PERIODIC SOLUTIONS BY PICARD'S APPROXIMATIONS

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1. Introduction. In [1] Demidovic considered a system of linear differential equations

$$(1) \qquad x' = A(t)x \qquad (' = d/dt)$$

with $A(t)$ continuous, T-periodic, odd, and skew symmetric. He proved that all solutions of (1) are either T-periodic or 2T-periodic. In [2] Epstein used Floquet theory to prove that all solutions of (1) are T-periodic without the skew symmetric hypothesis. Epstein's results were then generalized by Muldowney in [7] using Floquet theory. Much of the above work can also be interpreted as being part of the general framework of autosynartetic systems discussed by Lewis in [5] and [6]. According to private correspondence with Lewis it seems that he was aware of these results well before they were published. However, it appears that these theorems were neither stated nor suggested in the papers by Lewis.

The point of this note is that such theorems may be viewed as simple consequences of Picard's successive approximations rather than as results from Floquet theory or autosynartetic theory. It is often customary to begin a study of differential equations by proving an existence and uniqueness theorem using successive approximations. Much of the above work as well as generalizations may then be obtained as simple examples of successive approximations.

2. Periodic solutions. Consider a system of nonlinear differential equations

$$(2) \qquad x' = F(x, t)$$

where F is a column vector function continuous for all (x, t).

THEOREM. *Suppose that*

(i) $F(x, t+T) = F(x, t)$ *for all* (x, t) *and some* $T > 0$,

(ii) *for every* (x_o, t_o) *the successive approximations* $\{X_n\}$ *defined by* $X_o = x_o$ *and* $X_{n+1} = x_o + \int_{t_o}^{t} F(X_n(s), s)ds$ *converge uniformly*

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to a unique solution for $t_o \le t \le t_o + T$, *and*

(iii) $F(x, -t) = -F(x, t)$ *for all* (x, t).

Then every solution of (2) *is T-periodic.*

Proof. It follows from (i) and (iii) that X_1 is even and T-periodic. Thus $F(X_1(t), t)$ is odd and T-periodic. An easy induction shows that X_n is even and T-periodic for every n. Since X_n converges uniformly on one period, this sequence converges uniformly on every finite interval. Thus we may pass the limit through the integral in (ii). The limit function is necessarily even and T-periodic since the X_n have these properties. This completes the proof.

Remark 1. If $F(x, t) = A(t)x$ where A is odd and T-periodic, then conditions (i), (ii), and (iii) hold so that Epstein's result is obtained as a corollary.

Remark 2. A sufficiently strong Lipschitz and boundedness condition on F will insure that (ii) holds as a consequence of Picard's successive approximations ([3, pp. 20 and 28] or [4, p. 31]).

Remark 3. A similar nonlinear theorem and proof may be formulated yielding some of Muldowney's results; however, the formulation is cumbersome.

Remark 4. Some condition such as (ii) is necessary as the following scalar example shows if a solution is to be defined for all t over a period. Let $x' = x^2 \sin t$. One solution is $x(t) = 2/(1 + 2\cos t)$ which has finite escape time.

REFERENCES

1. B.P. Demidovic, Moskov. Gos. Univ. Uč. Zap. 6 (1952) 123-132.

2. I.J. Epstein, Periodic solutions of systems of differential equations. Proc. Amer. Math. Soc. 13 (1962) 690-694.

3. W. Hurewicz, Lectures on ordinary differential equations. (Wiley, New York, 1958).

4. S. Lefschetz, Differential equations: geometric theory. (Interscience, New York, 1957).

https://doi.org/10.4153/CMB-1968-092-2 Published online by Cambridge University Press

5. D.C. Lewis, Autosynartetic solutions of differential equations Amer. J. Math. 83 (1961) 1-32.

6. D.C. Lewis, Int. Sym. nonlinear differential equations and nonlinear mechanics (Edited by J.P. LaSalle and S. Lefschetz). (Academic Press, New York, 1963) 99-104.

7. J.S. Muldowney, Linear systems of differential equations with periodic solutions. Proc. Amer. Math. Soc. 18 (1967) 22-27.

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