I
Locally integrable structures

In this chapter we introduce the main concepts which will be studied throughout the book. In order to do so we recall some standard notions such as differentiable manifolds, vector fields, differential forms, etc., with the purpose mainly of laying down the basis for the presentation and to establish the notations.

Nevertheless, we assume from the reader some familiarity with these concepts. In particular, we freely use some standard results on complex vector fields and complex differential forms on $\mathbb{R}^N$.

I.1 Complex vector fields

Let Ω be a Hausdorff topological space, with a countable basis of open sets. A *differentiable structure over* Ω *of dimension* N is a collection of pairs $\mathcal{F} = \{(U, \mathbf{x})\}$, where $U \subset \Omega$ is a nonempty open set, $\mathbf{x} : U \longrightarrow \mathbb{R}^N$ is a homeomorphism onto an open subset $\mathbf{x}(U)$ of $\mathbb{R}^N$ and the following properties are satisfied:

(1) $\bigcup_{(U,\mathbf{x})\in\mathcal{F}} U = \Omega$;

(2) $\mathbf{x}(U \cap U') \xrightarrow{\mathbf{x}'\circ\mathbf{x}^{-1}} \mathbf{x}'(U \cap U')$ is C^∞ for each pair $(U, \mathbf{x})$, $(U', \mathbf{x}') \in \mathcal{F}$ with $U \cap U' \neq \emptyset$;

(3) $\mathcal{F}$ is maximal with respect to (1) and (2), that is, if $\emptyset \neq V \subset \Omega$ is open and $\mathbf{y} : V \longrightarrow \mathbf{y}(V)$ is a homeomorphism over an open subset of $\mathbb{R}^N$ such that, for any $(U, \mathbf{x}) \in \mathcal{F}$ with $U \cap V \neq \emptyset$, the composition $\mathbf{x}(U \cap V) \xrightarrow{\mathbf{y}\circ\mathbf{x}^{-1}} \mathbf{y}(U \cap V)$ is C^∞, then $(V, \mathbf{y}) \in \mathcal{F}$.

It is easy to see that given any family $\mathcal{F}^* = \{(U, \mathbf{x})\}$ as above satisfying (1) and (2) there is a unique differentiable structure $\mathcal{F}$ over Ω, of dimension N, such that $\mathcal{F}^* \subset \mathcal{F}$.

Definition I.1.1. *A differentiable manifold (or smooth manifold) of dimension N is a Hausdorff topological space Ω, with a countable basis equipped with a differentiable structure of dimension N.*

If, in the above definitions, we replace C^∞ by real-analytic we obtain the concept of a *real-analytic manifold of dimension N*.

We give some examples:

(1) $\Omega = \mathbb{R}^N$, $\mathcal{F}^* = \{(\mathbb{R}^N, \text{identity map})\}$.
(2) Let Ω be a differentiable manifold of dimension N and let $W \subset \Omega$ be open. Then over W is defined a natural differentiable structure of dimension N, which is given by

$$\mathcal{F}_W = \{(W \cap U, \mathbf{x}|_{W\cap U}) : (U, \mathbf{x}) \in \mathcal{F}, W \cap U \neq \emptyset\}.$$

(3) Let $f : \mathbb{R}^{N+1} \to \mathbb{R}$ be a C^∞ function. Let

$$\Omega = \{x \in \mathbb{R}^{N+1} : f(x) = 0\}$$

and suppose that $\mathrm{d}f(x) \neq 0, \forall x \in \Omega$. Then a natural differentiable structure of dimension N is defined over Ω (as a consequence of the implicit function theorem).

Notation. An element $(U, \mathbf{x}) \in \mathcal{F}$ will be refered to as a *local chart* or as a *local system of coordinates*. If we write $\mathbf{x} = (x_1, \ldots, x_N)$ then for $p \in U$ its local coordinates (with respect to this given local chart) are given by $(x_1(p), \ldots, x_N(p))$.

From now on, unless otherwise stated, we shall fix a differentiable manifold Ω (of dimension N). We shall say that a function $f : \Omega \to \mathbb{C}$ is *smooth* if for every $(U, \mathbf{x}) \in \mathcal{F}$ the composition $f \circ \mathbf{x}^{-1}$ is C^∞ on $\mathbf{x}(U)$.[1] We shall denote by $C^\infty(\Omega)$ the set of all smooth functions on Ω. We observe that C^∞ is an algebra over $\mathbb{C}$ which contains, as an $\mathbb{R}$-subalgebra, the set $C^\infty(\Omega; \mathbb{R})$ of all smooth functions on Ω which are real-valued.

Definition I.1.2. *A (smooth) complex vector field over Ω is a $\mathbb{C}$-linear map*

$$L : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$$

[1] More generally, we say that a function $f : \Omega \to \mathbb{C}$ is C^k $(k \geq 0)$ if for every $(U, \mathbf{x}) \in \mathcal{F}$ the composition $f \circ \mathbf{x}^{-1}$ is C^k on $\mathbf{x}(U)$.

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which satisfies the Leibniz rule

$$L(fg) = fL(g) + gL(f), \quad f, g \in C^\infty(\Omega). \tag{I.1}$$

We shall denote by $\mathfrak{X}(\Omega)$ the set of all complex vector fields over Ω.

PROPOSITION I.1.3. *If $L \in \mathfrak{X}(\Omega)$ and if f is constant then $Lf = 0$. We also have*

$$\operatorname{supp} Lf \subset \operatorname{supp} f, \quad \forall f \in C^\infty(\Omega),\ L \in \mathfrak{X}(\Omega). \tag{I.2}$$

PROOF. For the first statement it suffices to show that $L1 = 0$ and this follows from (I.1) together with the fact that $1^2 = 1$. We shall now prove (I.2); we must show that if f vanishes on an open set $V \subset \Omega$ then the same is true for Lf.

Let $p \in V$ be arbitrary. We select a local chart $(U, \mathbf{x})$ with $p \in U \subset V$ and take $\varphi \in C_c^\infty(\mathbf{x}(U))$ such that $\varphi(\mathbf{x}(p)) = 1$. Then the function $g : \Omega \to \mathbb{R}$ defined by the rule

$$g(q) = \begin{cases} \varphi(\mathbf{x}(q)) & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

belongs to $C^\infty(\Omega; \mathbb{R})$ and vanishes on $\Omega \backslash V$. In particular,

$$f = (1 - g)f$$

and then

$$L(f)(p) = (1 - g(p))L(f)(p) + f(p)L(1 - g)(p) = 0,$$

since $g(p) = 1$. $\square$

A consequence of the preceding result is the possibility of defining the *restriction* of an element $L \in \mathfrak{X}(\Omega)$ to an open subset W of Ω. More precisely, there is a $\mathbb{C}$-linear map

$$\mathfrak{X}(\Omega) \ni L \longrightarrow L_W \in \mathfrak{X}(W)$$

which turns the diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \xrightarrow{L} & C^\infty(\Omega) \\ \downarrow & & \downarrow \\ C^\infty(W) & \xrightarrow{L_W} & C^\infty(W) \end{array}$$

commutative (the vertical arrows denote the restriction map). Indeed, if $p \in W$ and $f \in C^\infty(W)$ we set

$$L_W(f)(p) = L(\tilde{f})(p),$$

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where $\tilde{f}$ is any element in $C^\infty(\Omega)$ which coincides with f in a neighborhood of p. Such a definition is meaningful according to Proposition I.1.3 and it is very easy to check that L_W defines an element in $\mathfrak{X}(W)$. As usual we shall write L instead of L_W, since the meaning will always be clear from the context.

I.2 The algebraic structure of $\mathfrak{X}(\Omega)$

Given $g \in C^\infty(\Omega)$ and $L \in \mathfrak{X}(\Omega)$ we can define $gL \in \mathfrak{X}(\Omega)$ by

$$(gL)(f) = g \cdot L(f), \quad f \in C^\infty(\Omega).$$

Such external multiplication gives $\mathfrak{X}(\Omega)$ the structure of a $C^\infty(\Omega)$-*module*.

A very important (internal) operation in $\mathfrak{X}(\Omega)$ is the so-called *Lie bracket* (or *commutator*) between two vector fields. Given $L, M \in \mathfrak{X}(\Omega)$ we define

$$[L, M](f) = L\left(M(f)\right) - M\left(L(f)\right), \quad f \in C^\infty(\Omega). \tag{I.3}$$

It is a simple verification to check that $[L, M] \in \mathfrak{X}(\Omega)$. This bracket operation turns $\mathfrak{X}(\Omega)$ into a *Lie algebra*[2] *over* $\mathbb{C}$.

Let $(U, \mathbf{x})$ be a local chart in Ω and let also $L \in \mathfrak{X}(U)$. We fix $p \in U$ and write as before

$$\mathbf{x}(q) = (x_1(q), \ldots, x_N(q)), \quad q \in U.$$

Next we take $V \subset U$ open such that $\mathbf{x}(V)$ is an open ball centered at $\mathbf{x}(p) = a = (a_1, \ldots, a_N)$. Given $f \in C^\infty(U)$, write $f^* = f \circ \mathbf{x}^{-1}$. If $(x_1, \ldots, x_N) \in \mathbf{x}(V)$, the Fundamental Theorem of Calculus applied to the function $t \mapsto f^*(a_1 + t(x_1 - a_1), \ldots, a_N + t(x_N - a_N))$ gives

$$f^*(x_1, \ldots, x_N) = f^*(a_1, \ldots, a_N) + \sum_{j=1}^{N} h_j(x_1, \ldots, x_N)(x_j - a_j),$$

where $h_j \in C^\infty(\mathbf{x}(V))$ and $h_j(a) = (\partial f^* / \partial x_j)(a)$. If we further set $g_j = h_j \circ \mathbf{x} \in C^\infty(V)$, we obtain

$$f(q) = f(p) + \sum_{j=1}^{N} g_j(q)(x_j(q) - x_j(p)), \quad q \in V, \tag{I.4}$$

[2] Recall that a Lie algebra over $\mathbb{C}$ is a $\mathbb{C}$-vector space E over which is defined a bilinear form $E \times E \ni (v, w) \mapsto [v, w]$ which satisfies

$$[u, u] = 0, \quad [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \quad u, v, w \in E.$$

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and consequently the Leibniz rule gives

$$L(f)(p) = \sum_{j=1}^{N} g_j(p) \left(Lx_j\right)(p). \tag{I.5}$$

DEFINITION I.2.1. *The $\mathbb{C}$-linear map $C^\infty(U) \to C^\infty(U)$ given by*

$$f \mapsto \frac{\partial f^*}{\partial x_j} \circ \mathbf{x}$$

defines an element in $\mathfrak{X}(U)$, which will be denoted by $\frac{\partial}{\partial x_j}$.

Returning to the preceding argument and notation we can write

$$g_j(p) = h_j(\mathbf{x}(p)) = \frac{\partial f^*}{\partial x_j}(\mathbf{x}(p)) = \left(\frac{\partial}{\partial x_j}\right)(f)(p).$$

Inserting this in (I.5) gives

$$L(f)(p) = \sum_{j=1}^{N} \left(Lx_j\right)(p) \left(\frac{\partial}{\partial x_j}\right)(f)(p);$$

since p was an arbitrary point taken in U we obtain the *representation of L in the local coordinates* $(x_1, \ldots, x_N)$:

$$L = \sum_{j=1}^{N} \left(Lx_j\right) \frac{\partial}{\partial x_j}. \tag{I.6}$$

In particular this representation shows that the $C^\infty(U)$-module $\mathfrak{X}(U)$ is free, with basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_N\}$.

Observe that if $M \in \mathfrak{X}(U)$ then the representation of $[L, M]$ in the local coordinates $(x_1, \ldots, x_N)$ is given by

$$[L, M] = \sum_{j=1}^{N} \left\{L(Mx_j) - M(Lx_j)\right\} \frac{\partial}{\partial x_j}. \tag{I.7}$$

I.3 Formally integrable structures

Denote by $\mathcal{B}_p$ the set of all pairs (V, f), where V is an open neighborhood of p and $f \in C^\infty(V)$. In $\mathcal{B}_p$ we introduce the following equivalence relation: $(V_1, f_1) \sim (V_2, f_2)$ if there is an open neighborhood V of p, $V \subset V_1 \cap V_2$, such that f_1 and f_2 agree on V.

A *germ of a C^∞ function at p* is an element in the quotient space $C^\infty(p) \doteq \mathcal{B}_p / \sim$. We observe that $C^\infty(p)$ is also a $\mathbb{C}$-algebra. Given a C^∞ function f

defined in an open neighborhood of p, the germ at p defined by f will be denoted by $\underline{f}$. Notice that there is a natural $\mathbb{C}$-algebra homomorphism $C^\infty(p) \to \mathbb{C}$ defined by $\underline{f} \mapsto f(p)$.

Definition I.3.1. *A complex tangent vector (to Ω) at p is a $\mathbb{C}$-linear map*

$$\mathsf{v} : C^\infty(p) \longrightarrow \mathbb{C}$$

satisfying

$$\mathsf{v}(\underline{f}\,\underline{g}) = f(p)\mathsf{v}(\underline{g}) + g(p)\mathsf{v}(\underline{f}), \quad \underline{f}, \underline{g} \in C^\infty(p). \tag{I.8}$$

The set of all complex tangent vectors at p, denoted by $\mathbb{C}T_p\Omega$, has a structure of a $\mathbb{C}$-vector space and is called the *complex tangent space to Ω at p*.

If $L \in \mathfrak{X}(\Omega)$ then $L_p : C^\infty(p) \to \mathbb{C}$ defined by

$$L_p(\underline{f}) = L(f)(p), \quad \underline{f} \in C^\infty(p)$$

belongs to $\mathbb{C}T_p\Omega$. Conversely, suppose that for each $p \in \Omega$ an element $\mathsf{v}_p \in \mathbb{C}T_p\Omega$ is given such that

$$p \mapsto \mathsf{v}_p(\underline{f}) \in C^\infty(\Omega), \quad \forall f \in C^\infty(\Omega).$$

Then there is $L \in \mathfrak{X}(\Omega)$ such that $L_p = \mathsf{v}_p$ for all $p \in \Omega$.

Suppose now that $p \in U$ and that $(U, \mathbf{x})$ is a local chart. If $\mathsf{v} \in \mathbb{C}T_p\Omega$ then, according to (I.5),

$$\mathsf{v}(\underline{f}) = \sum_{j=1}^{N} g_j(p)\mathsf{v}(\underline{x_j}) = \sum_{j=1}^{N} \mathsf{v}(\underline{x_j}) \left(\frac{\partial}{\partial x_j}\right)_p (\underline{f}), \quad \underline{f} \in C^\infty(p).$$

In particular we conclude that $\{(\frac{\partial}{\partial x_j})_p : j = 1, \ldots, N\}$ is a basis of $\mathbb{C}T_p\Omega$.

The *complexified tangent bundle of* Ω is defined as the disjoint union

$$\mathbb{C}T\Omega = \bigcup_{p\in\Omega} \mathbb{C}T_p\Omega.$$

We shall also need the notion of a *complex vector sub-bundle of* $\mathbb{C}T\Omega$ *of rank n and corank $N-n$*. By this we mean a disjoint union

$$\mathcal{V} = \bigcup_{p\in\Omega} \mathcal{V}_p \subset \mathbb{C}T\Omega$$

satisfying the following conditions:

(a) For each $p \in \Omega$, $\mathcal{V}_p$ is a vector subspace of $\mathbb{C}T_p\Omega$ of dimension n.
(b) Given $p_0 \in \Omega$ there are an open set U_0 containing p_0 and vector fields $L_1, \ldots, L_n \in \mathfrak{X}(U_0)$ such that $L_{1p}, \ldots, L_{np}$ span $\mathcal{V}_p$ for every $p \in U_0$.

The vector space $\mathcal{V}_p$ is called the *fiber* of $\mathcal{V}$ at p.

Given a complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T\Omega$ and an open subset W of Ω, a *section of* $\mathcal{V}$ *over* W is an element L of $\mathfrak{X}(W)$ such that $L_p \in \mathcal{V}_p$ for all $p \in W$. We are now in a position to introduce our main object of study:

Definition I.3.2. *A formally integrable structure over* Ω *is a complex vector sub-bundle* $\mathcal{V}$ *of* $\mathbb{C}T\Omega$ *satisfying the involutive (or Frobenius) condition:*

- *If* $W \subset \Omega$ *is open and* $L, M \in \mathfrak{X}(W)$ *are sections of* $\mathcal{V}$ *over* W *then* $[L, M]$ *is also a section of* $\mathcal{V}$ *over* W.

The rank (resp. corank) of $\mathcal{V}$ will be referred to as the *rank* (resp. *corank*) of the formally integrable structure $\mathcal{V}$. Let $\mathcal{V}$ be a formally integrable structure over Ω and fix $p \in \Omega$. There is a local chart $(U, \mathbf{x})$ with $p \in U$ and vector fields $L_1, \ldots, L_n \in \mathfrak{X}(U)$ such that $\{L_{1q}, \ldots, L_{nq}\}$ is a basis of $\mathcal{V}_q$ for every $q \in U$. If we write $\mathbf{x} = (x_1, \ldots, x_N)$ and

$$L_j = \sum_{k=1}^{N} a_{jk}(x)\frac{\partial}{\partial x_k}$$

then the matrix (a_{jk}) has rank equal to n at every point; moreover, there are $c_{jk}^{\nu} \in C^\infty(U)$, $j, k, \nu = 1, \ldots, n$, such that

$$[L_j, L_k] = \sum_{\nu=1}^{n} c_{jk}^{\nu} L_\nu, \quad j, k = 1, \ldots, n.$$

Definition I.3.3. *A (classical) solution for the formally integrable structure* $\mathcal{V}$ *over* Ω *is a* C^1*-function* u *on* Ω *such that* $Lu = 0$ *for every section* L *of* $\mathcal{V}$ *defined in an open subset of* Ω.

More generally, we can consider the concept of *(weak) solutions* for the formally integrable structure $\mathcal{V}$ over Ω: it suffices to consider u, in the preceding definition, belonging to the space of distributions on Ω (we refer to [**H2**] for the theory of distributions on manifolds).

I.4 Differential forms

We shall denote by $\mathfrak{N}(\Omega)$ the dual of the $C^\infty(\Omega)$-module $\mathfrak{X}(\Omega)$ and shall refer to its elements as *differential forms over* Ω *of degree one* (or *one-forms* for short). In other words, a one-form on Ω is a $C^\infty(\Omega)$-linear map

$$\omega : \mathfrak{X}(\Omega) \to C^\infty(\Omega).$$

Let $\omega \in \mathfrak{N}(\Omega)$, $L \in \mathfrak{X}(\Omega)$ and suppose that L vanishes on an open subset $V \subset \Omega$. Then $\omega(L)$ also vanishes on V. Indeed, let $p \in V$ and let $g \in C^\infty(\Omega, \mathbb{R})$ be equal to one at p and vanish on $\Omega \backslash V$. Then $L = (1-g)L$ and consequently

$$\omega(L) = (1-g)\omega(L)$$

vanishes at p. In fact, we have a more precise result:

LEMMA I.4.1. *Let $\omega \in \mathfrak{N}(\Omega)$, $L \in \mathfrak{X}(\Omega)$ and suppose that $L_p = 0$. Then $\omega(L)(p) = 0$.*

PROOF. By the preceding discussion it is clear that we can restrict a one-form on Ω to an open set $W \subset \Omega$, that is, given $\omega \in \mathfrak{N}(\Omega)$ there is $\omega|_W \in \mathfrak{N}(W)$ which makes the diagram

$$\begin{array}{ccc} \mathfrak{X}(\Omega) & \xrightarrow{\omega} & C^\infty(\Omega) \\ \downarrow & & \downarrow \\ \mathfrak{X}(W) & \xrightarrow{\omega|_W} & C^\infty(W) \end{array}$$

commutative (the vertical arrows denote restriction homomorphisms). Let then $(U, \mathbf{x})$ be a local chart with $p \in U$. Then, if $\mathbf{x} = (x_1, \ldots, x_N)$ we have by (I.6)

$$\omega(L)(p) = \omega_U(L_U)(p) = \sum_{j=1}^{N} (Lx_j)(p)\omega_U\left(\frac{\partial}{\partial x_j}\right)(p) = 0.$$

The proof of Lemma I.4.1 is complete. □

If we then define

$$\mathbb{C}T_p^*\Omega \doteq \text{dual of } \mathbb{C}T_p\Omega,$$

to each $\omega \in \mathfrak{N}(\Omega)$ we can associate an element $\omega_p \in \mathbb{C}T_p^*\Omega$ by the formula

$$\omega_p(\mathrm{v}) = \omega(L)(p),$$

where $L \in \mathfrak{X}(\Omega)$ is such that $L_p = \mathrm{v}$.

As in the case for vector fields, we have a converse: if for every $p \in \Omega$ an element $\eta_p \in \mathbb{C}T_p^*\Omega$ is given such that

$$p \mapsto \eta_p(L_p) \in C^\infty(\Omega), \quad \forall\, L \in \mathfrak{X}(\Omega),$$

then there is $\omega \in \mathfrak{N}(\Omega)$ such that $\omega_p = \eta_p$, for every $p \in \Omega$.

PROPOSITION I.4.2. $\mathbb{C}T_p^*\Omega = \{\omega_p : \omega \in \mathfrak{N}(\Omega)\}$.

PROOF. Let $(U, \mathbf{x})$ be a local chart with $p \in U$. Formula (I.6) allows one to define $\mathrm{d}x_j \in \mathfrak{N}(U)$, $j = 1, \ldots, N$, by the rule

$$\mathrm{d}x_j\left(\frac{\partial}{\partial x_k}\right) = \delta_{jk}, \quad j, k = 1, \ldots, N.$$

Hence, if $\omega \in \mathfrak{N}(U)$ we have

$$\omega = \sum_{j=1}^{N} \omega\left(\frac{\partial}{\partial x_j}\right) \mathrm{d}x_j, \tag{I.9}$$

where $\omega(\partial/\partial x_j) \in C^\infty(U)$. If we now observe that $\{(\mathrm{d}x_j)_p\} \subset \mathbb{C}T_p^*\Omega$ is the dual basis of $\{(\partial/\partial x_j)_p\} \subset \mathbb{C}T_p\Omega$ then the conclusion will follow easily. $\square$

Definition I.4.3. *Given $f \in C^\infty(\Omega)$ we define $\mathrm{d}f \in \mathfrak{N}(\Omega)$ by the formula*

$$\mathrm{d}f(L) = L(f), \quad L \in \mathfrak{X}(\Omega). \tag{I.10}$$

From (I.9) we obtain the usual representation in local coordinates

$$\mathrm{d}f = \sum_{j=1}^{N} \mathrm{d}f\left(\frac{\partial}{\partial x_j}\right) \mathrm{d}x_j = \sum_{j=1}^{N} \frac{\partial f}{\partial x_j} \mathrm{d}x_j.$$

We now introduce the *complexified cotangent bundle of* Ω as being the disjoint union

$$\mathbb{C}T^*\Omega \doteq \bigcup_{p\in\Omega} \mathbb{C}T_p^*\Omega.$$

As before we can also introduce the notion of a *complex vector sub-bundle of* $\mathbb{C}T^*\Omega$ *of rank* m as being a disjoint union

$$\mathcal{W} = \bigcup_{p\in\Omega} \mathcal{W}_p,$$

where each $\mathcal{W}_p$ is a vector subspace of $\mathbb{C}T_p^*\Omega$ of dimension m, satisfying the following property:

- Given $p_0 \in \Omega$ there are an open set U_0 containing p_0 and one-forms $\omega_1, \ldots, \omega_m \in \mathfrak{N}(U_0)$ such that $\omega_{1p}, \ldots, \omega_{mp}$ span $\mathcal{W}_p$ for every $p \in U_0$.

As before we shall refer to the space $\mathcal{W}_p$ as the *fiber* of $\mathcal{W}$ at the point p.

Proposition I.4.4. *Let $\mathcal{V} = \cup_{p\in\Omega}\mathcal{V}_p$ be a complex vector sub-bundle of $\mathbb{C}T\Omega$ and set, for each $p \in \Omega$,*

$$\mathcal{V}_p^\perp \doteq \{\lambda \in \mathbb{C}T_p^*\Omega : \lambda = 0 \text{ on } \mathcal{V}_p\}.$$

Then $\mathcal{V}^\perp \doteq \cup_{p\in\Omega}\mathcal{V}_p^\perp$ is a complex vector sub-bundle of $\mathbb{C}T^*\Omega$.

Proof. Given $p_0 \in \Omega$ there is a local chart

$$(U_0, \mathbf{x}), \qquad \mathbf{x} = (x_1, \dots, x_N),$$

with $p_0 \in U_0$, and vector fields on U_0

$$L_j = \sum_{k=1}^{N} a_{jk} \frac{\partial}{\partial x_k}, \quad j = 1, \dots, n,$$

such that $\{L_{1p}, \dots, L_{np}\}$ spans $\mathcal{V}_p$ for all $p \in U_0$. After a contraction of U_0 around p_0 and a relabeling of the indices we can assume that the matrix $(a_{jk})_{j,k=1,\dots,n}$ is invertible in U_0. Let $(b_{jk})_{j,k=1,\dots,n}$ be its inverse and set

$$L_j^{\#} = \sum_{\nu=1}^{n} b_{j\nu} L_\nu, \quad j = 1, \dots, n.$$

Then $\{L_{1p}^{\#}, \dots, L_{np}^{\#}\}$ also spans $\mathcal{V}_p$ for all $p \in U_0$. Moreover, we have

$$L_j^{\#} = \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} c_{jk} \frac{\partial}{\partial x_{n+k}}, \quad j = 1, \dots, n,$$

where c_{jk} are smooth in U_0 and $m = N - n$. Set

$$\omega_\ell = \mathrm{d}x_{n+\ell} - \sum_{\gamma=1}^{n} c_{\gamma\ell} \mathrm{d}x_\gamma, \quad \ell = 1, \dots, m.$$

Then $\omega_{1p}, \dots, \omega_{mp}$ are linearly independent for all $p \in U_0$ and furthermore

$$\omega_\ell(L_j^{\#}) = \mathrm{d}x_{n+\ell}(L_j^{\#}) - c_{j\ell} = 0.$$

Hence $\{\omega_{1p}, \dots, \omega_{mp}\}$ is a basis for $\mathcal{V}_p^\perp$ for each $p \in U_0$. □

Remark I.4.5. It is clear that the preceding argument can be reversed. If $\mathcal{V}^\perp$ is a vector sub-bundle of $\mathbb{C}T^*\Omega$ then it follows that $\mathcal{V}$ is a vector sub-bundle of $\mathcal{V}$.

When $\mathcal{V}$ is a formally integrable structure over Ω of dimension N we shall always denote the sub-bundle $\mathcal{V}^\perp$ by T'. We shall also always denote by n the rank of $\mathcal{V}$ and by m the rank of T'. In particular, $n + m = N$.

We shall also use the standard notation:

$$T_p\Omega \doteq \{\mathrm{v} \in \mathbb{C}T_p\Omega : \mathrm{v} \text{ is real}\};$$

$$T_p^*\Omega \doteq \{\xi \in \mathbb{C}T_p^*\Omega : \xi \text{ is real}\};$$

$$T\Omega \doteq \bigcup_{p\in\Omega} T_p\Omega;$$

$$T^*\Omega \doteq \bigcup_{p\in\Omega} T_p^*\Omega.$$

Given $L \in \mathfrak{X}(\Omega)$ its *(complex)-conjugate* is the vector field $\overline{L} \in \mathfrak{X}(\Omega)$ defined by

$$\overline{L}(f) = \overline{L(\overline{f})}, \quad f \in C^\infty(\Omega).$$

In particular we shall say that L is a *real vector field* if $L = \overline{L}$, that is, if $LC^\infty(\Omega, \mathbb{R}) \subset C^\infty(\Omega, \mathbb{R})$. In the same way we can define the (complex)-conjugate of an element in $\mathbb{C}T_p\Omega$. Given a subspace $\mathcal{V}_p \subset \mathbb{C}T_p\Omega$ we define

$$\overline{\mathcal{V}}_p \doteq \{\overline{\mathrm{v}} : \mathrm{v} \in \mathcal{V}_p\}.$$

It is clear from the definitions that if $\mathcal{V}$ is a complex vector sub-bundle of $\mathbb{C}T\Omega$ then the same is true for $\overline{\mathcal{V}} \doteq \cup_{p\in\Omega}\overline{\mathcal{V}}_p$. We shall refer to $\overline{\mathcal{V}}$ as the *(complex)-conjugate* of the sub-bundle $\mathcal{V}$. Analogous definitions and results can be introduced and obtained for $\mathbb{C}T^*\Omega$ and its fibers $\mathbb{C}T_p^*\Omega$. It is also important to mention the equality

$$\overline{\mathcal{V}}^\perp = \overline{\mathcal{V}^\perp},$$

which is valid for every complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T\Omega$.

I.5 The Frobenius theorem

We start by considering a *real* vector field

$$L = \sum_{j=1}^N a_j(x)\frac{\partial}{\partial x_j}$$

defined in a neighborhood of the origin in $\mathbb{R}^N$. Assume that $L \neq 0$. Then it is possible to find local coordinates $y_1, y_2, \ldots, y_N$, defined near the origin, such that

$$L = \frac{\partial}{\partial y_1}.$$

The proof of this result is very simple and will be recalled here.

We assume that $a_1(0) \neq 0$ and solve, in some neighborhood of the origin, the following Cauchy problem:

$$\begin{cases} \partial x_j/\partial y_1 = a_j(x_1, \ldots, x_N) & j = 1, \ldots, N \\ x_1(0, y_2, \ldots, y_N) = 0 & \\ x_j(0, y_2, \ldots, y_N) = y_j & j = 2, \ldots, N. \end{cases}$$

The fact that $a_1(0) \neq 0$ implies that $(y_1, \dots, y_N) \mapsto (x_1, \dots, x_N)$ is a smooth diffeomorphism at the origin and a simple computation shows our claim.

The generalization of this result to a larger number of vector fields is the classical *Frobenius theorem*:

THEOREM I.5.1. *Let $L_1, \dots, L_n$ be linearly independent, real vector fields defined in a neighborhood V of the origin in $\mathbb{R}^N$. Assume that the sub-bundle $\mathcal{V}$ of $\mathbb{C}TV$ generated by $L_1, \dots, L_n$ is a formally integrable structure. Then there are local coordinates $y_1, y_2, \dots, y_N$, defined near the origin, such that $\mathcal{V}$ is generated by $\partial/\partial y_1$,..., $\partial/\partial y_n$.*

PROOF. We shall proceed by induction on N. The case $N = 1$ is trivial. We then suppose that the result was proved for values $< N$. Applying the procedure described at the beginning of this section we can make a change of variables and assume that the given vector fields have the form:

$$L_1 = \frac{\partial}{\partial x_1}, \;\; L_j = \sum_{k=1}^{N} a_{jk} \frac{\partial}{\partial x_k}, \;\; j = 2, \dots, n.$$

We then introduce a new set of generators for the bundle $\mathcal{V}$:

$$L_1^{\#} = L_1, \;\; L_j^{\#} = L_j - a_{j1} L_1, \;\; j = 2, \dots, n.$$

Notice that when $j \geq 2$ the vector field $L_j^{\#}$ does not involve differentiation in the x_1-variable. Thus, in a neighborhood of the origin, we have

$$[L_j^{\#}, L_k^{\#}] = \sum_{\nu=2}^{n} C_{jk}^{\nu} L_\nu^{\#}, \quad j, k = 2, \dots, n.$$

If we then consider, in a neighborhood W of the origin in $\mathbb{R}^{N-1}$, the vector fields

$$M_j = \sum_{k=2}^{N} a_{jk}(0, x_2, \dots, x_N) \frac{\partial}{\partial x_k}, \;\; j = 2, \dots, n,$$

as well as the sub-bundle $\mathcal{V}'$ of $\mathbb{C}TW$ defined by them, we conclude the existence of a coordinate system $y_2, \dots, y_N$ defined near the origin in $\mathbb{R}^{N-1}$ for which $\mathcal{V}'$ is spanned by

$$\partial/\partial y_2, \dots, \partial/\partial y_n.$$

This argument has the following consequence: returning to the original coordinates $(x_1, \dots, x_N)$, the induction hypothesis allows us to assume from the beginning that

$$a_{jk}(0, x_2, \dots, x_N) = 0, \quad j = 2, \dots, n, \;\; k > n.$$

Now, the coefficient of $\partial/\partial x_\ell$ in the commutator $[L_1^\#, L_j^\#]$ is equal to $\partial a_{j\ell}/\partial x_1$. On the other hand,

$$[L_1^\#, L_j^\#] = \sum_{\nu=1}^{n} C_{1j}^{\nu} L_\nu^\# = C_{1j}^{1}\frac{\partial}{\partial x_1} + \sum_{\nu=2}^{n} C_{1j}^{\nu} \sum_{k=2}^{N} a_{\nu k}\frac{\partial}{\partial x_k},$$

and thus

$$\frac{\partial a_{j\ell}}{\partial x_1} = \sum_{\nu=2}^{n} C_{1j}^{\nu} a_{\nu\ell}, \quad \ell = 2, \ldots, N,\ j = 2, \ldots, n.$$

Hence for each fixed ℓ the vector $(a_{2\ell}, \ldots, a_{n\ell})$ satisfies a linear system of ordinary differential equations with trivial initial condition. By the uniqueness theorem for such systems we conclude that $a_{j\ell} = 0$ if $j = 2, \ldots, n$ and $\ell > n$. Thus we have

$$L_j^\# = \sum_{k=2}^{n} a_{jk}\frac{\partial}{\partial x_k}, \quad j = 2, \ldots, n,$$

which conclude the proof. $\square$

We now discuss the holomorphic version of the Frobenius theorem. Write the complex coordinates in $\mathbb{C}^\mu$ as $z_1, \ldots, z_\mu$, where $z_j = x_j + iy_j$, and identify $\mathbb{C}^\mu \simeq \mathbb{R}^{2\mu}$ by

$$z = (z_1, \ldots, z_\mu) \mapsto (x_1, y_1, \ldots, x_\mu, y_\mu).$$

Given an open set $\Omega \subset \mathbb{C}^\mu$ denote by $\mathcal{O}(\Omega)$ the algebra of holomorphic functions on Ω. An element $L \in \mathfrak{X}(\Omega)$ is said to be a *holomorphic vector field* if given any $f \in \mathcal{O}(\Omega)$ we have $Lf \in \mathcal{O}(\Omega)$ and $L\overline{f} = 0$. Introducing the standard notation

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left\{\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right\}, \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left\{\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right\}$$

it is clear that every vector field $L \in \mathfrak{X}(\Omega)$ can be written as

$$L = \sum_{j=1}^{\mu}\left\{a_j\frac{\partial}{\partial z_j} + b_j\frac{\partial}{\partial \overline{z}_j}\right\}, \tag{I.11}$$

where $a_j, b_j \in C^\infty(\Omega)$; (I.11) is then a holomorphic vector field if and only if $b_j = 0$ and $a_j \in \mathcal{O}(\Omega)$, $j = 1, \ldots, \mu$.

We now state the holomorphic version of the Frobenius theorem, whose proof is the same as that of Theorem I.5.1, working now in the holomorphic category.

THEOREM I.5.2. *Let $L_1, \dots, L_n$ be linearly independent, holomorphic vector fields defined in a neighborhood V of the origin in $\mathbb{C}^\mu$. Assume that the sub-bundle $\mathcal{V}$ of $\mathbb{C}TV$ generated by $L_1, \dots, L_n$ is a formally integrable structure. Then there are local holomorphic coordinates $w_1, w_2, \dots, w_\mu$, defined near the origin in $\mathbb{C}^\mu$ such that $\mathcal{V}$ is generated by $\partial/\partial w_1,\dots, \partial/\partial w_n$.*

I.6 Analytic structures

Let Ω be a real-analytic manifold, defined by the differentiable (real-analytic) structure $\mathcal{F} = \{(V, \mathbf{x})\}$. A function $f : \Omega \to \mathbb{C}$ is *real-analytic* if for every $(V, \mathbf{x}) \in \mathcal{F}$ the composition $f \circ \mathbf{x}^{-1}$ is *real-analytic* on $\mathbf{x}(V)$. Given $U \subset \Omega$ an open set, we shall denote by $\mathcal{A}(U)$ the space of real-analytic functions on U. An element $L \in \mathfrak{X}(\Omega)$ is said to be a *real-analytic vector field on* Ω if

$$L\mathcal{A}(U) \subset \mathcal{A}(U), \quad \forall U \subset \Omega \text{ open.}$$

If L is given in local coordinates as in (I.6) then L is real-analytic if and only if its coefficients Lx_j, $j = 1, \dots, N$, are real-analytic functions.

Analogously, we shall say that $\omega \in \mathfrak{N}(\Omega)$ is a *real-analytic one-form* on Ω if $\omega(L) \in \mathcal{A}(U)$ for every $U \subset \Omega$ open and every real-analytic vector field L.

From such definitions it is clear that one can introduce the notions of complex *analytic* vector sub-bundles of $\mathbb{C}T\Omega$ and of $\mathbb{C}T^*\Omega$; in particular we can refer to the notion of an *analytic* formally integrable structure over Ω.

REMARK I.6.1. Suppose that Ω is now an open subset of $\mathbb{R}^N$ and let $L \in \mathfrak{X}(\Omega)$ be real-analytic. Write

$$L = \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j}.$$

Let also $u \in \mathcal{A}(\Omega)$ and take an open set $\Omega^{\mathbb{C}} \subset \mathbb{C}^N$, where the holomorphic coordinates are written as $(z_1, \dots, z_N)$, such that

- $\Omega^{\mathbb{C}} \cap \mathbb{R}^N = \Omega$;
- u, a_j extend as holomorphic functions $\tilde{u}$, $\tilde{a}_j$ on $\Omega^{\mathbb{C}}$.

Then

$$Lu = (\tilde{L}\tilde{u})|_{\Omega}, \tag{I.12}$$

where $\tilde{L}$ is the holomorphic vector field

$$\tilde{L} = \sum_{j=1}^{N} \tilde{a}_j(z) \frac{\partial}{\partial z_j}.$$

I.7 The characteristic set

Let $\mathcal{V} \subset \mathbb{C}T\Omega$ be a formally integrable structure over Ω. The *characteristic set of* $\mathcal{V}$ is the subset of $T^*\Omega$ defined by

$$T^0 \doteq T' \cap T^*\Omega. \tag{I.13}$$

We shall also write $T^0_p = T'_p \cap T^*_p\Omega$ if $p \in \Omega$. If we recall that the *symbol* of a vector field $L \in \mathfrak{X}(\Omega)$ is the function

$$\sigma(L) : T^*\Omega \to \mathbb{C}, \quad \sigma(L)(\xi) = \xi(L_p) \quad \text{if } \xi \in T^*_p\Omega,$$

then we see that $\xi \in T^0_p$ if and only if $\sigma(L)(\xi) = 0$ for every section L of $\mathcal{V}$.

Let $(U, \mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_N)$ be a local chart on Ω. Take $p \in U$ and $\xi \in T^*_p\Omega$. If we write $\xi = \sum_{j=1}^N \xi_j \mathrm{d}x_{jp}$ $(\xi_j \in \mathbb{R})$ and $L = \sum_{j=1}^N a_j(\partial/\partial x_j)$ then

$$\sigma(L)(\xi) = \sum_{j=1}^N a_j(p)\xi_j.$$

Thus, if $L_j = \sum_{k=1}^N a_{jk}(\partial/\partial x_k)$ are n linearly independent sections of $\mathcal{V}$ over U we can describe $T^0 \cap T^*U$ by the system of equations

$$\sum_{k=1}^N a_{jk}(p)\xi_k = 0, \quad p \in U,\ \xi_k \in \mathbb{R},\ j = 1, \ldots, n.$$

EXAMPLE I.7.1 (The Mizohata operator). If we write the coordinates in $\Omega = \mathbb{R}^2$ as (x, t) then

$$M \doteq \frac{\partial}{\partial t} - it\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2) \tag{I.14}$$

is called the *Mizohata vector field* or *Mizohata operator*. We now describe the characteristic set of the formally integrable structure defined by M. From the equation $\tau - it\xi = 0$ we get

$$T^0_{(x,t)} = \begin{cases} 0 & \text{if } t \neq 0 \\ \{\xi \mathrm{d}x_p : \xi \in \mathbb{R}\} & \text{if } p = (x, 0). \end{cases}$$

This example in particular shows that T^0 is not, in general, a vector sub-bundle of $T^*\Omega$.

I.8 Some special structures

Let $\mathcal{V}$ be a formally integrable structure over Ω. We shall say that $\mathcal{V}$ defines

- an *elliptic structure* if $T^0_p = 0$, $\forall p \in \Omega$;
- a *complex structure* if $\mathcal{V}_p \oplus \overline{\mathcal{V}}_p = \mathbb{C}T_p\Omega$, $\forall p \in \Omega$;

- a *Cauchy–Riemann (CR) structure* if $\mathcal{V}_p \cap \overline{\mathcal{V}}_p = 0$, $\forall p \in \Omega$;
- an *essentially real structure* if $\mathcal{V}_p = \overline{\mathcal{V}}_p$, $\forall p \in \Omega$.

Before we proceed further we state some easy consequences of the preceding definitions.

PROPOSITION I.8.1. *Every essentially real structure is locally generated by real vector fields.*

PROOF. Given $p_0 \in \Omega$ we take vector fields $L_1, \dots, L_n$ which generate $\mathcal{V}$ in a neighborhood of p_0. By hypothesis the real vector fields $\Re L_j$, $\Im L_j$ are also sections of $\mathcal{V}$. Moreover,

$$\operatorname{span}\{(\Re L_j)_{p_0}, (\Im L_j)_{p_0} : j = 1, \dots, n\} = \mathcal{V}_{p_0}$$

and consequently n of the tangent vectors $(\Re L_j)_{p_0}$, $(\Im L_j)_{p_0}$ are linearly independent. Since this remains true in a neighborhood of p_0 the result is proved. $\square$

Next we recall a very elementary but useful result.

LEMMA I.8.2. *If V is a vector subspace of $\mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N$ and if $V^0 = V \cap \mathbb{R}^N$ then $V^0 \otimes_{\mathbb{R}} \mathbb{C} \simeq V^0 + iV^0 = V \cap \overline{V}$.*

PROOF. We only verify the equality. If $x, y \in V^0$ then $x \pm iy \in V$ and so $V^0 + iV^0 \subset V \cap \overline{V}$. For the reverse inclusion take $z \in V \cap \overline{V}$. Then

$$z = \frac{1}{2}(z + \overline{z}) - \frac{i}{2}(iz - i\overline{z}) \in V^0 + iV^0. \qquad \square$$

As a consequence, given any formally integrable structure $\mathcal{V}$ over Ω we have

$$T_p^0 \otimes_{\mathbb{R}} \mathbb{C} \simeq T'_p \cap \overline{T}'_p, \quad \forall p \in \Omega. \tag{I.15}$$

Since for a complex structure we also have

$$T'_p \oplus \overline{T}'_p = \mathbb{C}T^*_p\Omega, \quad \forall p \in \Omega$$

we obtain:

COROLLARY I.8.3. *Every complex structure is elliptic.*

Unlike what happens with Mizohata structures we have:

PROPOSITION I.8.4. *If $\mathcal{V}$ defines a CR structure over Ω then T^0 is a vector sub-bundle of $T^*\Omega$ of rank $d \doteq N - 2n$.*

PROOF. If $\mathcal{V}_p \cap \overline{\mathcal{V}}_p = 0$, for all $p \in \Omega$, then

$$\mathcal{W} = \mathcal{V} \oplus \overline{\mathcal{V}} \doteq \bigcup_{p \in \Omega} (\mathcal{V}_p \oplus \overline{\mathcal{V}}_p)$$

is a vector sub-bundle of $\mathbb{C}T\Omega$ (of rank $2n$) which defines an essentially real structure over Ω. By Proposition I.4.4, $\mathcal{W}^\perp$ is a vector sub-bundle of $\mathbb{C}T^*\Omega$ of rank d which of course satisfies $\mathcal{W}_p^\perp = \overline{\mathcal{W}_p^\perp}$ for all $p \in \Omega$. The same argument used in the proof of Proposition I.8.1 shows that $\mathcal{W}^\perp$ has local *real* generators. Since these generators span T^0 the proof is complete. □

In order to obtain appropriate local generators for a formally integrable structure we shall need an elementary result:

LEMMA I.8.5. *Let V be a complex subspace of $\mathbb{C}^N$ of dimension m. Let $V_0 = V \cap \mathbb{R}^N$, $d \doteq \dim_{\mathbb{R}} V_0$, $\nu \doteq m - d$. Let also $V_1 \subset \mathbb{C}^N$ be a subspace such that $(V_0 \oplus iV_0) \oplus V_1 = V$ and take:*

$$\{\zeta_1, \ldots, \zeta_\nu\} : \textit{ basis for } V_1; \quad \{\xi_{\nu+1}, \ldots, \xi_m\} : \textit{ real basis for } V_0.$$

If we write $\zeta_j = \xi_j + i\eta_j$, $j = 1, \ldots, \nu$, then:

$$\{\zeta_1, \ldots, \zeta_\nu, \xi_{\nu+1}, \ldots, \xi_m\} \quad \textit{is a basis for } V; \tag{I.16}$$

$$\{\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_\nu\} \quad \textit{is linearly independent over } \mathbb{R}; \tag{I.17}$$

$$\nu + m \le N. \tag{I.18}$$

PROOF. Notice that (I.16) is trivial since $\{\xi_{\nu+1}, \ldots, \xi_m\}$ is also a basis for $V_0 \oplus iV_0$.

Next we notice that $V \cap \overline{V}_1 = 0$. Indeed, let $z \in V \cap \overline{V}_1$. Then $\overline{z} \in V_1 \subset V$ and consequently $\Re z, \Im z \in V_0$, which gives $\overline{z} \in (V_0 \oplus iV_0) \cap V_1 = 0$. Hence

$$\{\zeta_1, \ldots, \zeta_\nu, \overline{\zeta}_1, \ldots, \overline{\zeta}_\nu, \xi_{\nu+1}, \ldots, \xi_m\}$$

is linearly independent. In particular, $2\nu + d = \nu + m \le N$ and (I.17) holds. □

Given a formally integrable structure $\mathcal{V}$ over Ω and fixing $p \in \Omega$ we shall apply Lemma I.8.5 with the choices

$$V = T'_p, \quad V_0 = T^0_p.$$

If $\{\zeta_1, \ldots, \zeta_\nu, \xi_{\nu+1}, \ldots, \xi_m\}$ is the basis given in (I.16) we first take a system of local coordinates

$$x_1, \ldots, x_\nu, y_1, \ldots, y_\nu, s_1, \ldots, s_d, t_1, \ldots, t_{n'}$$

vanishing at p such that, writing $z_j = x_j + iy_j$ we have

$$\mathrm{d}z_j|_p = \zeta_j, \ \mathrm{d}s_k|_p = \xi_{\nu+k}, \quad j = 1, \ldots, \nu, \ k = 1, \ldots, d.$$

Afterwards we take one-forms $\omega_1, \ldots, \omega_\nu, \theta_1, \ldots, \theta_d$ which spanT' in a neighborhood of p and such that

$$\omega_j|_p = \mathrm{d}z_j|_p, \quad \theta_k|_p = \mathrm{d}s_k|_p, \quad j = 1, \ldots, \nu, \ k = 1, \ldots, d.$$

If L is a complex vector field on Ω defined near p we can write it in the form

$$L = \sum_j A_j \frac{\partial}{\partial z_j} + \sum_j B_j \frac{\partial}{\partial \overline{z}_j} + \sum_k C_k \frac{\partial}{\partial s_k} + \sum_\ell D_\ell \frac{\partial}{\partial t_\ell}.$$

If, furthermore, L is a section of $\mathcal{V}$ we necessarily must have $A_j = C_k = 0$ at p for all j and k. Since $\nu + n' = n$, it follows that after a linear substitution we can find a set of local generators of the sub-bundle $\mathcal{V}$ in a neighborhood of p of the form

$$L_j = \frac{\partial}{\partial \overline{z}_j} + \sum_{j'=1}^{\nu} a_{jj'} \frac{\partial}{\partial z_{j'}} + \sum_{k=1}^{d} b_{jk} \frac{\partial}{\partial s_k}, \quad j = 1, \ldots, \nu, \tag{I.19}$$

$$\tilde{L}_\ell = \frac{\partial}{\partial t_\ell} + \sum_{j'=1}^{\nu} \tilde{a}_{\ell j'} \frac{\partial}{\partial z_{j'}} + \sum_{k=1}^{d} \tilde{b}_{\ell k} \frac{\partial}{\partial s_k}, \quad \ell = 1, \ldots, n', \tag{I.20}$$

where the coefficients $a_{jj'}$, $\tilde{a}_{\ell j'}$, b_{jk}, $\tilde{b}_{\ell k}$ all vanish at p.

We notice that the elliptic case corresponds to the situation when $d = 0$, the complex case to the one when $d = n' = 0$, and the CR case to the one when $n' = 0$.

Next we introduce a generalization of the structure defined by the Mizohata operator (*cf.* Example I.7.1).

DEFINITION I.8.6. *We shall say that a formally integrable structure $\mathcal{V}$ over Ω is a generalized Mizohata structure at $p_0 \in \Omega$ if $\mathcal{V}_{p_0} = \overline{\mathcal{V}_{p_0}}$.*

Thus in the case of generalized Mizohata structures the coordinates vanishing at p_0 can be taken as $(s_1, \ldots, s_m, t_1, \ldots, t_n)$ [$d = m$, $n = n'$ in this case] and $\mathcal{V}$ is spanned by the vector fields

$$L_\ell = \frac{\partial}{\partial t_\ell} + \sum_{k=1}^{d} \tilde{b}_{\ell k}(s,t) \frac{\partial}{\partial s_k}, \quad \ell = 1, \ldots, n,$$

where $b_{\ell k} = 0$ at the origin for every ℓ, k.

Finally we recall the classical notion of the so-called *CR functions*:

Definition I.8.7. *Given a CR formally integrable structure $\mathcal{V}$ over Ω, any classical solution (for the formally integrable structure $\mathcal{V}$) is called a CR function.*

Needless to add, we can also introduce the concept of CR distributions, etc.

I.9 Locally integrable structures

A complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T\Omega$, of rank n, is said to define a *locally integrable structure* if given an arbitrary point $p_0 \in \Omega$ there are an open neighborhood U_0 of p_0 and functions $Z_1, \ldots, Z_m \in C^\infty(U_0)$, with $m = N - n$, such that

$$\operatorname{span}\{\mathrm{d}Z_{1p}, \ldots, \mathrm{d}Z_{mp}\} = \mathcal{V}_p^\perp, \quad \forall p \in U_0. \tag{I.21}$$

If one observes that the differential of a smooth function g is a section of $\mathcal{V}^\perp$ if and only if $Lg = 0$ for every section of $\mathcal{V}$, it follows easily that every locally integrable structure satisfies the Frobenius condition. Hence, *every locally integrable structure defines a formally integrable structure*.

We have:

- *The formally integrable structure $\mathcal{V}$ is locally integrable if and only if, given $p_0 \in \Omega$ and vector fields $L_1, \ldots, L_n$ which span $\mathcal{V}$ in an open neighborhood U_0 of p_0, there are an open neighborhood $V_0 \subset U_0$ of p_0 and smooth functions $Z_1, \ldots, Z_m \in C^\infty(V_0)$ such that*:

$$\mathrm{d}Z_1 \wedge \ldots \wedge \mathrm{d}Z_m \neq 0 \quad \text{in} \quad V_0;$$

$$L_j Z_k = 0, \quad j = 1, \ldots, n,\ k = 1, \ldots, m.$$

Thus, checking local integrability is equivalent to looking for a maximal number of nontrivial solutions to the (in general overdetermined) homogeneous system defined by a fixed set of independent sections of $\mathcal{V}$.

Theorem I.9.1. *Every essentially real structure is locally integrable.*

Proof. By Frobenius Theorem I.5.1, in conjunction with Proposition I.8.1, given $p \in \Omega$ we can find a local chart $(U, \mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_N)$, with $p \in U$, such that

$$\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n$$

are sections of $\mathcal{V}$ over U. It suffices to take

$$Z_k = x_{k+n}, \quad k = 1, \dots, m.$$

□

THEOREM I.9.2. *Every analytic formally integrable structure is locally integrable.*

PROOF. We shall prove that if $L_1, \dots, L_n$ are linearly independent, real-analytic vector fields in an open ball B centered at the origin in $\mathbb{R}^N$ such that

$$[L_\alpha, L_\beta] = \sum_{\gamma=1}^{n} C^\gamma_{\alpha\beta} L_\gamma,$$

where $C^\gamma_{\alpha\beta} \in \mathcal{A}(B)$, then we can find real-analytic functions $Z_1, \dots, Z_m$ defined in a neighborhood of the origin and satisfying

$$L_j Z_\ell = 0, \quad j = 1, \dots, n, \ \ell = 1, \dots, m;$$

$$\mathrm{d}Z_1 \wedge \dots \wedge \mathrm{d}Z_m \neq 0.$$

We write

$$L_j = \sum_{k=1}^{N} a_{jk} \frac{\partial}{\partial x_k}$$

and take an open, connected set $U \subset \mathbb{C}^N$ such that $U \cap \mathbb{R}^N = B$ and such that there are $\tilde{a}_{jk}$, $\tilde{C}^\gamma_{\alpha\beta} \in \mathcal{O}(U)$ satisfying

$$\tilde{a}_{jk} = a_{jk}, \quad \tilde{C}^\gamma_{\alpha\beta} = C^\gamma_{\alpha\beta} \quad \text{in } B.$$

Consider then the holomorphic vector fields in U:

$$\tilde{L}_j = \sum_{k=1}^{N} \tilde{a}_{jk} \frac{\partial}{\partial z_k}.$$

By analytic continuation the coefficients of the holomorphic vector fields

$$[\tilde{L}_\alpha, \tilde{L}_\beta] - \sum_{\gamma=1}^{n} \tilde{C}^\gamma_{\alpha\beta} \tilde{L}_\gamma$$

must vanish identically in U since they vanish on B and the former is connected. By the holomorphic version of the Frobenius theorem we can find holomorphic functions $W_1, \dots, W_m$ defined in an open neighborhood $V \subset U$ of the origin in $\mathbb{C}^N$ such that

$$\tilde{L}_j W_\ell = 0, \quad j = 1, \dots, n, \ \ell = 1, \dots, m;$$

$$\mathrm{d}W_1 \wedge \ldots \wedge \mathrm{d}W_m \neq 0.$$

It suffices then to set $Z_k \doteq W_k|_{V \cap B}$ in order to obtain the desired solutions (*cf.* (I.12)). □

EXAMPLE I.9.3. For the Mizohata vector field (I.14) we have $MZ = 0$ in $\mathbb{R}^2$, where $Z(x, t) = x + it^2/2$. Notice that $\mathrm{d}Z \neq 0$ everywhere.

I.10 Local generators

In this section we shall construct appropriate local coordinates and local generators of the sub-bundle T' when the structure $\mathcal{V}$ is locally integrable. Once more we shall apply Lemma I.8.5.

Let $p \in \Omega$ and let also $G_1, \ldots, G_m$ be smooth functions defined in a neighborhood of p such that $\mathrm{d}G_1, \ldots, \mathrm{d}G_m$ span T'. As in Section I.8 we make the choices: $V = T'_p$, $V_0 = T^0_p$. If $\{\zeta_1, \ldots, \zeta_\nu, \xi_{\nu+1}, \ldots, \xi_m\}$ is the basis given in (I.16) then we can find $(c_{jk}) \in \mathrm{GL}(m, \mathbb{C})$ such that

$$\sum_{k=1}^{m} c_{jk} \mathrm{d}G_k(p) = \zeta_j, \quad j = 1, \ldots, \nu,$$

$$\sum_{k=1}^{m} c_{jk} \mathrm{d}G_k(p) = \xi_j, \quad j = \nu + 1, \ldots, m.$$

We then set

$$Z_j = \sum_{k=1}^{m} c_{jk} \{G_k - G_k(p)\}, \quad j = 1, \ldots, \nu,$$

$$W_\ell = \sum_{k=1}^{m} c_{\nu+\ell,k} \{G_k - G_k(p)\}, \quad \ell = 1, \ldots, d.$$

It is clear that $\mathrm{d}Z_1, \ldots, \mathrm{d}Z_\nu, \mathrm{d}W_1, \ldots, \mathrm{d}W_d$ also span T' in a neighborhood of p. If we further set

$$x_j = \Re Z_j, \; y_j = \Im Z_j, \; s_\ell = \Re W_\ell$$

then (I.17) gives that

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_\nu, \mathrm{d}y_1, \ldots, \mathrm{d}y_\nu, \mathrm{d}s_1, \ldots, \mathrm{d}s_d$$

are linearly independent at p. We are now ready to state and prove the following important result:

THEOREM I.10.1. *Let $\mathcal{V}$ be a locally integrable structure defined on a manifold Ω. Let $p \in \Omega$ and d be the real dimension of T_p^0. Then there is a coordinate system vanishing at p,*

$$\{x_1, \ldots, x_\nu, y_1, \ldots, y_\nu, s_1, \ldots, s_d, t_1, \ldots, t_{n'}\}$$

and smooth, real-valued functions $\phi_1, \ldots, \phi_d$ defined in a neighborhood of the origin and satisfying

$$\phi_k(0) = 0,\ \mathrm{d}\phi_k(0) = 0, \quad k = 1, \ldots, d,$$

such that the differentials of the functions

$$Z_j(x, y) = z_j \doteq x_j + iy_j, \quad j = 1, \ldots, \nu; \tag{I.22}$$

$$W_k(x, y, s, t) = s_k + i\phi_k(z, s, t), \quad k = 1, \ldots, d, \tag{I.23}$$

span T' in a neighborhood of the origin. In particular, we have $\nu + d = m$, $\nu + n' = n$ and also

$$T_p^0 = \operatorname{span}\{\mathrm{d}s_1|_0, \ldots, \mathrm{d}s_d|_0\}. \tag{I.24}$$

PROOF. The proof follows almost immediately from the preceding discussion: it suffices to take smooth, real-valued functions $t_1, \ldots, t_{n'}$ defined near p and vanishing at p such that

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_\nu, \mathrm{d}y_1, \ldots, \mathrm{d}y_\nu, \mathrm{d}s_1, \ldots, \mathrm{d}s_d, \mathrm{d}t_1, \ldots \mathrm{d}t_{n'}$$

are linearly independent. Notice that $\mathrm{d}W_k(p) = \xi_{\nu+k}$ is real, from which we derive that $\mathrm{d}\phi_k = 0$ at the origin. □

Since we have

$$\frac{\partial W_k}{\partial s_{k'}}(0, 0, 0) = \delta_{kk'}, \quad k, k' = 1, \ldots, d,$$

we can introduce, in a neighborhood of the origin in $\mathbb{R}^{2\nu+d+n'}$, the vector fields

$$M_k = \sum_{k'=1}^{d} \mu_{kk'}(z, s, t)\frac{\partial}{\partial s_{k'}}, \quad k = 1, \ldots, d \tag{I.25}$$

characterized by the relations

$$M_k W_{k'} = \delta_{kk'}. \tag{I.26}$$

Consequently the vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i\sum_{k=1}^{d} \frac{\partial \phi_k}{\partial \bar{z}_j}(z, s, t)M_k, \quad j = 1, \ldots, \nu, \tag{I.27}$$

$$\tilde{L}_\ell = \frac{\partial}{\partial t_\ell} - i \sum_{k=1}^{d} \frac{\partial \phi_k}{\partial t_\ell}(z, s, t) M_k, \quad \ell = 1, \ldots, n' \tag{I.28}$$

are linearly independent and satisfy

$$L_j Z_{j'} = \tilde{L}_\ell Z_{j'} = L_j W_k = \tilde{L}_\ell W_k = 0$$

for all $j, j' = 1, \ldots, \nu$, $\ell = 1, \ldots, n'$, and $k = 1, \ldots, d$. Hence

$$L_1, \ldots, L_\nu, \tilde{L}_1, \ldots, \tilde{L}_{n'} \text{ span } \mathcal{V} \text{ in a neighborhood of the origin.} \tag{I.29}$$

Notice that the one-forms

$$\mathrm{d}z_1, \ldots, \mathrm{d}z_\nu, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_\nu, \mathrm{d}W_1, \ldots, \mathrm{d}W_d, \mathrm{d}t_1, \ldots, \mathrm{d}t_{n'} \tag{I.30}$$

span $\mathbb{C}T^*\Omega$ near the origin. Moreover, the dual basis of (I.30) is given by

$$L_1^\flat, \ldots, L_\nu^\flat, L_1, \ldots, L_\nu, M_1, \ldots, M_d, \tilde{L}_1, \ldots, \tilde{L}_{n'}, \tag{I.31}$$

where

$$L_j^\flat = \frac{\partial}{\partial z_j} - i \sum_{k=1}^{d} \frac{\partial \phi_k}{\partial z_j}(z, s, t) M_k, \quad j = 1, \ldots, \nu. \tag{I.32}$$

Finally we observe that

$$\text{the vector fields (I.31) are pairwise commuting.} \tag{I.33}$$

Indeed it suffices to notice that if P, Q are any two of the vector fields (I.31) and if F is any one of the functions $\{Z_j, \overline{Z_j}, W_k, t_\ell\}$, the fact that (I.30) is dual to (I.31) gives

$$\mathrm{d}F\,([P, Q]) = [P, Q](F) = 0,$$

from which we obtain that $[P, Q] = 0$.

In many cases we do not need the precise information provided by Theorem I.10.1 and the following particular case is enough:

Corollary I.10.2. *Same hypotheses as in Theorem I.10.1. Then there is a coordinate system vanishing at p,*

$$\{x_1, \ldots, x_m, t_1, \ldots, t_n\}$$

and smooth, real-valued $\phi_1, \ldots, \phi_m$ defined in a neighborhood of the origin and satisfying

$$\phi_k(0, 0) = 0, \quad \mathrm{d}_x \phi_k(0, 0) = 0, \quad k = 1, \ldots, m,$$

such that the differentials of the functions

$$Z_k(x, t) = x_k + i\phi_k(x, t), \quad k = 1, \ldots, m, \tag{I.34}$$

span T' in a neighborhood of the origin. □

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If we write $Z(x, t) = (Z_1(x, t), \dots, Z_m(x, t))$ then $Z_x(0, 0)$ equals the identity $m \times m$ matrix. Hence we can introduce, in a neighborhood of the origin in $\mathbb{R}^N$, the vector fields

$$M_k = \sum_{\ell=1}^{m} \mu_{k\ell}(x, t) \frac{\partial}{\partial x_\ell}, \quad k = 1, \dots, m \tag{I.35}$$

characterized by the relations

$$M_k Z_\ell = \delta_{k\ell}. \tag{I.36}$$

Consequently the vector fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x, t) M_k, \quad j = 1, \dots, n \tag{I.37}$$

are linearly independent and satisfy $L_j Z_k = 0$, for $j = 1, \dots, n$, $k = 1, \dots, m$. The same argument as before gives:

$L_1, \dots, L_n$ span $\mathcal{V}$ in a neighborhood of the origin; (I.38)

$L_1, \dots, L_n, M_1, \dots, M_m$ are pairwise commuting and span $\mathbb{C}T\mathbb{R}^N$ in a neighborhood of the origin in $\mathbb{R}^N$. (I.39)

Let U be an open set of $\mathbb{R}^n$ and assume, given a smooth function $\Phi : U \to \mathbb{R}^m$, $\Phi(t) = (\phi_1(t), \dots, \phi_m(t))$. We shall call a *tube structure* on $\mathbb{R}^m \times U$ the locally integrable structure $\mathcal{V}$ on $\mathbb{R}^m \times U$ for which T' is spanned by the differentials of the functions

$$Z_k = x_k + i\phi_k(t), \quad k = 1, \dots, m.$$

A tube structure $\mathcal{V}$ has remarkably simple global generators. Indeed if we set, as usual, $Z = (Z_1, \dots, Z_m)$ we have $Z_x(x, t) = I$, the identity $m \times m$ matrix, for *every* $(x, t) \in \mathbb{R}^m \times U$. This gives $M_k = \partial/\partial x_k$ and consequently the vector fields (I.37) take the form

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(t) \frac{\partial}{\partial x_k} \quad j = 1, \dots, n. \tag{I.37$'$}$$

Observe that these vector fields span $\mathcal{V}$ on $\mathbb{R}^m \times U$.

I.11 Local generators in analytic structures

When $\mathcal{V}$ is real-analytic then the functions ϕ_k in Corollary I.10.2 can be taken real-analytic. We keep the notation established in the preceding section and consider the equation

$$Z(x, t) - z = 0$$

for $(x, t, z) \in \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^m$ in a neighborhood of the origin. Since

$$\frac{\partial Z}{\partial x}(0, 0) = I$$

we can find, by the implicit function theorem, a holomorphic function $x = H(z, t) = (H_1(z, t), \ldots, H_m(z, t))$ defined in a neighborhood of the origin in $\mathbb{C}^m \times \mathbb{C}^n$ satisfying

$$H(0, 0) = 0, \quad H(Z(x, t), t) = x.$$

We set

$$Z_k^\#(x, t) \doteq H_k(Z(x, t), 0), \quad k = 1, \ldots, m.$$

Then we also have

$$L_j Z_k^\# = 0, \quad j = 1, \ldots, n, k = 1, \ldots, m,$$
$$\mathrm{d}Z_1^\# \wedge \ldots \wedge \mathrm{d}Z_m^\# \neq 0.$$

Moreover, $Z_k^\#(x, 0) = x_k$ for every k. Hence, if we consider the real-analytic diffeomorphism

$$(x, t) \mapsto (X, T) = (\Re Z^\#(x, t), t)$$

in these new variables we can write $Z_k^\#(X, T) = X_k + i\Phi_k^\#(X, T)$ where now we have $\Phi_k^\#(X, 0) = 0$ for every k. Summing up we can state:

Corollary I.11.1. *Let $\mathcal{V}$ be a locally integrable real-analytic structure defined on a real-analytic manifold Ω. Let $p \in \Omega$. Then there is a real-analytic coordinate system vanishing at p,*

$$\{x_1, \ldots, x_m, t_1, \ldots, t_n\}$$

and real-analytic, real-valued $\phi_1, \ldots, \phi_m$ defined in a neighborhood of the origin and satisfying

$$\phi_k(x, 0) = 0, \quad k = 1, \ldots, m$$

such that the differentials of the functions

$$Z_k = x_k + i\phi_k(x, t), \quad k = 1, \ldots, m \tag{I.40}$$

span T' in a neighborhood of the origin.

Remark I.11.2. We point out that in the coordinates (x, t) given by Corollary I.11.1 it is elementary to find the unique analytic solution u to the Cauchy problem:

$$\begin{cases} L_j u = 0 & j = 1, \dots, n, \\ u(x, 0) = h(x) \end{cases} \tag{I.41}$$

where h is real-analytic. Indeed,

$$u(x, t) = h(Z_1^{\#}(x, t), \dots, Z_m^{\#}(x, t))$$

solves (I.41) and in order to see that this is the unique analytic solution it suffices to notice that if v is analytic, if $v(x, 0) = 0$, and if $L_j v = 0$ for every j then v must vanish identically since all its derivatives vanish at the origin.

Uniqueness for the distribution solutions of (I.41) holds when the structure is only C^∞. This, though, is a much deeper result and its discussion will be postponed to Chapter II.

I.12 Integrability of complex and elliptic structures

The celebrated theorem of Newlander and Nirenberg ([**NN**]) states that *every complex structure is locally integrable*. We shall postpone the proof of this result to the appendix of this chapter and now we will apply it to prove the more general statement that in fact every elliptic structure is locally integrable. This result is due to L. Nirenberg.

Theorem I.12.1. *Let $\mathcal{V}$ be an elliptic structure over a smooth manifold Ω. Then $\mathcal{V}$ is locally integrable.*

Proof. By (I.15) we have $T'_p \cap \overline{T}'_p = 0$ for every $p \in \Omega$ and then

$$T' \oplus \overline{T}' \doteq \bigcup_{p \in \Omega} \left(T'_p \oplus \overline{T}'_p \right)$$

is a vector sub-bundle of $\mathbb{C}T^*\Omega$ of rank $2m$. In particular, if n is the dimension of Ω, we obtain that $2m \le n$. Thus

$$\mathcal{V} \cap \overline{\mathcal{V}} \doteq \bigcup_{p \in \Omega} (\mathcal{V}_p \cap \overline{\mathcal{V}}_p) = \bigcup_{p \in \Omega} \left(T'_p \oplus \overline{T}'_p \right)^\perp$$

is a vector sub-bundle of $\mathbb{C}T\Omega$. By the argument that led to the proof of Proposition I.8.1 we see then that

$$\mathcal{V} \cap T\Omega \doteq \bigcup_{p \in \Omega} (\mathcal{V}_p \cap T_p\Omega)$$

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is a vector sub-bundle of $T\Omega$. Notice that

$$n' \doteq \dim_{\mathbb{R}} \left(\mathcal{V}_p \cap T_p\Omega\right) = n - 2m, \quad p \in \Omega.$$

Let $p_0 \in \Omega$ be fixed. By the Frobenius Theorem I.5.1 we can find a coordinate system $(x_1, \dots, x_{2m}, t_1, \dots, t_{n'})$ around p_0 such that $\mathcal{V} \cap T\Omega$ is generated near p_0 by the vector fields

$$\frac{\partial}{\partial t_j}, \quad j = 1, \dots, n'.$$

Next we select m complex vector fields

$$L_k = \sum_{\ell=1}^{2m} a_{k\ell}(x, t) \frac{\partial}{\partial x_\ell}$$

in such a way that $L_1, \dots, L_m, \partial/\partial t_1, \dots, \partial/\partial t_{n'}$ span $\mathcal{V}$ in a neighborhood of p_0. After a linear substitution (as in the proof of Proposition I.4.4) we can assume that the vector fields L_k take the form

$$L_k = \frac{\partial}{\partial x_k} + \sum_{\ell=m+1}^{2m} b_{k\ell}(x, t) \frac{\partial}{\partial x_\ell}, \quad k = 1, \dots, m.$$

Since $\mathcal{V}$ is a formally integrable structure, we know $[\partial/\partial t_\alpha, L_k]$ must be a linear combination of $L_1, \dots, L_m, \partial/\partial t_1, \dots, \partial/\partial t_{n'}$. Due to the special form of the vector fields L_k these brackets must vanish identically, that is:

$$\sum_{\ell=m+1}^{2m} \frac{\partial b_{k\ell}}{\partial t_\alpha}(x, t) \frac{\partial}{\partial x_\ell} = 0, \quad \forall \alpha,\ k.$$

Consequently, the functions $b_{k\ell}$ do not depend on $t_1, \dots, t_{n'}$ in a full neighborhood of p_0. Since, moreover,

$$L_1, \dots, L_m, \overline{L_1}, \dots, \overline{L}_m, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_{n'}}$$

span $\mathbb{C}T\Omega$ it follows that $L_1, \dots, L_m, \overline{L_1}, \dots, \overline{L}_{n'}$ are linearly independent. We conclude then that $L_1, \dots, L_m$ define a complex structure (in the x-space) in a neighborhood of p_0. By the Newlander–Nirenberg theorem there are $Z_1(x), \dots, Z_m(x)$ with linearly independent differentials such that

$$L_k Z_\ell = 0, \quad k, \ell = 1, \dots, m.$$

Since, moreover,

$$\frac{\partial Z_\ell}{\partial t_j} = 0, \quad \ell = 1, \dots, m,\ j = 1, \dots, n'$$

the proof is complete. $\square$

Theorem I.10.1 gives a particularly simple local representation for an elliptic structure. Let $\mathcal{V}$ and Ω be as in Theorem I.12.1 and fix $p \in \Omega$. With the notation as in Theorem I.10.1 we have $d = 0$, $\nu = m$ and thus there is a coordinate system

$$(x_1, \ldots, x_m, y_1, \ldots, y_m, t_1, \ldots t_{n'})$$

vanishing at p such that, setting $z_j = x_j + iy_j$, the differentials $\mathrm{d}z_j$ span T' near p, and the vector fields $\partial/\partial\overline{z}_k$, $\partial/\partial t_j$ span $\mathcal{V}$ near p. Notice also that $n' = 0$ corresponds to the case when $\mathcal{V}$ defines a complex structure.

I.13 Elliptic structures in the real plane

In this section we depart a bit from the spirit we have adopted in the exposition up to now and make use of some standard results on Fourier analysis and pseudo-differential operators in order to study elliptic structures in two-dimensional manifolds. The results contained here are not necessary for the comprehension of the remaining parts of the chapter and the section can be avoided in a first reading.

If Ω is an open subset of $\mathbb{R}^2$ any sub-bundle $\mathcal{V}$ of $\mathbb{C}T\Omega$ of rank one defines a formally integrable structure over Ω, for the involutive condition is automatically satisfied. Suppose that Ω contains the origin and let L be a complex vector field that spans $\mathcal{V}$ in a neighborhood of 0. After division by a nonvanishing smooth factor it can be assumed that, in suitable coordinates (x_1, x_2), we can write

$$L = \frac{\partial}{\partial x_2} + a(x_1, x_2)\frac{\partial}{\partial x_1}.$$

As at the beginning of Section I.5 we can find a smooth diffeomorphism $(x, t) \mapsto (x_1, x_2)$, $x_2 = t$, which reduces $\Re L$ to $\partial/\partial t$. Since also $\partial/\partial x_1$ is a multiple of $\partial/\partial x$ in these new variables, L can be written as a nonvanishing multiple of

$$L_\bullet = \frac{\partial}{\partial t} + ib(x, t)\frac{\partial}{\partial x}, \tag{I.42}$$

where b is smooth and *real-valued*. Since both L and $L_\bullet$ span $\mathcal{V}$ in a neighborhood of the origin of $\mathbb{R}^2$, there is no loss of generality in assuming that our original L takes the form (I.42).

The structure $\mathcal{V}$ is elliptic if and only if L and $\overline{L}$ are linearly independent at every point. This is equivalent to saying that the function b in (I.42) never vanishes (in the p.d.e. terminology, L is an *elliptic* operator). We shall now recall the standard elliptic estimates satisfied by L and its transpose ${}^tL$ in a neighborhood of the origin. Let

$$L_0 = \frac{\partial}{\partial t} + ib(0,0)\frac{\partial}{\partial x}. \tag{I.43}$$

If $\varphi \in C_c^\infty(\mathbb{R}^2)$ then taking Fourier transforms gives

$$\mathcal{F}(L_0\varphi)(\xi,\tau) = (i\tau - b(0,0)\xi)\mathcal{F}(\varphi)(\xi,\tau).$$

Since $b(0,0) \neq 0$ we have

$$\tau^2 + \xi^2 \leq \max\left\{1, \frac{1}{b(0,0)^2}\right\} |i\tau - b(0,0)\xi|^2$$

and thus by Parseval's formula we obtain, in Sobolev norms,

$$\|\varphi\|_1 \leq C\left(\|L_0\varphi\|_0 + \|\varphi\|_0\right), \ \varphi \in C_c^\infty(\mathbb{R}^2), \tag{I.44}$$

where for any real s we denote by $\|\varphi\|_s$ the norm in the Sobolev space $L_s^2(\mathbb{R}^2)$ (see Section II.3.2 for the definition of Sobolev norms). We select an open neighborhood of the origin $U \subset \Omega$ such that $|b(x,t) - b(0,0)| \leq 1/(2C)$ for $(x,t) \in U$. If $\varphi \in C_c^\infty(U)$ then by (I.44)

$$\begin{aligned} \|\varphi\|_1 &\leq C\left(\|L\varphi\|_0 + \|\left(L - L_0\right)\varphi\|_0 + \|\varphi\|_0\right) \\ &= C\left(\|L\varphi\|_0 + \|\left(b(x,t) - b(0,0)\right)\varphi_x\|_0 + \|\varphi\|_0\right) \\ &\leq C\left(\|L\varphi\|_0 + \|\varphi\|_0\right) + \frac{1}{2}\|\varphi\|_1, \end{aligned}$$

and thus

$$\|\varphi\|_1 \leq 2C\left(\|L\varphi\|_0 + \|\varphi\|_0\right), \ \varphi \in C_c^\infty(U). \tag{I.45}$$

Let now $V \subset\subset U$ be an open set and let also $\theta \in C_c^\infty(U)$ be identically equal to one in V. We denote by Θ the operator 'multiplication by θ' and by Λ the operator $(1-\Delta)^{1/2}$. For a real number s and for $\varphi \in C_c^\infty(V)$, we obtain

$$\|\varphi\|_{s+1} = \|\Lambda^s\Theta(\varphi)\|_1 \leq \|\Theta\Lambda^s\varphi\|_1 + C_1\|\varphi\|_s$$

since the commutator between Λ^s and Θ has order $s-1$. If we now apply (I.45) we obtain

$$\begin{aligned} \|\varphi\|_{s+1} &\leq C_2\left\{\|L\Theta\Lambda^s\varphi\|_0 + \|\Theta\Lambda^s\varphi\|_0 + \|\varphi\|_s\right\} \\ &\leq C_3\left\{\|(\Theta\Lambda^s)L\varphi\|_0 + \|\varphi\|_s\right\} \end{aligned}$$

since both $\Theta\Lambda^s$ and its commutator with L have order s. We then obtain:

- For every $V \subset\subset U$ open and every $s \in \mathbb{R}$ there is $C^\bullet > 0$ such that

$$\|\varphi\|_{s+1} \leq C^\bullet\left\{\|L\varphi\|_s + \|\varphi\|_s\right\}, \ \varphi \in C_c^\infty(V). \tag{I.46}$$

PROPOSITION I.13.1. *If $u \in \mathcal{D}'(U)$ and $Lu \in L^{2,s}_{\text{loc}}(U)$ then $u \in L^{2,s+1}_{\text{loc}}(U)$. In particular, if $u \in \mathcal{D}'(U)$ and $Lu \in C^\infty(U)$ then $u \in C^\infty(U)$.*

PROOF. Let $W \subset\subset V \subset\subset U$ be open sets and let $\theta \in C_c^\infty(V)$ be identically equal to one in W. Since there is $\sigma \leq s$ such that $u \in L^{2,\sigma}_{\text{loc}}(V)$ it will suffice to show that $\theta u \in L^2_{\sigma+1}$, for iteration of the argument will give the result.

Let $B_\epsilon = \rho_\epsilon * \cdot$, where $\{\rho_\epsilon\}$ is the usual family of mollifiers in $\mathbb{R}^2$. We have $B_\epsilon(\theta u) \to \theta u$ in L^2_σ as $\epsilon \to 0$ and also

$$LB_\epsilon(\theta u) = B_\epsilon L(\theta u) + [L, B_\epsilon](\theta u) \xrightarrow{\epsilon \to 0} L(\theta u) \text{ in } L^2_\sigma$$

by Friedrich's lemma, since $L(\theta u) \in L^2_\sigma$. Thus, if take $\epsilon_n \to 0$ and if we apply (I.46) for $s = \sigma$ and $\varphi = B_{\epsilon_m}(\theta u) - B_{\epsilon_n}(\theta u)$ we conclude that $\{B_{\epsilon_n}(\theta u)\}$ is a Cauchy sequence in $L^2_{\sigma+1}$. Hence $\theta u \in L^2_{\sigma+1}$ and the proof is complete. $\square$

We shall now derive from (I.45) an estimate for the transpose of L which will lead us to a solvability result. If we notice that ${}^tL = -L - ib_x(x, t)$ then from (I.45) we obtain, for some constant $C' > 0$,

$$\|\varphi\|_1 \leq C' \left(\|{}^tL\varphi\|_0 + \|\varphi\|_0\right), \ \varphi \in C_c^\infty(U). \tag{I.47}$$

Now, it is elementary that

$$\|\varphi\|_0 \leq 2\delta \|\varphi_t\|_0, \quad \varphi \in C_c^\infty(U),$$

where $\delta = \sup\{|t| : (x, t) \in U\}$. Consequently, if we further contract U about the origin in order to achieve $2\delta C' \leq 1/2$, from (I.47) we finally obtain

$$\|\varphi\|_0 \leq 2C' \|{}^tL\varphi\|_0, \ \varphi \in C_c^\infty(U). \tag{I.48}$$

PROPOSITION I.13.2. *For every $f \in L^2(U)$ there is $u \in L^2(U)$ such that $Lu = f$ in U.*

PROOF. Given $f \in L^2(U)$ consider the functional

$${}^tL\varphi \mapsto \int f(x, t)\varphi(x, t)\mathrm{d}x\mathrm{d}t \tag{I.49}$$

defined on $\{{}^tL\varphi : \varphi \in C_c^\infty(U)\}$, where the latter is considered as a subspace of $L^2(U)$. By (I.48) it follows that (I.49) is well-defined and continuous. By the Hahn–Banach theorem we extend (I.49) to a continuous functional λ on $L^2(U)$ and by the Riesz representation theorem we find $u \in L^2(U)$ such that

$$\lambda(g) = \int g(x, t)u(x, t)\mathrm{d}x\mathrm{d}t, \quad g \in L^2(U).$$

In particular, if $\varphi \in C_c^\infty(U)$

$$\lambda({}^tL\varphi) = \int f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t,$$

which is precisely the meaning of the equality $Lu = f$ in the weak sense. □

Corollary I.13.3. *Let $D \subset\subset U$ be an open disk centered at the origin. Then*

$$L\,C^\infty(\overline{D}) = C^\infty(\overline{D}). \tag{I.50}$$

Proof. Given $f \in C^\infty(\overline{D})$ we extend it to an element $\tilde{f} \in C_c^\infty(U)$ and by Proposition I.13.2 we find $u \in L^2(U)$ solving $Lu = \tilde{f}$ in U. Finally, by Proposition I.13.1, we have $u \in C^\infty(U)$ and thus its restriction to D belongs to $C^\infty(\overline{D})$. □

Still under the assumption that L is elliptic we apply (I.50) in order to find $v \in C^\infty(\overline{D})$ such that

$$Lv = -ib_x. \tag{I.51}$$

If we set

$$u(x,t) = \int_0^x \mathrm{e}^{v(x',t)}\mathrm{d}x'$$

we get

$$\begin{aligned} Lu(x,t) &= \int_0^x v_t(x',t)\mathrm{e}^{v(x',t)}\mathrm{d}x' + ib(x,t)\mathrm{e}^{v(x,t)} \\ &= \int_0^x (-ibv_x - ib_x)\,(x',t)\mathrm{e}^{v(x',t)}\mathrm{d}x' + ib(x,t)\mathrm{e}^{v(x,t)} \\ &= -i\int_0^x \partial_x\{b\mathrm{e}^v\}\,(x',t)\mathrm{d}x' + ib(x,t)\mathrm{e}^{v(x,t)} \\ &= ib(0,t)\mathrm{e}^{v(0,t)}. \end{aligned}$$

Then if we set

$$Z(x,t) = u(x,t) - i\int_0^t b(0,t')\mathrm{e}^{v(0,t')}\mathrm{d}t' \tag{I.52}$$

we obtain

$$LZ = 0, \quad Z_x = \mathrm{e}^v \neq 0, \tag{I.53}$$

that is, our original elliptic structure $\mathcal{V}$ is locally integrable. We have thus obtained a proof of the *Newlander–Nirenberg theorem in the particular case when* $N = 2$. We emphasize for this situation the conclusion that we have reached at the end of Section I.12:

COROLLARY I.13.4. *If L is an elliptic operator in an open subset $\Omega \subset \mathbb{R}^2$ and if $p \in \Omega$ then we can find local coordinates (x, y) vanishing at p such that L can be written, in a neighborhood of p, as*

$$L = g(x, y)\left\{\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right\},$$

where g never vanishes. $\square$

REMARK I.13.5. Our discussion indeed leads to a general criterion that characterizes when a rank one formally integrable structure $\mathcal{V} \subset \mathbb{C}T\Omega$, $\Omega \subset \mathbb{R}^2$ open, is locally integrable. Suppose that $\mathcal{V}$ is spanned, in a neighborhood of the origin, by the vector field (I.42).

PROPOSITION I.13.6. *The following properties are equivalent:*

(†) *there is $Z \in C^\infty$ near the origin solving $LZ = 0$, $Z_x \neq 0$;*
(‡) *there is $v \in C^\infty$ near the origin solving* (I.51).

PROOF. We have already presented the argument that (‡) $\Rightarrow$ (†). For the reverse implication we notice that

$$0 = (LZ)_x = L(Z_x) + ib_x Z_x$$

and consequently

$$L\left(\log(Z_x)\right) = Z_x^{-1} L(Z_x) = -ib_x. \qquad \square$$

I.14 Compatible submanifolds

Let Ω be a smooth manifold. A subset $\mathcal{M}$ of Ω is called an *embedded submanifold* (or *submanifold* for short) *of* Ω if there is $r \in \{0, 1, \ldots, N\}$ for which the following is true:

- Given $p_0 \in \mathcal{M}$ arbitrary there is a local chart $(U_0, \mathbf{x})$, with $p_0 \in U_0$ and $\mathbf{x} = (x_1, \ldots, x_N)$, such that

$$U_0 \cap \mathcal{M} = \{q \in U_0 : x_{r+1}(q) = x_{r+1}(p_0), \ldots, x_N(q) = x_N(p_0)\}.$$

When p_0 runs over $\mathcal{M}$ the pairs $(U_0, \mathbf{x}_0)$, where

$$\mathbf{x}_0 = (x_1|_{U_0\cap\mathcal{M}}, \ldots, x_r|_{U_0\cap\mathcal{M}}),$$

make up a family $\mathcal{F}^*$ that satisfies properties (1) and (2) of Section I.1 Hence $\mathcal{M}$ is a smooth manifold of dimension r. We shall refer to the number $N - r$ as the *codimension of $\mathcal{M}$ (in Ω).*

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Let $p \in \mathcal{M}$ and denote by $C^\infty_{\mathcal{M}}(p)$ the space of germs of smooth functions on $\mathcal{M}$ at p. It is clear that the *restriction to* $\mathcal{M}$ defines a surjective homomorphism o f $\mathbb{C}$-algebras $C^\infty(p) \to C^\infty_{\mathcal{M}}(p)$ which gives us then a natural injection

$$\iota_p : \mathbb{C}T_p\mathcal{M} \hookrightarrow \mathbb{C}T_p\Omega. \tag{I.54}$$

By transposition we thus obtain a surjection

$$(\iota_p)^* : \mathbb{C}T^*_p\Omega \longrightarrow \mathbb{C}T^*_p\mathcal{M}, \tag{I.55}$$

whose kernel will be denoted by $\mathbb{C}N^*_p\mathcal{M}$. We shall sometimes refer to the disjoint union

$$\mathbb{C}N^*\mathcal{M} \doteq \bigcup_{p\in\mathcal{M}} \mathbb{C}N^*_p\mathcal{M} \tag{I.56}$$

as the *complex conormal bundle of* $\mathcal{M}$ *in* Ω.

Let now $U \subset \Omega$ be open and let $\omega \in \mathfrak{N}(U)$. Given $L \in \mathfrak{X}(U \cap \mathcal{M})$ the map

$$p \mapsto \left(\iota^*_p(\omega_p)\right)(L_p)$$

is easily seen to be smooth on $U \cap \mathcal{M}$. By the discussion that precedes Proposition I.4.2, there is a form $\omega^\bullet \in \mathfrak{N}(U \cap \mathcal{M})$ such that

$$\omega^\bullet_p = (\iota_p)^*(\omega_p)$$

for every $p \in U \cap \mathcal{M}$. We shall denote $\omega^\bullet$ by $\iota^*\omega$ and shall refer to it as the *pullback of* ω *to* $U \cap \mathcal{M}$. It is clear that ι^* is a homomorphism which is moreover surjective when $U \cap \mathcal{M}$ is closed in U. Observe also that

$$\iota^*(\mathrm{d}f) = \mathrm{d}(f|_{U\cap\mathcal{M}}), \quad f \in C^\infty(U). \tag{I.57}$$

Let now $\mathcal{V}$ be a formally integrable structure over Ω, with $T' = \mathcal{V}^\perp$, and let $\mathcal{M} \subset \Omega$ be a submanifold. If $p \in \mathcal{M}$ we set

$$\mathcal{V}(\mathcal{M})_p \doteq \mathcal{V}_p \cap \mathbb{C}T_p\mathcal{M}, \quad \mathcal{V}(\mathcal{M}) \doteq \bigcup_{p\in\mathcal{M}} \mathcal{V}(\mathcal{M})_p. \tag{I.58}$$

With orthogonal now taken in the duality $(\mathbb{C}T_p\mathcal{M}, \mathbb{C}T^*_p\mathcal{M})$ we have

$$(\iota_p)^*(T'_p) = \mathcal{V}(\mathcal{M})^\perp_p, \tag{I.59}$$

since the left-hand side is the image of the composition

$$T'_p \hookrightarrow \mathbb{C}T^*_p\Omega \xrightarrow{(\iota_p)^*} \mathbb{C}T^*_p\mathcal{M}$$

and consequently is equal to the orthogonal to the kernel of the composition

$$\mathbb{C}T_p\mathcal{M} \hookrightarrow \mathbb{C}T_p\Omega \longrightarrow \mathbb{C}T_p\Omega/\mathcal{V}_p.$$

Definition I.14.1. *We shall say that $\mathcal{M}$ is compatible with the formally integrable structure $\mathcal{V}$ if $\mathcal{V}(\mathcal{M})$ defines a formally integrable structure over $\mathcal{M}$.*

When $\mathcal{M}$ is compatible with $\mathcal{V}$ then, according to our previous notation,

$$T'(\mathcal{M})_p \doteq \mathcal{V}(\mathcal{M})_p^{\perp} = (\iota_p)^*(T'_p)$$

(*cf.* (I.59)). The next result gives a very useful criterion:

Proposition I.14.2. *The submanifold $\mathcal{M}$ is compatible with $\mathcal{V}$ if (and only if)*

$$p \mapsto \dim \mathcal{V}(\mathcal{M})_p \text{ is constant on } \mathcal{M}. \tag{I.60}$$

Proof. We must prove that (I.60) implies that $\mathcal{V}(\mathcal{M})$ is a vector sub-bundle of $\mathcal{V}$ which satisfies the Frobenius condition.

First we observe that (I.60) and (I.59) give the existence of α such that

$$\dim(\iota_p)^*(T'_p) = \alpha, \ \forall p \in \mathcal{M}. \tag{I.61}$$

Let $p_0 \in \mathcal{M}$ and take $\omega_1, \ldots, \omega_m \in \mathfrak{N}(U_0)$, where U_0 is an open subset of Ω that contains p_0, such that $(\omega_1)_q, \ldots, (\omega_m)_q$ span T'_q for every $q \in U_0$. Select $j_1, \ldots, j_\alpha$ such that

$$\left\{(\iota^*\omega_{j_1})_{p_0}, \ldots, (\iota^*\omega_{j_\alpha})_{p_0}\right\}$$

form a basis for $(\iota_{p_0})^*(T'_{p_0})$. Then

$$\left\{(\iota^*\omega_{j_1})_{p}, \ldots, (\iota^*\omega_{j_\alpha})_{p}\right\}$$

will still be linearly independent when p belongs to an open neighborhood V_0 of p_0 in $\mathcal{M}$ and consequently, thanks to (I.61), will form a basis to $(\iota_p)^*(T'_p)$ for all such p. By the remark that follows Proposition I.4.4 we conclude that $\mathcal{V}(\mathcal{M})$ is a vector sub-bundle of $\mathcal{V}$.

To conclude the argument it suffices to observe that if U is an open subset of Ω and if $L, M \in \mathfrak{X}(U)$ are such that $L_p, M_p \in \mathbb{C}T_p\mathcal{M}$ for every $p \in U \cap \mathcal{M}$ then $[L, M]_p \in \mathbb{C}T_p\mathcal{M}$ also for every $p \in U \cap \mathcal{M}$. This property will easily imply that $\mathcal{V}(\mathcal{M})$ satisfies the Frobenius condition. □

Proposition I.14.3. *If $\mathcal{V}$ is a locally integrable structure over Ω and if $\mathcal{M}$ is a submanifold of Ω which is compatible with $\mathcal{V}$ then $\mathcal{V}(\mathcal{M})$ is a locally integrable structure over $\mathcal{M}$.*

Proof. It follows from the proof of Proposition I.14.2 in conjunction with (I.57). □

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Example I.14.4. *Generic submanifolds of complex space.* As in Section I.5 we shall write the complex coordinates in $\mathbb{C}^\mu$ as $z_1, \ldots, z_\mu$, where $z_j = x_j + iy_j$. If f is a smooth function on an open subset of $\mathbb{C}^\mu$ we shall write, as usual,

$$\partial f = \sum_{j=1}^{\mu} \frac{\partial f}{\partial z_j}\, \mathrm{d}z_j, \tag{I.62}$$

$$\overline{\partial} f = \sum_{j=1}^{\mu} \frac{\partial f}{\partial \overline{z}_j}\, \mathrm{d}\overline{z}_j. \tag{I.63}$$

Definition I.14.5. *Let $\mathcal{M}$ be a submanifold of $\mathbb{C}^\mu$ of codimension d. We shall say that $\mathcal{M}$ is generic if given $p_0 \in \mathcal{M}$ there are an open neighborhood U_0 of p_0 in $\mathbb{C}^\mu$ and real-valued functions $\rho_1, \ldots, \rho_d \in C^\infty(U_0)$ such that*

$$\mathcal{M} \cap U_0 = \{z \in U : \rho_k(z) = 0,\ k = 1, \ldots, d\}$$

and

$$\overline{\partial}\rho_1, \ldots, \overline{\partial}\rho_d \text{ are linearly independent at each point of } \mathcal{M} \cap U_0.$$

Notice that every one-codimensional submanifold of $\mathbb{C}^\mu$ is automatically generic. Denote by $\mathcal{V}^{0,1}$ the sub-bundle of $\mathbb{C}T\mathbb{C}^\mu$ which defines the complex structure on $\mathbb{C}^\mu$, that is, the sub-bundle spanned by the vector fields $\partial/\partial\overline{z}_j$, $j = 1, \ldots, \mu$.

Proposition I.14.6. *If $\mathcal{M}$ is a generic submanifold of $\mathbb{C}^\mu$ of codimension d then $\mathcal{M}$ is compatible with $\mathcal{V}^{0,1}$. Moreover, $\mathcal{V}^{0,1}(\mathcal{M})$ is a locally integrable, CR structure for which n and m satisfy:*

$$\dim \mathcal{M} = 2n + d, \quad m = \mu = n + d.$$

The sub-bundle $T'(\mathcal{M})$ is spanned by the differentials of the restriction to $\mathcal{M}$ of the complex coordinate functions on $\mathbb{C}^\mu$.

Proof. Let $p \in \mathcal{M}$. A vector $\sum_{j=1}^{\mu} a_j(\partial/\partial\overline{z}_j)_p$ belongs to $\mathbb{C}T_p\mathcal{M} \cap \mathcal{V}_p^{0,1}$ if and only if

$$\sum_{j=1}^{\mu} a_j \frac{\partial \rho_k}{\partial \overline{z}_j}(p) = 0, \quad k = 1, \ldots, d.$$

Since $\mathcal{M}$ is generic it follows that

$$\dim_{\mathbb{C}} \left(\mathbb{C}T_p\mathcal{M} \cap \mathcal{V}_p^{0,1}\right) = \mu - d, \quad \forall p \in \mathcal{M}.$$

By Propositions I.14.2 and I.14.3 we conclude that $\mathcal{M}$ is compatible with $\mathcal{V}^{0,1}$ and that $\mathcal{V}^{0,1}(\mathcal{M})$ is locally integrable. Moreover, since $(\mathcal{V}^{0,1})_p \cap (\overline{\mathcal{V}^{0,1}})_p = 0$ for every $p \in \mathbb{C}^\mu$ we obtain

$$\mathcal{V}^{0,1}(\mathcal{M})_p \cap \overline{\mathcal{V}^{0,1}(\mathcal{M})}_p = 0, \quad \forall p \in \mathcal{M},$$

which shows that $\mathcal{V}^{0,1}(\mathcal{M})$ defines a CR structure over $\mathcal{M}$. Finally, we have $n \doteq \operatorname{rank} \mathcal{V}^{0,1}(\mathcal{M}) = \mu - d$ and thus $\dim \mathcal{M} = 2\mu - d = 2n + d$ and $m = \dim \mathcal{M} - n = n + d$.

The last statement follows immediately from the proof of Proposition I.14.2. $\square$

I.15 Locally integrable CR structures

When $\mathcal{V}$ defines a locally integrable CR structure over Ω then, according to Proposition I.8.4, $d \doteq \dim T_p^0 = N - 2n$, for all $p \in \Omega$. Using Theorem I.10.1 we obtain $m = N - n = n + d$, $\nu = m - d = n$ and $n' = N - 2\nu - d = 0$. We summarize:

- *Given $p \in \Omega$ there is a coordinate system vanishing at p,*

$$\{x_1, \dots, x_n, y_1, \dots, y_n, s_1, \dots, s_d\}$$

and smooth, real-valued functions $\phi_1, \dots, \phi_d$ defined in a neighborhood of the origin and satisfying

$$\phi_k(0) = 0, \; \mathrm{d}\phi_k(0) = 0, \quad k = 1, \dots, d, \tag{I.64}$$

such that the differentials of the functions

$$Z_j = x_j + iy_j, \quad j = 1, \dots, n; \tag{I.65}$$

$$W_k = s_k + i\phi_k(z, s), \quad k = 1, \dots, d \tag{I.66}$$

span T' in a neighborhood of the origin.

Notice that $\mathcal{V}$ is spanned, in a neighborhood of the origin, by the pairwise commuting vector fields (I.27), where $\nu = n$ and there is no t-variable.

Suppose that $\phi = (\phi_1, \dots, \phi_d)$ is defined in a neighborhood U of the origin in $\mathbb{C}^n \times \mathbb{R}^d$. Then the map

$$\mathfrak{F} : U \to \mathbb{C}^{n+d}, \quad \mathfrak{F}(z, s) = (z, s + i\phi(z, s)) \tag{I.67}$$

has rank $2n + d$ and consequently $\mathfrak{F}(U)$ is an embedded submanifold of $\mathbb{C}^{n+d}$ of dimension $2n + d$ (and of codimension d).

Now we write the coordinates in $\mathbb{C}^{n+d}$ as

$$(z_1, \dots, z_n, w_1, \dots, w_d),$$

where $w = s + it$, $w_j = s_j + it_j$. Then $\mathfrak{F}(U)$ is defined by the equations

$$\rho_k(z, w) \doteq \phi_k(z, s) - t_k = 0, \quad k = 1, \dots, d.$$

Since

$$\frac{\partial \rho_k}{\partial w_\ell} = \frac{1}{2}\left\{\frac{\partial \phi_k}{\partial s_\ell} + i\delta_{k\ell}\right\}, k, \ell = 1, \dots, d,$$

we conclude, taking into account (I.64), that $\mathfrak{F}(U)$ is generic if U is taken small enough so that

$$\left\|\frac{\partial \phi}{\partial s}(z, s)\right\| \le \frac{1}{2}, \quad (z, s) \in U. \tag{I.68}$$

By Proposition I.14.6 the complex structure $\mathcal{V}^{0,1}$ on $\mathbb{C}^{n+d}$ defines a locally integrable CR structure $\mathcal{V}(\mathfrak{F}(U))$ on $\mathfrak{F}(U)$ for which the sub-bundle $T'(\mathfrak{F}(U)$ is spanned by the differentials of the restrictions of the functions $z_1, \dots, z_n$, $w_1, \dots, w_d$ to $\mathfrak{F}(U)$. Since in the local coordinates (z, s) we have

$$z_j|_{\mathfrak{F}(U)} = Z_j(z, s), \quad w_k|_{\mathfrak{F}(U)} = W_k(z, s)$$

(*cf.* (I.65), (I.66)), we can state:

PROPOSITION I.15.1. *Every locally integrable CR structure can be locally realized as the CR structure induced by the complex structure on a generic submanifold of the complex space.* □

REMARK I.15.2. Let $\mathcal{V}$ be a tube structure on $\mathbb{R}^m \times U$ (*cf.* Section I.10). Thus U is an open subset of $\mathbb{R}^n$ and we assume given smooth, real-valued functions $\phi_1, \dots, \phi_m$ on U such that T' is spanned by the differential of the functions $Z_k = x_k + i\phi_k(t)$, $k = 1, \dots, m$. Recall that $\mathcal{V}$ is then spanned on $\mathbb{R}^m \times U$ by the vector fields (I.37′). Let us now assume that $\mathcal{V}$ is also a CR structure. Let $d = m - n$ be the rank of the characteristic set T^0 (*cf.* Proposition I.8.4). Since $\mathcal{V}$ being CR demands that $T'_{(x,t)} + \overline{T'}_{(x,t)} = \mathbb{C}T^*_{(x,t)}(\mathbb{R}^m \times U)$ for every $(x, t) \in \mathbb{R}^m \times U$, we must then have

$$\operatorname{rank} \Phi'(t) = n, \quad \forall t \in U,$$

where $\Phi = (\phi_1, \dots, \phi_m)$. This implies that $\mathcal{M} \doteq \Phi(U)$ is an embedded submanifold of $\mathbb{R}^m$ of dimension n and it is clear that $\mathcal{V}$ can be realized as the CR structure induced by the complex structure on the generic submanifold $\mathbb{R}^m + i\mathcal{M}$ of $\mathbb{R}^m + i\mathbb{R}^m = \mathbb{C}^m$.

One very important model of a CR structure is the *Hans Lewy* structure. We take as Ω the space $\mathbb{C} \times \mathbb{R}$, where the coordinates are written as $z = x + iy$ and s, and consider the formally integrable structure $\mathcal{V}$ spanned by the Hans Lewy vector field (or operator)

$$L = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}. \tag{I.69}$$

Since L and $\overline{L}$ are linearly independent at every point it follows that $\mathcal{V}$ defines a CR structure which is furthermore locally integrable, since the differential of the functions z and $W = s + i|z|^2$ span T' on $\mathbb{C}\times\mathbb{R}$. Notice also that the Hans Lewy structure can be globally realized as the CR structure induced on the hyperquadric

$$\mathfrak{Q} \doteq \{(z, w) \in \mathbb{C}^2 : w = s + it,\ t = |z|^2\} \tag{I.70}$$

by the complex structure on $\mathbb{C}^2$.

More generally, given $\epsilon_j \in \{-1, 1\}$, $j = 1, \ldots, n$, we can consider the CR structure $\mathcal{V}$ on $\mathbb{C}^n \times \mathbb{R}$ spanned by the pairwise commuting vector fields

$$L_j = \frac{\partial}{\partial \overline{z}_j} - i\epsilon_j z_j \frac{\partial}{\partial s}, \quad j = 1, \ldots, n. \tag{I.71}$$

Such a structure is also locally integrable for the differential of the functions $z_1, \ldots, z_n$ and $W = s + i\phi(z)$, with

$$\phi(z) = \sum_{j=1}^{n} \epsilon_j |z_j|^2,$$

span T' on $\mathbb{C}^n \times \mathbb{R}$.

I.16 A CR structure that is not locally integrable

In this section we shall prove the following quite involved result:

Proposition I.16.1. *Let*

$$\epsilon_1 = 1, \quad \epsilon_j = -1,\ j = 2, \ldots, n. \tag{I.72}$$

There is a smooth function $g(z, s)$ defined in an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^n \times \mathbb{R}$ and vanishing to infinite order at $z_1 = 0$, such that if we set

$$L_j^{\#} = \frac{\partial}{\partial \overline{z}_j} - i\epsilon_j z_j (1 + g(z, s)) \frac{\partial}{\partial s}, \quad j = 1, \ldots, n, \tag{I.73}$$

then the following is true:

(a) the vector fields $L_j^{\#}$ are pairwise commuting;
(b) if h is a C^1 function near the origin satisfying $L_j^{\#} h = 0$ $(j = 1, \ldots, n)$ then $(\partial h/\partial s)(0, 0) = 0$.

Before we embark on the proof we shall state and prove the important consequence of this result:

Corollary I.16.2. *The vector fields* (I.73) *span a CR structure which is not locally integrable in any neighborhood of the origin.*

Indeed, first we notice that $L_1^\#, \ldots, L_n^\#, \overline{L_1^\#}, \ldots, \overline{L_n^\#}$ are linearly independent over $\mathcal{O}$ which together with property (a) shows that (I.73) define a CR structure over $\mathcal{O}$.

Now, given any smooth solution h to the system

$$L_j^\# h = 0, \quad j = 1, \ldots, n, \tag{I.74}$$

we necessarily have $(\partial h/\partial \bar{z}_j)(0,0) = 0$ for all $j = 1, \ldots, n$. By property (b) we then obtain $\mathrm{d}h = \sum_{j=1}^n a_j \mathrm{d}z_j$ at the origin and hence any set $h_1, \ldots, h_{n+1}$ of smooth solutions to (I.74) must have linearly dependent differentials at the origin. In particular, the CR structure defined by the vector fields (I.73) cannot be locally integrable.

Proof of Proposition I.16.1. The first step in the proof is the construction of the function g. In the complex plane we denote the variable by $w = s + it$ and consider a sequence of closed, disjoint disks $\{D_j\}$, all of them contained in the sector $\{w : |s| < t\}$ and such that $D_j \to \{0\}$ as $j \to \infty$.

Let $F \in C^\infty(\mathbb{C}, \mathbb{R})$ have support contained in the union of the disks D_j and satisfy

$$F(w) > 0, \ \forall w \in \mathrm{int}\,(D_j), \ \forall j. \tag{I.75}$$

As before we shall write $W(z,s) = s + i\phi(z)$, with

$$\phi(z) = |z_1|^2 - |z_2|^2 - \ldots - |z_n|^2. \tag{I.76}$$

Lemma I.16.3. *The function $F \circ W$ vanishes to infinite order at $z_1 = 0$.*

Proof. Denote by H the Heaviside function. For every $\ell \in \mathbb{Z}_+$ there is $C_\ell > 0$ such that

$$|F(w)| \le C_\ell (tH(t))^\ell.$$

Then

$$|F(W(z,s))| \le C_\ell \left(\phi(z)H(\phi(z))\right)^\ell.$$

Since moreover $\phi(z)H(\phi(z)) \le |z_1|^2$, the lemma is proved. □

We then set

$$g(z,s) \doteq \frac{F(W(z,s))}{z_1 - F(W(z,s))}. \tag{I.77}$$

Since

$$g(z,s) = \frac{F(W(z,s))}{z_1} \frac{1}{1 - F(W(z,s))/z_1}$$

it follows from Lemma I.16.3 that g is smooth in an open neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}$ and that g vanishes to infinite order at $z_1 = 0$.

We shall now proceed to the proof of (a). We shall write

$$L_j^\# = L_j - i\epsilon_j z_j g(z,s) \frac{\partial}{\partial s},$$

(*cf.* (I.76), (I.73)). Since $[L_j, L_k] = 0$ and $L_j z_k = 0$ for all j and k we obtain

$$[L_j^\#, L_k^\#] = -i \left\{ \epsilon_k z_k L_j g - \epsilon_j z_j L_k g \right\} \frac{\partial}{\partial s}. \tag{I.78}$$

Now

$$L_j g = \frac{z_1}{(z_1 - F \circ W)^2} L_j(F \circ W)$$

and an easy computation making use of the chain rule gives

$$L_j \{F(W(z,s))\} = -2i\epsilon_j z_j \frac{\partial F}{\partial \overline{w}} (W(z,s)).$$

Hence from (I.78) we obtain

$$\begin{aligned} [L_j^\#, L_k^\#] &= \frac{-iz_1}{(z_1 - F \circ W)^2} \left\{ \epsilon_k z_k L_j(F \circ W) - \epsilon_j z_j L_k(F \circ W) \right\} \frac{\partial}{\partial s} \\ &= \frac{-2z_1}{(z_1 - F \circ W)^2} \left\{ \epsilon_k z_k \epsilon_j z_j - \epsilon_j z_j \epsilon_k z_k \right\} \frac{\partial}{\partial s} = 0. \end{aligned}$$

We now start to prove (b). For this we set

$$\chi(z,s) = h(z, 0, \ldots, 0, s)$$

and will show that $(\partial\chi/\partial s)(0,0) = 0$. We assume that χ is C^1 in a set of the form

$$V = \{(z,s) \in \mathbb{C} \times \mathbb{R} : |z| < r, \ |s| < \delta\}$$

and observe that

$$L\chi - izf(z,s) \frac{\partial \chi}{\partial s} = 0, \tag{I.79}$$

where L is the Hans Lewy operator given in (I.79) and

$$f(z,s) = \frac{F(s + i|z|^2)}{z - F(s + i|z|^2)}$$

is smooth in V (contracting V if necessary).

Let $U \doteq \{w = s + it \in \mathbb{C} : |s| < \delta,\ 0 < t < r^2\}$ and assume that $D_j \subset U$ for all j. Define

$$I(w) = \int_{|z|=\sqrt{t}} \chi(z, s)\mathrm{d}z, \quad w \in U. \tag{I.80}$$

By Stokes' theorem we have

$$I(w) = \int_{|z|\le\sqrt{t}} \frac{\partial\chi}{\partial\bar{z}}(z, s)\mathrm{d}\bar{z} \wedge \mathrm{d}z = 2i \int_0^{2\pi} \int_0^{\sqrt{t}} \frac{\partial\chi}{\partial\bar{z}}(\rho \mathrm{e}^{i\theta}, s)\rho\mathrm{d}\rho\mathrm{d}\theta$$

from where we obtain

$$\frac{\partial I}{\partial t}(w) = i \int_0^{2\pi} \frac{\partial\chi}{\partial\bar{z}}(\sqrt{t}\mathrm{e}^{i\theta}, s)\mathrm{d}\theta = \int_{|z|=\sqrt{t}} \frac{1}{z}\frac{\partial\chi}{\partial\bar{z}}(z, s)\mathrm{d}z.$$

Consequently,

$$\frac{\partial I}{\partial\bar{w}}(w) = \frac{i}{2} \int_{|z|=\sqrt{t}} \frac{1}{z}(L\chi)(z, s)\mathrm{d}z \tag{I.81}$$

(*cf.* (I.79)). From (I.79), (I.81) and from the fact that F is supported in the union of the disks D_j we conclude that I is a holomorphic function of w in the connected open set $U \backslash \cup_j D_j$. Since, moreover, $I(w) \to 0$ when $t \to 0^+$ the Schwarz reflection principle implies that I vanishes identically in $U \backslash \cup_j D_j$. In particular,

$$I \equiv 0 \quad \text{on} \quad \partial D_j, \quad \forall j. \tag{I.82}$$

Next we consider, for each j, the map

$$\partial D_j \times S^1 \longrightarrow \mathbb{R}^3, \quad (w, \theta) \mapsto (\sqrt{t}\mathrm{e}^{i\theta}, s) \tag{I.83}$$

whose image defines a torus $T_j \subset V$. If we set

$$u \doteq \chi\, \mathrm{d}z \wedge \mathrm{d}W^\flat, \tag{I.84}$$

where $W^\flat(z, s) = s + i|z|^2$, we have $\int_{T_j} u = 0$ for all j, as a consequence of (I.82). Consequently,

$$\int_{S_j} \mathrm{d}u = 0, \quad \forall j, \tag{I.85}$$

where S_j is the solid torus whose boundary is equal to T_j.

We shall now exploit property (I.85). Since $\mathrm{d}z, \mathrm{d}\bar{z}, \mathrm{d}W^\flat$ are linearly independent we can write

$$\mathrm{d}\chi = A\mathrm{d}z + B\mathrm{d}\bar{z} + C\mathrm{d}W^\flat, \tag{I.86}$$

where A, B and C are continuous functions. If we apply both sides of (I.86) to L we obtain that $B = L\chi$, since $Lz = LW^\flat = 0$. Hence, from (I.84) we obtain

$$\begin{aligned} \mathrm{d}u &= (L\chi)\mathrm{d}\bar{z} \wedge \mathrm{d}z \wedge \mathrm{d}W^\flat \\ &= izf(z,s)\frac{\partial \chi}{\partial s}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z \wedge \mathrm{d}s \\ &= -2zf(z,s)\frac{\partial \chi}{\partial s}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}s, \end{aligned}$$

which in conjunction with (I.85) gives

$$\int_{S_j} zf(z,s)\frac{\partial \chi}{\partial s}\mathrm{d}x\mathrm{d}y\mathrm{d}s = 0, \quad \forall j. \tag{I.87}$$

Now we observe that $zf(z,s) = F(s + i|z|^2)\psi(z,s)$, where ψ is smooth and satisfies $\psi(0,0) = 1$. From (I.87) we conclude the existence of points $P_j, Q_j \in S_j$ such that

$$\Re\left\{\psi(P_j)\frac{\partial \chi}{\partial s}(P_j)\right\} = \Im\left\{\psi(Q_j)\frac{\partial \chi}{\partial s}(Q_j)\right\} = 0$$

for all j. It suffices to let $j \to \infty$ to obtain that $(\partial\chi/\partial s)(0,0) = 0$ and hence to conclude the proof of the proposition. □

I.17 The Levi form on a formally integrable structure

Let $\mathcal{V}$ be a formally integrable structure over a smooth manifold Ω and let $\xi \in T_p^0$, $\xi \neq 0$ be fixed (recall that in particular $\xi \in T_p^*\Omega \subset \mathbb{C}T_p^*\Omega$). We start with the following result:

Lemma I.17.1. *Let L and M be sections of $\mathcal{V}$ in a neighborhood of p. If either $L_p = 0$ or $M_p = 0$ then $\xi\left([L, \overline{M}]_p\right) = 0$.*

Proof. We take complex vector fields $L_1, \ldots, L_n$ which span $\mathcal{V}$ at each point in a neighborhood of p.

Assume for instance that $M_p = 0$ (for the other case the argument is analogous). Then we can write

$$M = \sum_{j=1}^n g_j L_j,$$

where g_j are smooth functions and $g_j(p) = 0$ for all $j = 1, \ldots, n$. We have

$$[L, \overline{M}] = \sum_{j=1}^{n} \left\{ \left(L\overline{g_j}\right) \overline{L_j} + \overline{g_j}[L, \overline{L_j}] \right\}$$

and thus $\xi\left([L, \overline{M}]_p\right) = 0$ since $\xi(\overline{L_j}|_p) = 0$ (because ξ is real) and $\overline{g_j}(p) = 0$. $\square$

From Lemma I.17.1 it follows that the following definition is meaningful:

DEFINITION I.17.2. *The Levi form of the formally integrable structure $\mathcal{V}$ at the characteristic point $\xi \in T^0_p$, $\xi \neq 0$ is the hermitian form on $\mathcal{V}_p$ defined by*

$$\mathfrak{L}_{(p,\xi)}(\mathsf{v}, \mathsf{w}) = \frac{1}{2i}\xi\left([L, \overline{M}]_p\right), \tag{I.88}$$

where L and M are smooth sections of $\mathcal{V}$ defined in a neighborhood of p and satisfying $L_p = \mathsf{v}$, $M_p = \mathsf{w}$.

Given a hermitian form $\mathfrak{H}$ on a finite-dimensional complex vector space V, its main invariants are the subspaces V^+, V^- and $V^\perp$ of V, which give a decomposition

$$V = V^+ \oplus V^- \oplus V^\perp$$

and are characterized by:

- $v \mapsto \mathfrak{H}(v, v)$ is positive definite on V^+;
- $v \mapsto \mathfrak{H}(v, v)$ is negative definite on V^-;
- $V^\perp = \{v \in V\colon \mathfrak{H}(v, w) = 0,\ \forall w \in V\}$.

Thus $\mathfrak{H}$ is itself positive definite (resp. positive negative) if $V = V^+$ (resp. $V = V^-$). More generally, $\mathfrak{H}$ is said to be *positive* (resp. *negative*) if $V^- = \{0\}$ (resp. $V^+ = \{0\}$). Also, $\mathfrak{H}$ is said to be *nondegenerate* if $V^\perp = \{0\}$. Finally, we recall that it is common to call the positive integer $|\dim V^+ - \dim V^-|$ the *signature* of $\mathfrak{H}$. Notice that the signature does not change after multiplication of $\mathfrak{H}$ by a nonzero real number.

A formally integrable structure $\mathcal{V}$ over Ω is *nondegenerate* if given any $\xi \in T^0_p$, $\xi \neq 0$ the Levi form $\mathfrak{L}_{(p,\xi)}$ is a nondegenerate hermitian form.

We now describe the Levi form for a formally integrable CR structure over Ω. Let $p \in \Omega$, $\xi \in T^0_p$, $\xi \neq 0$. According to the results described in Section I.8 we can find a system of coordinates

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, s_1, \ldots, s_d)$$

vanishing at p and vector fields of the form

$$L_j = \frac{\partial}{\partial \overline{z}_j} + \sum_{j'=1}^{d} a_{jj'}(z,s)\frac{\partial}{\partial z_{j'}} + \sum_{k=1}^{d} b_{jk}(z,s)\frac{\partial}{\partial s_k}, \quad j = 1,\dots,n,$$

with $a_{jj'}(0,0) = b_{jk}(0,0) = 0$ for all j, j', k, which span $\mathcal{V}$ in a neighborhood of the origin in $\mathbb{R}^{2n+d}$. Notice, moreover, that T^0_p is equal to the span of $\{\mathrm{d}s_1|_0, \dots, \mathrm{d}s_d|_0\}$.

Write $\xi = \xi_1 \mathrm{d}s_1|_0 + \ldots + \xi_d \mathrm{d}s_d|_0$ and denote by $(A_{jj'})$ the matrix of the Levi form $\mathfrak{L}_{(p,\xi)}$ with respect to the basis $\{(\partial/\partial\overline{z}_1)_p, \dots, (\partial/\partial\overline{z}_n)_p\}$ of $\mathcal{V}_p$. Thus, by definition, $A_{jj'} = \mathfrak{L}_{(p,\xi)}\left((\partial/\partial\overline{z}_j)_p, (\partial/\partial\overline{z}_{j'})_p\right)$ and then

$$\begin{aligned} A_{jj'} &= \frac{1}{2i}(\xi_1 \mathrm{d}s_1|_0 + \ldots + \xi_d \mathrm{d}s_d|_0)([L_j, \overline{L}_{j'}]_p) \\ &= \frac{1}{2i}\sum_{k=1}^{d} \xi_k \left\{ L_j \overline{b_{j'k}} - \overline{L}_{j'} b_{jk} \right\}(0,0), \end{aligned}$$

that is

$$A_{jj'} = \frac{1}{2i}\sum_{k=1}^{d} \xi_k \left\{ \overline{\frac{\partial b_{j'k}}{\partial z_j}}(0,0) - \frac{\partial b_{jk}}{\partial z_{j'}}(0,0) \right\}. \tag{I.89}$$

As an example, let us consider the CR structure defined by the vector fields $L^\#_j$ given by (I.73). In this case $d = 1$ and we take $\xi = \mathrm{d}s|_0$. We also have $b_j = -i\epsilon_j z_j (1 + g(z,s))$, where g vanishes to infinite order at $z_1 = 0$. Then

$$\frac{\partial b_{j'}}{\partial z_j}(0,0) = -i\epsilon_j \delta_{jj'}$$

and (I.89) gives

$$(A_{jj'}) = \operatorname{diag}\{\epsilon_1, \dots, \epsilon_n\}.$$

Thus, Corollary I.16.2 has provided an example of a nondegenerate CR structure, defined in a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}$, for which the signature of the Levi form at $\lambda \mathrm{d}s|_0 \in T^0_0$, $\lambda \neq 0$ is equal to $n - 1$.

In connection with this example we mention the following deep result which gives a positive answer to the problem of local integrability (or local realizability, as we have seen in Proposition I.15.1) for certain classes of CR structures. It shows that the value of the signature of the Levi form plays a crucial role.

Recall that by Proposition I.8.4 the characteristic set of a CR structure is a sub-bundle of the cotangent bundle.

THEOREM I.17.3. *Let $\mathcal{V}$ be a nondegenerate CR structure over a smooth manifold Ω and assume that its characteristic set has rank equal to one. Let*

n denote the rank of $\mathcal{V}$ (and thus the dimension of Ω is equal to $2n+1$). Suppose that for some $p \in \Omega$ the signature of the Levi form at $\xi \in T_p^0$, $\xi \neq 0$, is equal to n. If $n \geq 3$ then $\mathcal{V}$ is locally integrable in a neighborhood of p.

Finally, we shall compute the expression of the matrix $(A_{jj'})$ of the Levi form when $\mathcal{V}$ is locally integrable and CR. Invoking the local coordinates described at the beginning of Section I.15, and in particular the functions (I.66) satisfying (I.64), we see that we can take the vector fields L_j in the form (*cf.* (I.27))

$$L_j = \frac{\partial}{\partial \overline{z}_j} - i \sum_{k=1}^{d} \frac{\partial \phi_k}{\partial \overline{z}_j}(z, s) M_k, \quad j = 1, \dots, n,$$

where

$$M_k = \sum_{k'=1}^{d} \mu_{kk'}(z, s) \frac{\partial}{\partial s_{k'}}, \quad k = 1, \dots, d,$$

characterized by the relations $M_k \{s_{k'} + i\phi_{k'}\} = \delta_{kk'}$. In particular, (I.64) gives

$$\mu_{kk'}(0, 0) = \delta_{kk'}. \tag{I.90}$$

According to our previous notation, we have $a_{jj'} \equiv 0$ for all j, j' and

$$b_{jk} = -i \sum_{k'=1}^{d} \frac{\partial \phi_{k'}}{\partial \overline{z}_j} \mu_{k'k}.$$

Again by (I.64) and by (I.90) we have

$$\frac{\partial b_{jk}}{\partial z_{j'}}(0, 0) = -i \frac{\partial^2 \phi_k}{\partial z_{j'} \partial \overline{z}_j}(0, 0)$$

and then by (I.89) we obtain

$$A_{jj'} = \sum_{k=1}^{d} \xi_k \frac{\partial^2 \phi_k}{\partial z_{j'} \partial \overline{z}_j}(0, 0). \tag{I.91}$$

Example I.17.4. The following discussion justifies our terminology and makes a connection with the theory of several complex variables.

Let U be an open subset of $\mathbb{C}^{n+1}$ with a smooth boundary. Let $\rho \in C^\infty(\mathbb{C}^{n+1}, \mathbb{R})$ be such that $U = \{z : \rho(z) < 0\}$ and that $d\rho \neq 0$ on $\partial U = \{z : \rho(z) = 0\}$. We say that U satisfies the *Levi condition* at the point $p \in \partial U$ if the restriction of the hermitian form

$$\zeta \mapsto \sum_{j,k=1}^{n+1} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) \zeta_j \overline{\zeta_k}$$

to the space $\mathfrak{T}_p = \{\zeta \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} (\partial \rho / \partial z_j)(p) \zeta_j = 0\}$ is positive.

The Levi condition is independent of the choice of the defining function ρ: it is also a holomorphic invariant. After a translation and a $\mathbb{C}$-linear tranformation we can assume that $0 \in \partial\Omega$ and that the tangent space to $\partial\Omega$ at the origin is given by the real-hyperplane $\Im w = 0$, where now we are writing the complex coordinates as $(z_1, \dots, z_n, w)$. We can also assume that the exterior normal to Ω at the origin is the vector $(0, \dots, 0, -i) \in \mathbb{C}^{n+1}$.

By the implicit function theorem we conclude the existence of a smooth, real-valued function ϕ satisfying $\phi(0,0) = 0$, $\mathrm{d}\phi(0,0) = 0$ such that ρ can be written, near the origin and in these new complex variables, as

$$\rho(z, w) = \phi(z, \Re w) - \Im w. \tag{I.92}$$

Since then $\mathfrak{T}_0 = \{\zeta_{n+1} = 0\}$, the Levi condition at the origin can be written as:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}(0,0)\zeta_j \overline{\zeta_k} \geq 0, \quad \forall \zeta \in \mathbb{C}^n. \tag{I.93}$$

The boundary of U is a one-codimensional submanifold of $\mathbb{C}^{n+1}$ and consequently it is generic. The complex structure $\mathcal{V}^{0,1}$ of $\mathbb{C}^{n+1}$ induces on ∂U a CR structure $\mathcal{V}^{0,1}(\partial U)$ and, according to the discussion in Section I.15, the differentials of the functions

$$Z_j = z_j, \quad j = 1, \dots, n, \;\; W(z, s) = s + i\phi(z, s)$$

span $T'(\partial U)$ near the origin [we are writing $s = \Re w$ and considering (z, s) as local coordinates in ∂U]. From (I.91) we obtain the following equivalent statement to (I.93): *the Levi form of the CR structure* $\mathcal{V}^{0,1}(\partial U)$ *at the characteristic point* $\mathrm{d}s|_0$ *is positive.*

To obtain an invariant statement let us first denote by $T^{1,0}$ the orthogonal sub-bundle $\left(\mathcal{V}^{0,1}\right)^{\perp}$. Given an open set U with a smooth boundary ∂U as above, and given $p \in \partial U$, the map $\iota_p^* : \mathbb{C}T_p^*\mathbb{C}^{n+1} \to \mathbb{C}T_p^*\partial U$ induces an isomorphism

$$\gamma_p : T_p^{1,0} \xrightarrow{\sim} T_p'(\partial U).$$

Let $\xi \in T_p^0(\partial U)$, $\xi \neq 0$. We shall say that ξ is *inward pointing* if

$$\Im \left(\gamma_p^{-1}(\xi)\right)(\mathsf{v}) > 0$$

for every $\mathsf{v} \in T_p\mathbb{C}^{n+1}$ which is inward pointing toward U. In the preceding set-up, when $p = 0$ and ρ is given by (I.92), then $\gamma_0^{-1}(\lambda \mathrm{d}s|_0) = \lambda \mathrm{d}w|_0$ and then $\xi = \lambda \mathrm{d}s|_0$ is inward pointing if and only if $\lambda > 0$. Summing up we can state:

PROPOSITION I.17.5. *Let $U \subset \mathbb{C}^{n+1}$ be an open set with a smooth boundary. Then U satisfies the Levi condition at $p \in \partial U$ if and only if the Levi form associated with the CR structure $\mathcal{V}(\partial U)$ is positive at every $\xi \in T_p^0(\partial U)$, $\xi \neq 0$ which is inward pointing.*

Appendix: Proof of the Newlander–Nirenberg theorem

In this appendix we shall present an argument due to B. Malgrange ([**Mal**]) which leads to the proof of the Newlander–Nirenberg theorem. We start by recalling some of the results we need from the theory of nonlinear elliptic equations.

Let us consider then an *overdetermined* system of nonlinear partial differential equations

$$\Phi\left[x, \vec{u}, \partial_{x_1}\vec{u}, \ldots, \partial^\alpha \vec{u}, \ldots\right] = 0, \quad |\alpha| \leq M, \tag{I.94}$$

where x varies in an open subset Ω of $\mathbb{R}^N$,

$$\vec{u} = (u_1, \ldots, u_q) \in C^M(\Omega; \mathbb{R}^q),$$

$\Phi = (\phi_1, \ldots, \phi_p)$ is smooth and real-valued and $q \leq p$. The system (I.94) is elliptic at $\vec{u}_0 \in C^M(\Omega; \mathbb{R}^q)$ in Ω if the *linear* differential operator

$$\vec{v} \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi\left[x, \vec{u}_0 + \lambda\vec{v}, \partial_{x_1}(\vec{u}_0 + \lambda\vec{v}), \ldots, \partial^\alpha(\vec{u}_0 + \lambda\vec{v}), \ldots\right]|_{\lambda=0} \tag{I.95}$$

is elliptic in the following sense: if

$$\sigma : \Omega \times (\mathbb{R}^N \backslash \{0\}) \to L(\mathbb{R}^q, \mathbb{R}^p)$$

denotes the principal symbol of (I.95) then

$$\operatorname{rank} \sigma(x, \xi) = q, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^N \backslash \{0\}.$$

We call (I.95) the *linearization of (I.94) at $\vec{u}_0$.*

Here is an important remark that will be quite important in what follows: if $x_0 \in \Omega$ and if

$$\vec{v} \mapsto \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi\left[x_0, \vec{u}_0(x_0) + \lambda\vec{v}, \partial_{x_1}(\vec{u}_0)(x_0) + \lambda\partial_{x_1}(\vec{v}), \ldots\right]|_{\lambda=0} \tag{I.96}$$

is an elliptic linear system (with constant coefficients!) then (I.94) is elliptic at $\vec{u}_0$ in a neighborhood of x_0. Accordingly, we shall call (I.96) the *linearization of (I.94) at $\vec{u}_0$ at the point x_0.*

The two main results that are essential for Malgrange's argument are:

- If $\vec{u}$ is a C^M-solution of (I.94), if (I.94) is elliptic at $\vec{u}$ in the sense just defined, and if the function Φ is real-analytic then $\vec{u}$ is real-analytic.
- Now assume that $q = p$ and that (I.94) is elliptic at $\vec{u}_0 \in C^M(\Omega; \mathbb{R}^q)$. Let $x_0 \in \Omega$ be such that

$$\Phi\left[x_0, \vec{u}_0(x_0), \partial_{x_1}\vec{u}_0(x_0), \ldots, \partial^\alpha \vec{u}_0(x_0), \ldots\right] = 0.$$

Then there are $\epsilon_0 > 0$, $C > 0$ and $0 < \mu < 1$ such that for every $0 < \epsilon \le \epsilon_0$ there is a smooth solution $\vec{u}_\epsilon$ to (I.94) on $|x - x_0| < \epsilon$ satisfying the bounds

$$|\partial^\alpha \left(\vec{u}_\epsilon(x) - \vec{u}_0(x)\right)| \le C\epsilon^{M-|\alpha|+\mu}, \quad |x - x_0| < \epsilon,\ |\alpha| \le M.$$

We now embark on the proof of the Newlander–Nirenberg theorem. The starting point is the description of the special generators presented after Lemma I.8.5, particularly the vector fields given by (I.19), taking into account that when the structure is complex then $d = n' = 0$. In other words, we can assume that our (complex) formally integrable structure is defined, in an open neighborhood of the origin in $\mathbb{C}^m$, by the pairwise commuting vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{m} a_{jk}(z)\frac{\partial}{\partial z_k}, \quad j = 1, \ldots, m, \tag{I.97}$$

where $a_{jk} = 0$ at the origin. For technical reasons, which are going to be clear in the argument, it is convenient to assume that $a_{jk}(z) = \mathrm{O}(|z|^2)$, and this property can be achieved after performing a local diffeomorphism of the form $z' = z + Q(z, \bar{z})$, where Q is a homogeneous polynomial of degree two in $(z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m)$ chosen suitably. We leave the details of this (simple) computation to the reader.

Malgrange's key idea is to show the existence of a local diffeomorphism $w = H(z)$, defined near the origin in $\mathbb{C}^m$, such that, in the new variables $w_1, \ldots, w_m$, the structure has a set of generators which have *real-analytic* coefficients. This implies the sought-for conclusion thanks to Theorem I.9.2.

In order to shorten the notation and make the computations more apparent, we shall describe all the systems involved in vector and matrix notation. Thus we set

$$\vec{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}, \quad \frac{\partial}{\partial z} = \begin{bmatrix} \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_m} \end{bmatrix}, \quad \frac{\partial}{\partial \bar{z}} = \begin{bmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_m} \end{bmatrix}$$

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and rewrite the system (I.97) as

$$\vec{L} = \frac{\partial}{\partial \overline{z}} + A(z)\frac{\partial}{\partial z},$$

where $A(z)$ denotes the matrix $\{a_{jk}(z)\}$.

Let $w = H(z)$ be a local diffeomorphism near the origin in $\mathbb{C}^m$ satisfying

$$H_z(0) \text{ is invertible.} \tag{I.98}$$

Since

$$\frac{\partial}{\partial \overline{z}} = {}^tH_{\overline{z}}\frac{\partial}{\partial w} + {}^t\overline{H}_{\overline{z}}\frac{\partial}{\partial \overline{w}}, \quad \frac{\partial}{\partial z} = {}^tH_z\frac{\partial}{\partial w} + {}^t\overline{H}_z\frac{\partial}{\partial \overline{w}},$$

a new set of generators for the structure is defined, in the new variables $w_1, \ldots, w_m$, by the system

$$\vec{L}^\bullet = \frac{\partial}{\partial \overline{w}} + B(w)\frac{\partial}{\partial w}, \tag{I.99}$$

where

$$B(w) = \left({}^t\overline{H}_{\overline{z}} + A\,{}^t\overline{H}_z\right)^{-1}\left({}^tH_{\overline{z}} + A\,{}^tH_z\right)|_{z=H^{-1}(w)}. \tag{I.100}$$

If $L_1^\bullet, \ldots, L_m^\bullet$ denote the components of $\vec{L}^\bullet$ then a fortiori we must have

$$[L_j^\bullet, L_k^\bullet] = 0, \quad \forall j, k = 1, \ldots, m,\ j < k.$$

Writing $B = \{b_{jk}\}$ this property is equivalent to

$$\frac{\partial b_{k\ell}}{\partial \overline{w}_j} - \frac{\partial b_{j\ell}}{\partial \overline{w}_k} - \sum_{r=1}^m \left\{ b_{kr}(w)\frac{\partial b_{j\ell}}{\partial w_r} - b_{jr}(w)\frac{\partial b_{k\ell}}{\partial w_r} \right\} = 0,\ \forall j, k, \ell,\ j < k. \tag{I.101}$$

We emphasize: given any local diffeomorphism H satisfying (I.98) then equations (I.101) are satisfied by $B = \{b_{jk}\}$ defined by (I.100).

The system (I.101) together with the additional equations

$$\sum_{j=1}^m \frac{\partial b_{jk}}{\partial w_j} = 0, \quad k = 1, \ldots, m \tag{I.102}$$

make up a system of quasi-linear partial differential equations in the unknowns $\{b_{jk}\}$. Let us write $\vec{V} = (\Re b_{1,1}, \Re b_{1,2}, \ldots, \Im b_{m,m-1}, \Im b_{m,m}) \in \mathbb{R}^{2m^2}$. Then systems (I.101) and (I.102) can be written as

$$\mathcal{P}\vec{V} + \Gamma(\vec{V}, \vec{\nabla}\vec{V}) = 0, \tag{I.103}$$

where $\mathcal{P}$ is an elliptic linear operator with constant coefficients and Γ is a bilinear form in its arguments. It then follows that there is a small number $\sigma > 0$ such that if $|B(0)| \le \sigma$ then (I.101), (I.102) is elliptic at B in an open neighborhood of the origin. Hence any such B is a real-analytic function of w and the argument will be complete if we can show that a diffeomorphism H satisfying (I.98) can be chosen in such a way that B, defined by (I.100), is a solution of (I.102) satisfying $|B(0)| \leq \sigma$.

We are left to solve the determined system

$$\sum_{j=1}^{m} \frac{\partial}{\partial w_j} \left[\left({}^t\overline{H}_{\overline{z}} + A\, {}^t\overline{H}_z \right)^{-1} \left({}^tH_{\overline{z}} + A\, {}^tH_z \right) \right]_{jk} = 0, \quad k = 1, \ldots, m, \qquad \text{(I.104)}$$

whose unknown is $(\Re H(z), \Im H(z))$ (we look at (I.104) as a determined system of $2m$ *real* equations). It is important to emphasize that these equations are now being considered in the $z_1, \ldots, z_m$ variables.

Since $A(z) = \mathrm{O}(|z|^2)$ it is easily seen that $H_0(z) = z$ satisfies (I.104) at the origin. Furthermore, taking

$$H(z) = H_0(z) + \lambda G(z)$$

then for $\lambda \in \mathbb{R}$, $|\lambda|$ small we have

$$\left({}^t\overline{H}_{\overline{z}} + A\, {}^t\overline{H}_z \right)^{-1} \left({}^tH_{\overline{z}} + A\, {}^tH_z \right) = A + \lambda \left[{}^tG_{\overline{z}} + A\,F \right] + \mathrm{O}(\lambda^2)$$

for some F smooth. Furthermore, since

$$\frac{\partial}{\partial w} = {}^tH_z^{-1} \frac{\partial}{\partial z} - {}^tH_z^{-1}\, {}^t\overline{H}_z \frac{\partial}{\partial \overline{w}}$$

we obtain

$$\frac{\partial}{\partial w_j} = \frac{\partial}{\partial z_j} + \mathrm{O}(\lambda)\,.$$

Hence, using once more the fact that $A(z) = \mathrm{O}(|z|^2)$, we can easily conclude that the linearization of (I.104) at H_0 at the origin can be identified, in a natural way, with the *complex* operator

$$\begin{aligned} G \;\mapsto\; & \left(\sum_{j=1}^{m} \frac{\partial}{\partial z_j} \left[{}^tG_{\overline{z}} \right]_{j1}, \ldots, \sum_{j=1}^{m} \frac{\partial}{\partial z_j} \left[{}^tG_{\overline{z}} \right]_{jm} \right) \\ = \; & \left(\sum_{j=1}^{m} \frac{\partial^2 G_1}{\partial z_j \partial \overline{z_j}}, \ldots, \sum_{j=1}^{m} \frac{\partial^2 G_m}{\partial z_j \partial \overline{z_j}} \right), \end{aligned}$$

which is clearly elliptic (in the usual sense). We conclude that there are $\epsilon_0 > 0$, $C > 0$ and $\mu < 1$ such that for every $0 < \epsilon \leq \epsilon_0$ there is a smooth solution H_ϵ to (I.104) satisfying

$$\|H_\epsilon - H_0\|_{C^2\{z:|z|\leq\epsilon\}} \leq C\epsilon^{2+\mu}, \quad \epsilon \leq \epsilon_0. \qquad \text{(I.105)}$$

In particular, if $\epsilon > 0$ is small enough we can ensure that:

- H_ϵ is a local diffeomorphism near the origin satisfying (I.98);
- B defined by (I.100) satisfies $|B(0)| \leq \sigma$.

The proof is complete. $\square$

Notes

The first treatment of formally and locally integrable structures as presented here appeared in [**T4**], the main point for this being the discovery of the Approximation Formula by M. S. Baouendi and F. Treves in 1981 ([**BT1**]); such structures were then studied extensively in [**T5**]. The pioneering work though seems to be the article by Andreotti-Hill ([**AH1**]), where the concept of what we now call a real-analytic locally integrable structure was introduced in its full generality.

This introductory chapter contains mainly results that have already been presented in standard textbooks. We mention, for instance, the Frobenius theorem, whose proof was taken from L. Hörmander's book [**H4**] and the integrability of elliptic vector fields in the plane, of which we give an almost self-contained proof, depending only on very simple facts concerning commutators of certain pseudo-differential operators that can be found, for instance, in [**Fo**].

As mentioned in the text, Theorem I.12.1 is due to L. Nirenberg ([**N2**]) and the proof we present was taken from [**T5**].

Proposition I.16.1 is a particular case of a more general result due to H. Jacobowitz and F. Treves ([**JT1**]). We also refer to [**JT2**] where the same authors study, via a category argument, the set of all formally integrable CR structures of rank n on an open subset of $\mathbb{R}^{2n+1}$ whose Levi form has, at each nonzero characteristic point, signature $n-1$.

Theorem I.17.3 was originally due to M. Kuranishi ([**Ku1**], [**Ku2**]) in the case $n \geq 4$. Later, T. Akahori ([**Ak**]) presented an improvement to Kuranishi's argument which allowed him to prove Theorem I.17.3 also for the case $n = 3$. The case $n = 2$ is still an open problem, whereas when $n = 1$ the conclusion is false, according to [**N3**] (see also Theorem I.12.1). A proof of Theorem I.17.3 can also be found in [**W3**].

Finally, Malgrange's proof of the Newlander–Nirenberg theorem that we presented in the appendix was taken from [**N1**], where the use of a solvability result on elliptic determined systems of nonlinear partial differential equations makes the argument a bit simpler.