CATEGORY OF SEQUENCES OF ZEROS AND ONES IN SOME *FK* SPACES

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Let s denote the space of all complex valued sequences and let E^∞ be all eventually zero sequences. An *FK* space is a locally convex vector subspace of s which is also a Fréchet space (complete linear metric) with continuous coordinates. A *BK* space is a normed *FK* space. Some discussion of *FK* spaces is given in [**11**]. Well-known examples of *BK* spaces are the spaces m, c, c_0 of bounded, convergent, null sequences respectively, all with $\|x\|_\infty = \sup_k |x_k|$, and

$$\ell^p = \left\{x \in s : \|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty\right\} \qquad (1 \le p < \infty).$$

Let N_0 be all sequences of 0's and 1's. For each $t \in (0, 1]$, write t in its non-terminating binary decimal expansion. $N_0 \setminus E^\infty$ is equivalent to $(0, 1]$ and N_0 contains only a countable number of eventually zero sequences; hence we can talk about subsets of N_0 having category. This is the classical definition of the category of subsets of N_0 given in [**5**]. The topology of $N_0 \setminus E^\infty$ induced on it by its equivalence with $(0, 1]$ is the same as the topology inherited as a subset of s.

All *FK* spaces considered will contain E^∞. Let A be an infinite matrix, E an *FK* space, $E_A = \{x \in s : Ax \in E\}$ is well known to be an *FK* space.

In 1945, J. D. Hill [**5**] proved that if A is a regular matrix, then $c_A \cap N_0$ is a first category subset of N_0. This result was extended by T. A. Keagy in [**6**], where he shows that if $c_A \supseteq E^\infty$ then either $c_A \supseteq m$ or $c_A \cap N_0$ is a first category subset of N_0.

G. Bennett and N. Kalton in [**1**] have shown that if an *FK* space E contains N_0 then $E \supseteq m$. We conjecture that if an *FK* space E contains a second category subset of N_0 then it must contain m. We are able to prove the conjecture for some *FK* spaces and certain summability domains.

THEOREM 1. *Let E be a separable FK space, with $E \subseteq m$. Then $E \cap N_0$ is a countable subset of N_0.*

This follows since the topology of E is stronger than that of m.

The β dual of a sequence x is defined by

$$x^\beta = \left\{y \in s : \sum_{i=1}^{\infty} x_i y_i \text{ converges}\right\}.$$

LEMMA 1. *If $x \notin \ell^1$, then $x^\beta \cap N_0$ is of first category in N_0.*

Proof. Let $O_r = \left\{y \in N_0 : \exists m, \ell \ge r \text{ such that } \left|\sum_{i=m}^{\ell} x_i y_i\right| > 1\right\}$.

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By its definition O_r is a non-empty open subset of N_0. It is dense in N_0, for if we prescribe the first p slots, there is a sequence in O_r with those entries in the first p slots. $N_0 \cap x^\beta \subseteq N_0 \setminus \bigcap_{r=1}^{\infty} O_r$ and thus is of first category.

Since for any FK space E, and any matrix A, $E_A \subseteq s_A$ and s_A is the intersection of the β duals of the rows of A, we have the following result.

Corollary 1. *Let E be an FK space, A a matrix with some row not in ℓ^1. Then $E_A \cap N_0$ is a first category subset of N_0.*

Let $e = (1, 1, 1, \dots)$, $e^j = (0, \dots 0, 1, 0, \dots)$ (with 1 in rank j). We denote the nth section of an element $x \in E$ by $P_n x = \sum_{i=1}^{n} x_i e^i$ and say that x has AK provided that $P_n x \to x$ in E. $S_E = \{x \in E : x \text{ has } AK\}$. E has AK provided $S_E = E$.

Theorem 2. *Let E be an FK space such that $E \cap m \subseteq S_E$. Then $E \supseteq m$ or $E \cap N_0$ is a first category subset of N_0.*

Proof. Let q be the paranorm of E and suppose there exists an $x \in N_0 \setminus E$. Hence $P_r(x)$ is not a Cauchy sequence in E. So there exists an $\varepsilon > 0$ and increasing sequences of integers $(m(n))$ and $(\ell(n))$ such that $0 < m(1) < \ell(1) < m(2) \dots$ and $q([P_{\ell(n)} - P_{m(n)}]x) > \varepsilon$. Let

$$O_r = \{z \in N_0 : (P_{\ell(n)} - P_{m(n)})(x - z) \text{ is the zero sequence for some } n \geq r\}.$$

By definition each O_r is open and dense. $E \cap N_0 \subseteq N_0 \setminus \bigcap_{r=1}^{\infty} O_r$ and hence is of first category in N_0.

Theorem 3. *Let E be an FK space with AK, $E \supseteq \ell^1$ and A a matrix. Then $E_A \supseteq m$ or $E_A \cap N_0$ is a first category subset of N_0.*

Proof. By Corollary 1, we may assume the rows of A are in ℓ^1. Let q be the paranorm of E. Since $E \supseteq \ell^1$ we may assume for $x \in \ell^1$ that $q(x) \leq \|x\|_1$.

If $E_A \not\supseteq N_0$, then there exists an $x \in N_0$ such that $Ax \notin E$. Hence $P_r(Ax)$ is not a Cauchy sequence in E. So there exists an $\varepsilon > 0$ and increasing sequences of integers $(m(n))$ and $(\ell(m))$ such that $0 < m(1) < \ell(1) < m(2) \dots$ and $q([P_{\ell(n)} - P_{m(n)}]Ax) > \varepsilon$. Let

$$O_r = \{z \in N_0 : q([P_{\ell(n)} - P_{m(n)}]Az) > \varepsilon/2 \text{ for some } n \geq r\}.$$

O_r is open. Let $w \in O_r$. Then there exists an $n \geq r$ and a positive real number b such that $q([P_{\ell(n)} - P_{m(n)}]Aw) - b > \varepsilon/2$. Since the rows of A are in ℓ^1, there exists a positive integer c such that

$$\sum_{j=m(n)+1}^{\ell(n)} \sum_{i=c+1}^{\infty} |a_{ji}| < \frac{b}{2}.$$

Hence, for each $v \in N_0$, $q([P_{\ell(n)} - P_{m(n)}]A(v - P_c v)) < \frac{b}{2}$. Let $u \in N_0$ with $P_c u = P_c w$. We

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have

$$\begin{aligned}q([P_{\ell(n)}-P_{m(n)}]Au) &\geq q([P_{\ell(n)}-P_{m(n)}]AP_cu) \\ &\quad - q([P_{\ell(n)}-P_{m(n)}]A(u-P_cu)) \\ &\geq q([P_{\ell(n)}-P_{m(n)}]Aw)-q([P_{\ell(n)}-P_{m(n)}]A(w-P_cw))-\frac{b}{2} \\ &> \frac{\varepsilon}{2}.\end{aligned}$$

Hence O_r is open.

Let $u \in N_0$ and $c \in \mathbb{Z}^+$. To show denseness, it suffices to show that there exists a $z \in O_r$ with $P_cz = P_cu$. Let α^n be the nth column of A. There exists a $t \geq r$ such that, for $m, \ell \geq t$, $\sum_{i=1}^{c} q([P_\ell - P_m]\alpha^i) < \frac{\varepsilon}{4}$. Let $z = P_cu + x - P_cx$. Then

$$\begin{aligned}q([P_{\ell(t)}-P_{m(t)}]Az) &\geq -q([P_{\ell(t)}-P_{m(t)}]AP_cu)-q([P_{\ell(t)}-P_{m(t)}]AP_cx) \\ &\quad + q([P_{\ell(t)}-P_{m(t)}]Ax) \\ &> \frac{-\varepsilon}{4}-\frac{\varepsilon}{4}+\varepsilon=\frac{\varepsilon}{2}.\end{aligned}$$

Hence O_r is dense.

$\bigcap_{r=1}^{\infty} O_r$ is a second category set in N_0 whose complement is of first category and $E_A \cap N_0 \subseteq N_0 \setminus \bigcap_{r=1}^{\infty} O_r$. Hence $E_A \cap N_0$ is a first category subset of N_0.

Keagy in [5] proved the same result for c_A. A modification of our proof of Theorem 3 will give us his result and also the same result for bv_A where bv is the set of all sequences of bounded variation.

All *FK* spaces considered so far have been separable. The assumption is not necessary, since we have the following result.

THEOREM 4. *If $m_A \supseteq E^\infty$, then $m_A \supseteq m$ or $m_A \cap N_0$ is a first category subset of N_0.*

Proof. Assuming the rows of A are in ℓ^1 and $m_A \not\supseteq m$, we have $\sup_n \sum_{i=1}^{\infty} |a_{ni}| = \infty$. Hence there exists a sequence $u_n \to 0$ such that $\sup_n u_n \left(\sum_{i=1}^{\infty} |a_{ni}|\right) = \infty$. Let $D = \operatorname{diag}(u_1, u_2, \dots)$ and $B = DA$. $m_B \not\supseteq m$ and $m_B \supseteq (c_0)_B \supseteq m_A$. Theorem 3 implies that $(c_0)_B \cap N_0$ is a first category subset of N_0; hence $m_A \cap N_0$ is also a first category subset of N_0.

Let $\mathbb{Z}^+$ denote the set of positive integers. Using characteristic functions, the set of subsets of $\mathbb{Z}^+$ is equivalent to N_0. Hence we can talk about the category of a set of subsets of $\mathbb{Z}^+$. The following theorem improves results of Bennett and Kalton [**2**], Lorentz [**7**], Mehdi [**9**], Peyeremhoff [**8**], and Zeller [**11**].

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THEOREM 5. *Let $1 \le p < \infty$. The following conditions are equivalent for any matrix A:*

(i) *A maps m into ℓ^p;*

(ii) $\sup_{J \in \mathcal{A}} \sum_{i=1}^{\infty} \left| \sum_{j \in J} a_{ij} \right|^p < \infty$ *for $\mathcal{A}$ any second category subset of the set of subsets of $\mathbb{Z}^+$ which contains all finite sets;*

(iii) $\sum_{i=1}^{\infty} \left| \sum_{j \in J} a_{ij} \right|^p < \infty$ *for $J \in \mathcal{A}$, where $\mathcal{A}$ is as in* (ii).

This follows easily from Theorem 3 and the fact that any matrix map between *FK* spaces is continuous.

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REFERENCES

1. G. Bennett and N. J. Kalton, *FK*-spaces containing c_0, *Duke Math. J.* **39** (1972), 561–582.

2. G. Bennett and N. J. Kalton, Inclusion theorems for *K*-Spaces, *Canad. J. Math.* **25** (1973), 511–524.

3. R. DeVos, Subsequences and rearrangements of sequences in *FK* spaces, *Pacific J. Math.* **64** (1976), 129–135.

4. J. A. Fridy, Summability of rearrangements of sequences, *Math. Z.* **143** (1975), 187–192.

5. J. D. Hill, Summability of sequences of 0's and 1's, *Ann. of Math.* **64** (1945), 556–562.

6. T. A. Keagy, Summability of certain category two classes, to appear.

7. F. R. Keogh and G. M. Petersen, A universal Tauberian theorem, *J. London Math. Soc.* **33** (1958), 121–123.

8. G. G. Lorentz, Direct theorems on methods of summability II, *Canad. J. Math.* **3** (1951), 236–256.

9. A. Peyeremhoff, Über ein Lemma von Herrn H. C. Chow, *J. London Math. Soc.* **32** (1957), 33–36.

10. W. L. C. Sargent, Some sequence spaces related to the ℓ^p spaces, *J. London Math. Soc.* **35** (1960), 161–171.

11. A. Wilansky, *Functional analysis* (Blaisdell, 1964).

12. K. Zeller, Matrixtransformationen von Folgenraumen, *Univ. Roma. 1st Naz. Alta. Mat. Rend. Mat. e Appl.* (5) **12** (1954), 340–346.

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