1 \leqslant i < k$ with $a_i = 2$, $a_{i+1} = 0$ and $P_{j_i} = [\tau_{j_i}^{-1}(a), \tau_{j_i}^{-1}(b)]$ for every $i = 1, \ldots, s$. Similarly in the second case

$$[U) = P \cup \bar{P} \cup P_{j_1} \cup \cdots \cup P_{j_s} \cup P_k,$$

where P_{j_i} for $i = 1, \ldots, s$ are the same as in the first case and $P_k = [\tau_k^{-1}(a), \tau_k^{-1}(b')]$ defining $b'_{k+1} = 0$ and $b'_n = b_n$ for $n \neq k+1$. Therefore $[U)$ is clopen. The rest of the proof can be left to the reader.

Lemma 2. *The partially ordered topological space $S = (S; \subseteq *)$ is compact and totally order-disconnected.*

Proof. We have only to prove that for all $x, y \in S$ with $x \nsubseteq * \, y$ there exists a clopen order-ideal U of S with $y \in U$ and $x \neq U$. Two cases can arise:

(i) $x, y \in C$ or $x, y \in \bar{C}$ or $x \in C$ and $y \in \bar{C}$,

(ii) $x \in \bar{C}$ and $y \in C$.

In the first event we can set U as follows: $U = V \cup \bar{C}$ for $x, y \in C$, where V is a clopen subset of C satisfying $y \in V$ and $x \notin V$; $U = V$, where V is a clopen subset of $\bar{C}$ satisfying $y \in V$ and $x \notin V$; $U = \bar{C}$.

In the second case we can assume $\bar{x} \in \bar{C}$, $y \in C$ and $\bar{x} \nsubseteq * \, y$. Again two cases can occur: $x < y$ or $y < x$.

For $x < y$ there exists a clopen set P from the base of C (see (2)) such that $y \in P$ and $[0, x] \cap P = \varnothing$. By Lemma 1 $(P]$ is clopen in S. Hence, $x < y$ implies $\bar{x} \notin (P]$ and $(P] = U$ is the required set.

Suppose now $y < x$. Then there exists k such that $y_n = x_n$ for every $n < k$ and $y_k < x_k$, that is $y_k = 0$ and $x_k = 2$. Since $\bar{x} \not\subseteq^* y$, we have $t = \tau_k(y) \neq x$. There exists again r such that $t_r \neq x_r$ and $t_n = x_n$ for every $n < r$. Evidently $r > k$. Now, take a clopen set $P = [a, b]$ from (2) defined as follows:

$$a_n = b_n = y_n \quad \text{for every } n < r,$$
$$a_n = 0 \quad \text{and} \quad b_n = 2 \quad \text{for every } n \geqslant r.$$

Clearly, $y \in P$ and $x \notin P$. By Lemma 1, $(P]$ is clopen in S. We have only to show that $\bar{x} \notin (P]$. This is equivalent with: $\tau_n(p) = x$ for no $p \in P$ and no $n < \omega$. Suppose to the contrary that $\tau_m(p) = x$ for some integer m and some $p \in P$. Clearly, $p_j = y_j$ for every $j \leqslant r - 1$ and $p_m = 0$. Since $p_k \neq x_k$, we have $m \leqslant k$. But $m < k$ is impossible, as $p_m = x_m$. The last case $m = k$ is also impossible, because

$$(\tau_k(p))_r = p_{r-1} = y_{r-1} = (\tau_k(y))_r = t_r \neq x_r,$$

by hypothesis. Thus $\bar{x} \notin (P]$ and the proof is complete.

A distributive double p-algebra is regular if and only if every totally ordered set in its dual space has length $\leqslant 1$. Thus, it follows from Lemmas 1 and 2 that the partially ordered topological space $S = (S; \subseteq^*)$ is a dual space to a regular double p-algebra.

Denote by C_2 the set of all $x \in C$ satisfying the following property: $x_n = 2$ for all but finitely many indices n. Analogously, the definition of $\overline{C}_2$.

LEMMA 3. *The set C_2 is dense in C.*

PROOF. Let $x \in C$ and an integer n be given. Take $y = C_2$ as follows:

$$y_i = x_i \quad \text{for every } i \leqslant n, \qquad y_i = 2 \quad \text{for } i > n.$$

Therefore $y - x \leqslant 1/3^n$ and every neighbourhood of x contains an element from C_2.

LEMMA 4. *Let $U \neq S$ be a clopen order-ideal of the ordered topological space $S = (S; \subseteq^*)$. Then $\bigwedge(U^{n(+*)}; n \leqslant \omega) = \varnothing$.*

PROOF. It is known that $W^+ = (S \backslash W]$ and $W^* = S \backslash [W)$ for every clopen order-ideal of S. Denote by $K = \bigcap(U^{n(+*)}; n \leqslant \omega)$ the set-theoretical intersection of $\{U^{n(+*)}; n < \omega\}$. First we prove that $[K) \subseteq K$. Take $d \subseteq^* c$ with $d \in K \cap \overline{C}$, $c \in C$. If $c \in S \backslash U^{n(+*)}$ for some n, then $d \in (S \backslash U^{n(+*)}]$ and consequently $d \notin U^{(n+1)(+*)}$, which is impossible. Thus $c \in K$. $(K] \subseteq K$ is trivial.

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Suppose to the contrary that there exists a clopen order-ideal $V \neq \varnothing$ with $V \subseteq K$. Hence, there exists $P = [a, b] \subseteq V$ from the base (2). We can assume $a_n = b_n$ for every $n \leqslant k$ and $a_n = 0$, $b_n = 2$ for every $n > k$. We claim that there exists $e \in C$ such that $[0, e] \subseteq V$. Without loss of generality we can assume $a_k = b_k = 0$. Namely, for $a_k = b_k = 2$ take a new element $a < b' < b$ defined as follows:

$b'_n = b_n$ for every $n \neq k + 1$ and $b'_{k+1} = 0$. Evidently $[a, b'] \subseteq P$, $a_n = b'_n$ for $n < k + 1$, $a_{k+1} = b'_{k+1} = 0$ and $a_n = 0$, $b_n = 2$ for $n > k + 1$. Let $a \neq 0$. Define the elements $c, d \in C$ as follows:

Let m be the largest integer with $a_m = b_m = 2$. Clearly, $m < k$. Put

$$c_n = d_n = a_n \quad \text{for } n < m \quad \text{and}$$
$$c_n = a_{n+1}, \qquad d_n = b_{n+1} \quad \text{for } n \geqslant m.$$

Since $x \to \tau_m(x)$ is a homeomorphism between $Q = [c, d]$ and P, we see that $Q \subseteq K$. As Q is clopen, we have $(Q]$ clopen (Lemma 1) and consequently $(Q] \subseteq V$. Repeating this procedure we obtain finally $[0, e] \subseteq V$.

Let us start with $[0, e] \subseteq V$, where $k \geqslant 2$ is the smallest integer with $e_n = 2$ for all $n \geqslant k$. We claim that $[0, e'] \subseteq V$ for $e'_{k-1} = 2$ and $e'_n = e_n$ for all $n \neq k - 1$. Really, $[0, \tau_{k-1}(e)] \subseteq V$, because $x \to \tau_{k-1}(x)$ is a homeomorphism between $[0, e]$ and $[\tau_{k-1}(0), \tau_{k-1}(e)]$, $[K), (K] \subseteq K$ and $[0, e] \cup [\tau_{k-1}(0), \tau_{k-1}(e)] = [0, \tau_{k-1}(e)]$. Continuing in this way we get a sequence $e^{(n)}$ for $n \geqslant 1$ with

$$e^{(1)} = e, e^{(2)} = \tau_{k-1}(e^{(1)}), \ldots, e^{(n+1)} = \tau_{k+n-2}(e^{(n)}), \ldots.$$

It is easy to see that $e^{(n)} \in K$ and $e^{(n)} \to e'$ in C. Since K is closed, we have $e' \in K$. Therefore, $[0, e'] \subseteq K$ and consequently $[0, e'] \subseteq V$ as claimed. Repeating this procedure we obtain $[0, 1] = C \subseteq V$, and this implies $S \subseteq V$, a contradiction with $S \neq U \supseteq V$.

LEMMA 5. *If $U \neq S$ is a clopen order-ideal of the ordered topological space $S = (S; \subseteq {}^*)$ then there exists a clopen order-ideal D of S such that*

$$D^{n(+*)} \not\subseteq U$$

for all $n < \omega$.

PROOF. By hypothesis $I = C \cap (S \backslash U) \neq \varnothing$ and clopen. There exists $b \in I \cap C_2$ (Lemma 3). We need two more elements $a^{(1)}, a^{(2)} \in C$:

$a_i^{(1)} = b_i$ for $i < k$ and $a_i^{(1)} = 0$ for $i \geqslant k$, where k is the smallest integer with $b_i = 2$ for all $i \geqslant k$;

$a^{(2)} = \tau_k(a^{(1)})$. Two cases can arise: (i) there exists a largest r with $b_r = 2$ and $b_{r+1} = 0$ (in fact $r < k$) or (ii) there exists no such r. In the first event consider the element $c \in C_2$ defined as follows: $c_i = b_i$ for $i < r$, $c_r = 0$ and $c_i = 2$ for

$i > r$. Evidently, $c < a^{(1)} < a^{(2)} < b$. Take the clopen set $R = [0, c] \cup [a^{(2)}, 1]$ in C. Therefore $S \neq D = (R]$ is a clopen order-ideal of S (Lemma 1). In the second case we take $S \neq D = ([a^{(2)}, 1]]$.

First we make some observations. For an element $e \in C_2$ let $l_0(e)$ (that is to say, the *zero-length* of e) denote the number of all indices n with the property $e_n = 0$. We claim that

$$x \in C_2, \qquad l_0(x) = l_0(b) \quad \text{implies } x \in D.$$

Really, let us consider the first event, that is $b_r = 2$ and $b_{r+1} = 0$, $x \in C_2$, $x \neq b$ and $l_0(x) = l_0(b)$. Let j be the smallest integer with $x_j \neq b_j$. If $j < r$ then $x \in D$. If $j = r$ then $x \leqslant c$ and, hence, $x \in D$. Assume $j > r$ and $x_j = 0$. Hence, $b_j = 2$. However, $j < k$ is impossible, because $b_{k-1} = 0$ and this yields a contradiction with $r < j$. Therefore $j \geqslant k$. This case cannot occur since $l_0(x) = l_0(b)$. Concluding we have $j > r$ and $x_j = 2$. Hence $x \geqslant b$ and, consequently, $x \in D$, as claimed. The second case is trivial.

It is easy to check that $x \in C_2$ implies $l_0(x) = l_0(\tau_n(x))$. Similarly, $\tau_n(y) = x$ implies $l_0(y) = l_0(x)$, that is $l_0(x) = l_0(\tau_n^{-1}(x))$. Summarizing, we can say that D contains all elements $x \in C_2 \cup \overline{C}_2$ with the property: $l_0(x) = l_0(b)$.

We claim that $x \in C_2 \cup \overline{C}_2$ with $l_0(x) = l_0(b)$ implies $x \in D^{n(+*)}$ for every $n < \omega$. For $n = 0$ (recall $D^{0(+*)} = D$) it is true. Suppose that $D^{n(+*)}$ possesses this property. Therefore, $D^{n(+*)+} = (S \setminus D^{n(+*)}]$ does not contain elements of zero-length equal to $l_0(b)$. The same is also true for $[D^{n(+*)+})$. Since

$$D^{(n+1)(+*)} = S \setminus [D^{n(+*)+}),$$

we see that $D^{(n+1)(+*)}$ contains all elements of zero-length $l_0(b)$, as claimed. Concluding, we have $b \in D^{n(+*)}$ for all $n < \omega$. Since $b \notin U$, we see that $D^{n(+*)} \nsubseteq U$ for all $n < \omega$ and the proof is complete.

In answer to Beazer's question:

LEMMA 6. *There exists a regular double p-algebra that is finitely subdirectly irreducible but not subdirectly irreducible.*

PROOF. Consider the regular double p-algebra dual to the topological ordered space S of Lemmas 1–5. By Lemma 4, it is finitely subdirectly irreducible (see R. Beazer [3, Theorem 11]). However, by Lemma 5, it is not subdirectly irreducible (see Theorem B).

THEOREM 1. *The conditions* (i), (ii′), (iii) *and* (iv) *from Theorem* B *are independent.*

PROOF. The five-element chain satisfies (ii′), (iii), (iv) but not (i).

In order to show the independence of (ii′) let us consider a subdirectly irreducible regular double p-algebra K with $|K| \geqslant 3$. Take the regular double p-algebra $L = \mathbf{2} \times K$, where $\mathbf{2}$ is the two-element chain. Evidently L is regular double p-algebra. Take $a = (1, 0) \in L$. It is easy to check that $a^{n(+*)} = a \neq 0$ for every n. Since $D(L) = \{(1, x) \in L : x \in D(K)\}$, we have

$$(1, x)^{n(+*)} \leqslant (1, d)$$

for some n and every $1 \neq x \in D(K)$, where $d \in D(K)$ is the element from K satisfying (iii). Thus L satisfies (i), (ii), and (iv) but not (ii′).

The regular double p-algebra dual to the topological ordered space S considered in Lemmas 1–5 satisfies (i), (ii′) and (iv) but not (iii).

Finally, we demonstrate that condition (iv) is independent of (i), (ii′) and (iii). We need the algebraic construction of regular double p-algebras from [6]. Let B denote the Boolean algebra of all finite and cofinite subsets of N (that is $\{n : n < \omega\}$). Set

$$\varphi(a) = \{x \in N : x \in a \text{ and } x + 1 \in a\},$$
$$\psi(a) = \{x \in N : x \in a \text{ or } x - 1 \in a\},$$

for every $a \in B$. It is easy to verify that $\varphi: B \to B$ is a $\{0, 1, \wedge\}$-homomorphism, $\psi: B \to B$ is a $\{0, 1, \vee\}$-homomorphism and both satisfy

$$\psi(\varphi(a)) \leqslant a, \qquad \varphi(\psi(a)) \geqslant a$$

for every $a \in B$. Then $K = \{(x, y) \in B^2 : \varphi(x) \geqslant y\}$ is a $\{0, 1\}$-sublattice of B^2. Moreover, K is a regular double p-algebra (see [6, Theorem 2]) in which for $t = (x, y) \in K$

$$t^* = (x', \varphi(x')) \quad \text{and} \quad t^+ = (\psi(y'), y')$$

is true. First we shall show that K satisfies (ii′). Let us consider $t = (x, y) \in K$. Therefore,

$$t^{+*} = ((\psi(y'))', \varphi((\psi(y'))')).$$

It is not difficult to see that $\varphi((\psi(y'))') \subseteq (\psi(y'))' \subset y$ for every $\varnothing \neq y \neq N$. Therefore, $t^{+*} < t$ for every $0 \neq t \neq 1$. Now it is easy to see that $\bigwedge(t^{n(+*)}; n < \omega) = 0$ for y finite. Assume y cofinite. Let r be the smallest integer such that $n \geqslant r$ implies $n \in y$. Hence $r - 1, r \in \psi(y')$ and consequently $r - 1, r \notin (\psi(y'))'$. Two cases can arise: there exists a largest integer k with $k \in y$ and $k + 1 \notin y$ or there is no such integer. In the first event $k + 1 \notin (\psi(y'))'$. Therefore $k \notin \varphi((\psi(y'))')$. Proceeding in this way one can show in both cases that

$$t^{n(+*)} \leqslant (N \setminus \{0, \ldots, r + n - 1\}, N \setminus \{0, \ldots, r + n - 1\})$$

for $n \geqslant k + 2$. Therefore $\bigwedge(t^{n(+*)}; n < \omega) = 0$ for every $1 \neq t$.

Now we shall construct a nearly regular double p-algebra L with $L/(\gamma \wedge \bar{\gamma}) \cong K$. It is easy to check that $D(K) = \{(N, y) \in K: y \in B\} \cong B$. Let R be an ideal of K generated by $\{(N, y) \in K: y \text{ finite}\}$. Since

$$R \cap D(K) = \{(N, y) \in K: y \text{ finite}\}$$

is a prime ideal of $D(K)$, there exists a filter $\bar{R}$ of K such that $(R, \bar{R})$ is an ideal-filter pair of K (see [7, Corollary to Theorem 3]). By [7, Theorem 2] there exists a nearly regular double p-algebra L with $L/(\gamma \wedge \bar{\gamma}) \cong K$ such that, for $x \in L$, $|G[x]| = 2$ if and only if $G[x] \in R \cap \bar{R}$. Clearly, L satisfies (i), (ii′) and (iii). Take $x = (N, N \setminus \{0\}) \in D(K)$. By induction,

$$x^{n(+*)} = (N \setminus \{0, \ldots, n\}, N \setminus \{0, \ldots, n\})$$

in K for every n. Therefore $x^{n(+*)} \nleqslant d$ for all n and every $d \in R \cap D(K)$. Thus, (iv) is not satisfied by L and the proof is finished.

LEMMA 7. *Let L be a distributive double p-algebra. Let $a = \bigwedge(x^{n(+*)}; n < \omega)$ for some $x \in L$. Then $a \in C(L)$.*

PROOF. It is well known (see [7]) that

$$x \geqslant x^{+*} \geqslant \cdots \geqslant x^{n(+*)} \geqslant \cdots \geqslant a.$$

Therefore $a^* \geqslant x^{(n+1)(+*)*} = x^{n(+*)+**} \geqslant x^{n(+*)+}$ for every n. Hence

$$a^{*+} \leqslant x^{n(+*)++} \leqslant x^{n(+*)}$$

for every $n < \omega$. Thus, $a^{*+} \leqslant \bigwedge(x^{n(+*)}; n < \omega)$. In a distributive double p-algebra we have (see [7])

$$a \leqslant a^{**} \leqslant a^{*+} \quad \text{and} \quad a^* \leqslant a^+$$

for every element a. Therefore $a = a^{*+}$ and a^* is a complement of a in L, that is $a \in C(L)$.

THEOREM 2. *The condition* (ii′) *from Theorem* B *implies the condition* (ii) *from Theorem* A. *The converse implication is not true.*

PROOF. Since $a^{n(+*)} = a$ for every $a \in C(L)$, the first statement follows from Lemma 7 and (ii′). The regular double p-algebra $L(2\omega)$ from [7, pages 209, 210] is an example with $C(L(2\omega)) = \{0, 1\}$ such that $L(2\omega)$ does not satisfy the condition (ii′).

4. Auxiliary algebras

Motivated by [7] and the last example in the proof of Theorem 1, we make the following definition.

DEFINITION. *A regular double p-algebra L is said to be auxiliary if there exists a subdirectly irreducible distributive double p-algebra K such that* $L \cong K/(\gamma \wedge \bar{\gamma})$.

Clearly, by way of example, every subdirectly irreducible regular double p-algebra is auxiliary.

The following theorem characterized the auxiliary algebras and, interestingly enough, again exhibits the dual rôle played by (ii) and (ii′) of Theorems A and B.

THEOREM 3. *A regular double p-algebra L is auxiliary if and only if it satisfies the following conditions*:

(i) *there exists a prime ideal P of* $D(L)$ *such that for every* $1 \neq x \in D(L)$ *there exists* $d \in P$ *and an integer n with*

$$x^{n(+*)} \leqslant d;$$

(ii) $C(L) = \{0, 1\}$ (*or* (ii′) $\bigwedge (x^{n(+*)}\colon n < \omega) = 0$ *for every* $1 \neq x \in L$).

PROOF. Let L be auxiliary. Then there exists a subdirectly irreducible distributive double p-algebra K with $L \cong K/(\gamma \wedge \bar{\gamma})$. If K is regular, then $L = K$ and there exists $1 \neq d \in D(L)$ satisfying

$$x^{n(+*)} \leqslant d$$

for every $1 \neq x \in D(L)$ and some n (see Theorem A). Take P a prime ideal of $D(L)$ with $d \in P$. Then (i), (ii) and (ii′) follow from Theorems A and B. Suppose now that K is not regular. Then K is nearly regular. By [7, Lemma 4] there exists an ideal-filter pair $(R, \bar{R})$ on L. We can take a prime ideal P of $D(L)$ satisfying $P \supseteq R \cap D(L)$. Since $L \cong K/(\gamma \wedge \bar{\gamma})$, we see by Theorems A and B that (ii) and (ii′) are true for L. It is known (see [7, Lemma 4]) that $|G[x]| \neq 1$ in K if and only if $x \in R \cap \bar{R}$ in L. By hypothesis and Theorem A (or Theorem B) for every $1 \neq x \in D(L) \setminus R$ there exists $d \in R \cap D(L) \subseteq R \cap \bar{R}$ such that $x^{n(+*)} \leqslant d$ for some n. This is of course true of $x \in R \cap D(L)$ since, for any $1 \neq x \in L$, $x \geqslant x^{n(+*)}$. Thus the condition (i) holds.

Conversely, let L satisfy (i) and (ii) (or (i) and (ii′)). Put $R = (P]$ in L. By the Corollary to [7, Theorem 3] there exists an ideal-filter pair $(R, \bar{R})$ of L. By [7, Theorem 2] there exists a nearly regular double p-algebra K such that $K/(\gamma \wedge \bar{\gamma}) \cong L$. It remains to show that K is subdirectly irreducible. This can be easily done by Theorem A or Theorem B.

COROLLARY. *Let L be a subdirectly irreducible regular double p-algebra. Then there exists a subdirectly irreducible distributive double p-algebra K such that $K \not\cong L$ and $K/(\gamma \wedge \bar{\gamma}) \cong L$.*

PROOF. L is auxiliary. There exists a prime ideal P of $D(L)$ satisfying the condition (i) of Theorem 3. As above, there exists an ideal-filter pair $(R, \overline{R})$ of L (see the Corollary to [7, Theorem 3]). By [7, Theorem 2] there exists a proper nearly regular double p-algebra K which is the desired algebra.

EXAMPLE. An example of a regular double p-algebra that is auxiliary but not subdirectly irreducible is the algebra L dual to the topological ordered space $S = (S; \subseteq {}^*)$ from Lemmas 1–5. By Lemma 4, L satisfies (ii) from Theorem 3. It is easy to see that $D(L) = \{U\colon U \text{ is a clopen order-ideal in } S \text{ and } \overline{C} \subseteq U\}$. Let $a \in C_0$, that is, $a_n = 0$ for all but finitely many indices n. Take $R = \{U\colon U \text{ is a clopen order-ideal in } S \text{ and } a \neq U\}$. Evidently R is a prime ideal of L. Put $P = R \cap D(L)$. Clearly, P is a prime ideal of $D(L)$. Now we shall show that for $1 \neq x \in D(L) \setminus P$ there exists $d \in P$ with $x^{n(+*)} \leqslant d$ for some n. Take a clopen order-ideal U in S with $a \in U$ and $\overline{C} \in U$. Evidently $U \in D(L) \setminus P$. Assume $U \neq S$. We claim that there exists n such that $a \notin U^{n(+*)}$, that is $U^{n(+*)} \in R$. Suppose to the contrary that $a \in \bigwedge(U^{n(+*)}; n < \omega)$. We shall show that $C_0 \subseteq \bigwedge(U^{n(+*)}; n < \omega)$. Since C_0 is dense in C, $U \supseteq U^{+*} \ldots$, and $U^{n(+*)}$ are clopen order-ideals in S, we get $U = S$, a contradiction. By a straightforward calculation it can be seen that, for $x \in C$,

$$x \in \bigwedge\left(U^{n(+*)}; n < \omega\right) \quad \text{implies} \quad \tau_k(x), \tau_k^{-1}(x) \in \bigwedge\left(U^{n(+*)}; n < \omega\right).$$

Using this fact we show first that $a \in \bigwedge(U^{n(+*)}; n < \omega)$ implies $0 \in \bigwedge(U^{n(+*)}; n < \omega)$. Assume that k is the largest integer with $a_k = 2$. Therefore $\tau_k^{-1}(a) \in C_0$, $(\tau_k^{-1}(a))_n = a_n$ for $n \neq k$ and $(\tau_k^{-1}(a))_k = 0$. Evidently, $\tau_k^{-1}(a) \in \bigwedge(U^{n(+*)}; n < \omega)$. Repeating this procedure we obtain $0 \in \bigwedge(U^{n(+*)}; n < \omega)$ by finitely many steps. It is not difficult to show that every $x \in C_0$ can be obtained from $0 \in C_0$ by superposition of finitely many functions τ_r. Namely, let $1 \leqslant i_1 < \cdots < i_s$ be the only indices j with $x_j = 2$. Then

$$x = \tau_{i_s}\left(\cdots\left(\tau_{i_1}(0)\right)\cdots\right).$$

Therefore, $C_0 \subseteq \bigwedge(U^{n(+*)}; n < \omega)$ and the proof is complete. Thus L is auxiliary by Theorem 3.

Concerning the independence of the conditions in Theorem 3, $\mathbf{2} \times \mathbf{3}$ is, for example, a non-trivial example of a regular double p-algebra that satisfies (i) but fails to satisfy either (ii) or (ii′). By Theorem 3, $L(2\omega)$ (see the proof of Theorem 2) is an example of a regular double p-algebra that satisfies (ii) but does not satisfy (i). Furthermore, by Theorem 2, (ii′) implies (ii); however, whether (ii′) implies (i) is, as yet, unresolved.

PROBLEM. *Find a regular double p-algebra L such that, for every* $1 \neq x \in L$, $\bigwedge(x^{n(+*)}; n < \omega) = 0$ *which is not auxiliary.*

References

[1] R. Beazer, 'The determination congruence on double p-algebras', *Algebra Universalis* **6** (1976), 121–129.
[2] R. Beazer, 'Subdirectly irreducible double Heyting algebras', *Algebra Universalis* **10** (1980), 220–224.
[3] R. Beazer, 'Finitely subdirectly irreducible algebras with pseudocomplementation', *Algebra Universalis* **12** (1981), 376–386.
[4] B. A. Davey, 'Subdirectly irreducible distributive double p-algebras', *Algebra Universalis*, **8** (1978), 73–88.
[5] G. Grätzer, *General Lattice Theory* (Birkhäuser Verlag, Basel, 1978).
[6] T. Katriňák, 'Construction of regular double p-algebras', *Bull. Soc. Roy. Sci. Liège* **43** (1974), 294–301.
[7] T. Katriňák, 'Subdirectly irreducible distributive double p-algebras', *Algebra Universalis* **10** *(1980), 195–219.*
[8] J. L. Kelley, *General Topology* (Van Nostrand Reinhold, New York, 1955).
[9] H. A. Priestley, 'The construction of spaces dual to pseudocomplemented distributive lattices', *Quart J. Math. Oxford Ser.* (2) **26** (1975), 215–228.

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