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DECOMPOSITIONS OF GENERALIZED COMPLETE GRAPHS

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A decomposition of a graph G is a collection of edge-disjoint subgraphs of G whose edges partition the edges of G. In the case where each of these subgraphs is isomorphic to some graph H, we say that G admits a decomposition into H. The problem of determining whether a given graph G admits a decomposition into a given graph H is the primary focus of this thesis. In particular, we are concerned with those cases in which H is a cycle.

Obvious necessary conditions for a graph G (having nonempty edge set) to admit a decomposition into cycles of length k are that: (i) G contains at least k vertices; (ii) every vertex in G has even degree; and (iii) the total number of edges in G is a multiple of the cycle length k. Recent papers by Alspach and Gavlas [1] and Šajna [6] have shown that these three necessary conditions are sufficient in the case where G is either K_{2n+1} (the complete graph of odd order), or $K_{2n} - F$ (the complete graph of even order minus a 1-factor). In this thesis we examine whether conditions (i), (ii) and (iii) above are also sufficient when G is one of λK_n (the λ-fold complete multigraph), $K_n * \overline{K}_m$ (the complete equipartite graph having n parts of size m) or $\lambda K_n * \overline{K}_m$ (the λ-fold complete equipartite graph).

In Chapter 3 we consider k-cycle decompositions of λK_n. We also consider directed cycle decompositions of λK_n^*, which is the graph obtained by replacing each edge in λK_n with a pair of oppositely directed arcs. We give various new cyclic and 1-rotational decompositions of these graphs, in particular for those cases in which the cycle length k is a factor of the multiplicity λ. Ultimately we establish necessary and sufficient conditions for the existence of a k-cycle decomposition of λK_n, and of a directed k-cycle decomposition of λK_n^*, in the cases where k is an odd prime.

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In Chapters 4, 5 and 6 we turn our attention to k-cycle decompositions of $K_n * \overline{K}_m$. In [5], Manikandan and Paulraja established necessary and sufficient conditions for the existence of such decompositions in the case where k is an odd prime. In Chapter 4, using new techniques, we extend this result to include cases in which k is either twice an odd prime or three times an odd prime. In Chapter 5 we consider complete equipartite graphs having a 'small' number of parts; in particular, we establish necessary and sufficient conditions for decomposing $K_n * \overline{K}_m$ into k-cycles in the cases where $n = 4, 5$ (the cases $n = 2$ and $n = 3$ having already been dealt with by Sotteau [7] and Cavenagh [3], respectively). We also consider k-path decompositions in these cases. In Chapter 6 we present a new technique—using edge labelled decompositions of multigraphs—for obtaining k-cycle decompositions of $K_n * \overline{K}_m$ in cases where k is a factor of m^2. We use this technique to establish necessary and sufficient conditions for $K_n * \overline{K}_m$ to admit a decomposition into cycles of length p^2 where p is an odd prime.

In Chapter 7 we examine so called *gregarious* k-cycle decompositions of $K_n * \overline{K}_m$; that is, those in which each cycle in the decomposition has each of its vertices lying in a different partite set of $K_n * \overline{K}_m$. We prove that the well-known necessary conditions for the existence of such a decomposition are sufficient for all n, m and even k if and only if they are sufficient for all n in the range $k \le n < 3k$ with m odd, and all n in the range $k \le n < 2k$ with m even. We then use this fact to deal with cases in which $k = 6, 8$. Using several new construction methods, we also establish necessary and sufficient conditions for $K_n * \overline{K}_m$ to admit a decomposition into gregarious cycles of length p where p is an odd prime.

Chapter 8 gives some generalizations of the results in earlier chapters to k-cycle decompositions of $\lambda K_n * \overline{K}_m$, the λ-fold complete equipartite graph. Necessary and sufficient conditions for the existence of such decompositions have been established in the case $k = 3$ by Hanani [4], and in the case $k = 5$ by Billington *et al.* [2]. In this chapter we extend these results to include all prime values of k.

Finally, in Chapter 9 we establish necessary and sufficient conditions for decomposing complete equipartite graphs into closed trails of length k.

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