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ANALYTIC FUNCTIONS ASSOCIATED WITH STRONG HAMBURGER MOMENT PROBLEMS

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Abstract We complete the investigation of growth properties of analytic functions connected with the Nevanlinna parametrization of the solutions of an indeterminate strong Hamburger moment problem.

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1. Introduction

A solution of the *strong* (or *two-point*) Hamburger moment problem for a given doubly infinite sequence $\{c_n\}_{n=-\infty}^{\infty}$ of real numbers is a positive measure σ on the real line $\mathbb{R}$ such that

$$c_n = \int_{-\infty}^{\infty} t^n \, \mathrm{d}\sigma(t) \quad \text{for } n = 0, \pm 1, \pm 2, \dots. \tag{1.1}$$

An important tool in the study of moment problems is the *Stieltjes transform* $F(z, \sigma)$ of a given measure σ, which we define here as

$$F(z, \sigma) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma(t)}{z - t}. \tag{1.2}$$

The correspondence between measures and their Stieltjes transform is one-to-one. We shall also need the concept of a *Pick function* (or *Nevanlinna function*), that is, a function which is holomorphic in the open upper half-plane of the complex plane $\mathbb{C}$ and maps this half-plane into the closed upper half-plane; the function with constant value ∞ (on the Riemann sphere) is included as a Pick function.

We shall in the following assume that the given moment problem is *indeterminate*, i.e. it has more than one (hence, infinitely many) solutions. There then exists a one-to-one correspondence (depending on a real parameter) between all Pick functions φ and all solutions σ of the moment problem described by the formula

$$F(z, \sigma) = \frac{\alpha(z)\varphi(z) - \gamma(z)}{\beta(z)\varphi(z) - \delta(z)}, \tag{1.3}$$

where α, β, γ, δ are functions which are holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfy the identity $\alpha(z)\delta(z) - \beta(z)\gamma(z) = 1$ (Nevanlinna parametrization of the strong moment problem). We refer the reader to [**8**, **11**, **14**] for this result. For the classical Hamburger moment problem and its associated Nevanlinna parametrization, see, for example, [**1**–**7**, **10**, **16**–**20**].

In [**14**] the following result was proved: for fixed numbers ε and η, with $\varepsilon > 0$ and $0 < \eta < \frac{1}{2}\pi$, there exists a constant $M(\varepsilon, \eta)$ such that

$$|F(z)| \leqslant M(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{1.4}$$

for $0 < |z| < \infty$, $\eta \leqslant |\arg z| \leqslant \pi - \eta$, where F is any of the functions α, β, γ, δ.

The analogous result for the classical moment problem provides an analogous inequality (where the exponential factor contains only the term $\varepsilon|z|$) valid in the whole complex plane (see, for example, [**1**–**3**, **16**–**18**]). Our aim in this note is to extend the inequality (1.4) to an inequality valid in the whole deleted complex plane $\mathbb{C} \setminus \{0\}$.

2. Orthogonal Laurent polynomials

For detailed treatments of the topics discussed in this section, see [**8**, **11**–**15**].

The linear space spanned by all the monomials z^n, $n = 0, \pm 1 \pm 2, \dots$, is denoted by Λ, and the elements of Λ are called *Laurent polynomials*. The doubly infinite sequence $\{c_n\}_{n=-\infty}^{\infty}$ defining the given strong moment problem gives rise to a linear functional S and an inner product $\langle \cdot, \cdot \rangle$ on Λ through the formulae

$$\langle f, g \rangle = S[f(z) \cdot \bar{g}(z)], \quad S[z^n] = c_n, \quad n = 0, \pm 1, \pm 2, \dots. \tag{2.1}$$

Let $\{\varphi_n\}_{n=0}^{\infty}$ be the essentially unique orthonormal sequence of Laurent polynomials with respect to this inner product corresponding to the ordering $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$. These functions have the form

$$\varphi_{2m}(z) = \frac{u_{2m}}{z^m} + \cdots + v_{2m} z^m, \qquad v_{2m} > 0, \tag{2.2}$$

$$\varphi_{2m+1}(z) = \frac{v_{2m+1}}{z^{m+1}} + \cdots + u_{2m+1} z^m, \quad v_{2m+1} > 0, \tag{2.3}$$

for $m = 0, 1, 2, \dots$. All the coefficients of φ_n are real.

The orthonormal Laurent polynomial φ_n is called *regular* if $u_n \neq 0$. At least one of the functions φ_n, φ_{n+1} is regular for every n; hence, there is always an infinite subsequence of $\{\varphi_n\}$ consisting of regular elements. *For simplicity we assume that all the φ_n are regular.* This does not restrict the validity of the final result.

The *associated Laurent polynomials* $\{\psi_n\}_{n=0}^{\infty}$ are defined by

$$\psi_n(z) = S_t\left[\frac{\phi_n(t) - \phi_n(z)}{t - z}\right]. \tag{2.4}$$

All the coefficients of ψ_n are real.

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Let x_0 be an arbitrary fixed point in $\mathbb{R} \setminus \{0\}$. We define the functions α_n, β_n, γ_n and δ_n (depending on x_0) by

$$\alpha_n(z) = (z - x_0) \sum_{k=0}^{n-1} \psi_k(x_0)\psi_k(z), \tag{2.5}$$

$$\beta_n(z) = -1 + (z - x_0) \sum_{k=0}^{n-1} \psi_k(x_0)\varphi_k(z), \tag{2.6}$$

$$\gamma_n(z) = 1 + (z - x_0) \sum_{k=0}^{n-1} \varphi_k(x_0)\psi_k(z), \tag{2.7}$$

$$\delta_n(z) = (z - x_0) \sum_{k=0}^{n-1} \varphi_k(x_0)\varphi_k(z). \tag{2.8}$$

These functions are Laurent polynomials with real coefficients. They satisfy the identity

$$\alpha_n(z)\delta_n(z) - \beta_n(z)\gamma_n(z) = 1 \quad \text{for } z \in \mathbb{C} \setminus \{0\}. \tag{2.9}$$

Furthermore, for a given x_0 (except possibly for one value) they are all regular in a sense analogous to that given above (see [**13**]). We fix a value of x_0 where this regularity property holds for all n.

In addition to the Laurent polynomials introduced above, we consider the functions ω_n and π_n given by

$$\omega_n(z) = \sum_{k=0}^{n-1} |\varphi_k(z)|^2, \tag{2.10}$$

$$\pi_n(z) = \sum_{k=0}^{n-1} |\psi_k(z)|^2. \tag{2.11}$$

The two sequences $\{\omega_n(z)\}$ and $\{\pi_n(z)\}$ converge or diverge simultaneously, and the moment problem is indeterminate if and only if these sequences converge for all (or, equivalently, for some) $z \in \mathbb{C} \setminus \mathbb{R}$. When the moment problem is indeterminate, the sequence $\{\omega_n(z)\}$ converges locally uniformly in $\mathbb{C} \setminus \{0\}$ to a function $\omega(z)$ and the sequence $\{\pi_n(z)\}$ converges locally uniformly in $\mathbb{C} \setminus \{0\}$ to a function $\pi(z)$. Furthermore, the sequences $\{\alpha_n(z)\}$, $\{\beta_n(z)\}$, $\{\gamma_n(z)\}$ and $\{\delta_n(z)\}$ converge locally uniformly in $\mathbb{C}\setminus\{0\}$ to functions $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$, which are then holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfy

$$\alpha(z)\delta(z) - \beta(z)\gamma(z) = 1 \quad \text{for } z \in \mathbb{C} \setminus \{0\}. \tag{2.12}$$

These functions are those appearing in the Nevanlinna parametrization of the strong Hamburger moment problem stated in § 1.

A further important fact is that when the moment problem is indeterminate the inequality

$$\int_{-\infty}^{\infty} \frac{\ln[\omega(t)]}{1 + t^2}\, \mathrm{d}t < \infty \tag{2.13}$$

holds. This is the *Riesz criterion for the strong Hamburger moment problem* (see [**12**]).

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3. Inequalities

For proofs of the results in this section, see [**14**].

The following inequalities hold (with $z = x + \mathrm{i}y$):

$$\omega_n(z) \leqslant \frac{|\beta_n(z)\delta_n(z)|}{|y|} \quad \text{for } y \neq 0, \tag{3.1}$$

$$|\alpha_n(z)| \leqslant \frac{c_0}{|y|}|\beta_n(z)|, \quad |\gamma_n(z)| \leqslant \frac{c_0}{|y|}|\delta_n(z)| \quad \text{for } y \neq 0, \tag{3.2}$$

$$|g_n(z)| \leqslant \frac{c_0}{|y|}[1 + \pi(x_0)|z - x_0|\sqrt{\omega(z)}] \quad \text{for } y \neq 0, \tag{3.3}$$

$$|h_n(z)| \leqslant 1 + \pi(x_0)|z - x_0|\sqrt{\omega(z)} \quad \text{for } z \neq 0, \tag{3.4}$$

where g_n is either of the functions α_n or γ_n and h_n is either of the functions β_n or δ_n. From the Poisson formula it follows that

$$\ln|h_n(z)| = \frac{|y|}{\pi}\int_{-\infty}^{\infty} \frac{\ln|h_n(\xi)|\,\mathrm{d}\xi}{(x-\xi)^2 + y^2} \tag{3.5}$$

for z outside the real axis and h_n is either of the functions β_n or δ_n.

We introduce the angular regions Ω_η given by

$$\Omega_\eta = \{z \in \mathbb{C} : \eta \leqslant |\arg z| \leqslant \pi - \eta,\ |z| > 0\}, \tag{3.6}$$

where $0 < \eta < \frac{1}{2}\pi$. By using the Riesz criterion (2.13) it can be shown that, for every $\varepsilon > 0$, there is a constant $B(\varepsilon, \eta)$ independent of n such that

$$|h_n(z)| \leqslant B(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{3.7}$$

for all z in Ω_η. From this result, together with the inequalities (3.1)–(3.4), it follows that there exists a constant $M(\varepsilon, \eta)$ independent of n such that

$$|F_n(z)| \leqslant M(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{3.8}$$

for all z in Ω_η, where F_n is any of the functions α_n, β_n, γ_n, δ_n, ω_n. From this the inequalities (1.4) follow.

4. The general growth theorem

Theorem 4.1. *For every positive ε there exists a constant $A(\varepsilon)$ such that*

$$|F(z)| \leqslant A(\varepsilon) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.1}$$

for all $z \in \mathbb{C} \setminus \{0\}$, where F is any of the functions α, β, γ, δ.

Proof. We first note that, because of the locally uniform convergence of the sequences $\{\alpha_n(z)\}$, $\{\beta_n(z)\}$, $\{\gamma_n(z)\}$, $\{\delta_n(z)\}$, there is a constant μ such that

$$|\alpha_n(z)| \leqslant \mu, \quad |\beta_n(z)| \leqslant \mu, \quad |\gamma_n(z)| \leqslant \mu, \quad |\delta_n(z)| \leqslant \mu$$

for all z on the unit circle and all n. In the following (until (4.11)) we assume that $\mathrm{Re}(z) > 0$.

Let F_n be any of the functions α_n, β_n, γ_n, δ_n, and let η be an arbitrary value in $(0, \frac{1}{2}\pi)$. Let $\varepsilon > 0$. Then, according to (3.8), there exists a constant $M(\varepsilon\cos\eta, \eta)$, independent of n, such that

$$|F_n(z)| \leqslant M(\varepsilon\cos\eta, \eta)\exp\left[\varepsilon\cos\eta\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.2}$$

for all $z \in \Omega_\eta$.

We let S_η denote the region given by

$$S_\eta = \{z \in \mathbb{C} \setminus \Omega_\eta : |z| > 1\} \tag{4.3}$$

and define the function Q_n by

$$Q_n(z) = F_n(z)\mathrm{e}^{-\varepsilon z}. \tag{4.4}$$

This function is holomorphic in $\mathbb{C} \setminus \{0\}$, and $|Q_n(z)| = |F_n(z)|\mathrm{e}^{-\varepsilon x}$ (with $z = x + \mathrm{i}y$). For z on the line segments of the boundary ∂S_η we have

$$|Q_n(z)| \leqslant M(\varepsilon\cos\eta, \eta)\exp\left[\varepsilon\cos\eta\left(|z| + \frac{1}{|z|}\right)\right]\mathrm{e}^{-\varepsilon|z|\cos\eta};$$

hence,

$$|Q_n(z)| \leqslant M(\varepsilon\cos\eta, \eta)\mathrm{e}^{\varepsilon\cos\eta} \quad (\text{since } |z| \geqslant 1).$$

For z on the circular arc of ∂S_η we have $|Q_n(z)| \leqslant \mu\mathrm{e}^{-\varepsilon\cos\eta}$ (since $\mathrm{e}^{\varepsilon x} \geqslant \mathrm{e}^{\varepsilon\cos\eta}$). Thus, for all $z \in \partial S_\eta$, we have $|Q_n(z)| \leqslant C(\varepsilon, \eta)$, where

$$C(\varepsilon, \eta) = \max[M(\varepsilon\cos\eta, \eta)\mathrm{e}^{\varepsilon\cos\eta}, \mu\mathrm{e}^{-\varepsilon\cos\eta}].$$

Furthermore, since F_n is a Laurent polynomial, there is a constant Γ_n such that $|F_n(z)| \leqslant \Gamma_n\mathrm{e}^{\varepsilon|z|\cos\eta}$ for sufficiently large $|z|$. Consequently, $|Q_n(z)| \leqslant \Gamma_n$ for all sufficiently large $|z|, z \in S_\eta$. (Again recall that $\mathrm{e}^{\varepsilon x} \geqslant \mathrm{e}^{\varepsilon|z|\cos\eta}$). Thus,

$$\lim_{r\to\infty}\frac{\ln[\ln M(r)]}{\ln r} = 0, \quad \text{where } M(r) = \max_{|z|=r,\, z\in S_\eta}|Q_n(z)|.$$

Then, according to the Phragmén–Lindelöf theorem (see, for example, [**9**, Part II, Theorem 7.5 with proof]),

$$|Q_n(z)| \leqslant C(\varepsilon, \eta) \quad \text{for } z \in S_\eta. \tag{4.5}$$

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Consequently (since $x < |z| + 1/|z|$),

$$|F_n(z)| \leqslant C(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.6}$$

for all z in S_η.

Next, let T_η denote the region given by

$$T_\eta = \{z \in \mathbb{C} \setminus \Omega_\eta : |z| < 1\} \tag{4.7}$$

and define the function R_n by

$$R_n(z) = F_n(z)\mathrm{e}^{-\varepsilon/z}. \tag{4.8}$$

The function is holomorphic in $\mathbb{C} \setminus \{0\}$, and $|R_n(z)| = |F_n(z)| \exp(-\varepsilon x|z|^{-2})$. For z on the line segments of $\partial T_\eta \setminus \{0\}$ we have

$$|R_n(z)| \leqslant M(\varepsilon \cos\eta, \eta) \exp\left[\varepsilon \cos\eta\left(|z| + \frac{1}{|z|}\right)\right] \exp(-\varepsilon|z|^{-1}\cos\eta)$$

(note that $\exp(\varepsilon|z|^{-2}) = \exp(\varepsilon|z|^{-1}\cos\eta)$).

Hence, $|R_n(z)| \leqslant M(\varepsilon\cos\eta, \eta)\mathrm{e}^{\varepsilon\cos\eta}$ (since $|z| \leqslant 1$). For z on the circular arc of ∂T_η we have $|R_n(z)| \leqslant C(\varepsilon, \eta)$. Furthermore, since F_n is a Laurent polynomial, there is a constant Δ_n such that

$$|F_n(z)| \leqslant \Delta_n \exp(\varepsilon|z|^{-1}\cos\eta)$$

for sufficiently small $|z|$. Consequently, $|R_n(z)| \leqslant \Delta_n$ for all sufficiently small $|z|, z \in T_\eta$. (Recall that $\exp(\varepsilon x|z|^{-2}) \geqslant \exp(|z|^{-1}\cos\eta)$.) Thus,

$$\overline{\lim_{z\to 0,\, z\varepsilon E_\eta}} |R_n(z)| \leqslant \Delta_n.$$

Then according to a version of the maximum principle (see, for example, [**9**, Part II, p. 208]) we have

$$|R_n(z)| \leqslant C(\varepsilon, \eta) \quad \text{for } z \in T_\eta. \tag{4.9}$$

Consequently (since $x/|z|^2 < |z| + 1/|z|$),

$$|F_n(z)| \leqslant C(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.10}$$

for all z in T_η.

In a similar way we obtain an estimate

$$|F_n(z)| \leqslant \tilde{C}(\varepsilon, \eta) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.11}$$

for all z in $\mathbb{C} \setminus \Omega_\eta$ with $\mathrm{Re}(z) < 0$.

Now recall that η is an arbitrary fixed value. Taking into account (4.2), (4.6), (4.10) and (4.11), we find that there exists a constant $A(\varepsilon)$ independent of n such that

$$|F_n(z)| \leqslant A(\varepsilon) \exp\left[\varepsilon\left(|z| + \frac{1}{|z|}\right)\right] \tag{4.12}$$

for all $z \in \mathbb{C} \setminus \{0\}$. From this we conclude that (4.1) holds. □

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Remark 4.2. The inequality (4.1) is equivalent to two inequalities of the form

$$|F(z)| \leqslant A_0(\varepsilon) \exp\left[\frac{\varepsilon}{|z|}\right] \quad \text{and} \quad |F(z)| \leqslant A_\infty(\varepsilon) \exp[\varepsilon|z|].$$

It may therefore be natural to state Theorem 4.1 in the following form: the functions α, β, γ and δ (which are holomorphic in $\mathbb{C} \setminus \{0\}$) are of order less than 1 or of order 1 and type 0 at the origin and at ∞.

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