A NOTE CONCERNING SIMULTANEOUS INTEGRAL EQUATIONS

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1. Introduction. In (**2**) Sinkhorn showed that corresponding to each positive $n \times n$ matrix A (i.e., every $a_{ij} > 0$) is a unique doubly stochastic matrix of the form D_1AD_2, where each D_k is a diagonal matrix with a positive main diagonal. The D_k themselves are unique up to a scalar multiple. In (**3**) the result was extended to show that D_1AD_2 could be made to have arbitrary positive row and column sums (with the reservation, of course, that the sum of the row sums equal the sum of the column sums) where A need no longer be square. The D_k are again unique (up to a scalar multiple). In either of these situations the D_1AD_2 can be obtained as a limit to the iteration of alternately scaling the rows and columns of A to have the appropriate sums. These results lead quite naturally to analogous questions in which the matrices are replaced by continuous functions. It is our intention here to consider some of these questions. Related problems have been studied by Hobby and Pyke in (**1**).

2. The main result.

THEOREM 1. *Let M and N be compact topological spaces and μ and ν non-negative regular Borel measures on M and N, respectively, such that $\mu(M) > 0$ and $\nu(N) > 0$. Let $h(x, y)$ and $k(x, y)$ be positive and continuous on $M \times N$. Then there exist functions $f(x)$ and $g(y)$ positive and continuous on M and N, respectively, and a positive number c such that $\int_M f(x)h(x, y)g(y)d\mu(x) = 1$ for all $y \in N$ and $\int_N f(x)k(x, y)g(y)d\nu(y) = c$ for all $x \in M$. The forms $f(x)h(x, y)g(y)$ and $f(x)k(x, y)g(y)$ can be obtained as limits to the iteration of alternately scaling the functions h and k to have the appropriate integrals over M and N. If, in addition, each non-void open set in M and N has positive measure, then the product $f(x)g(y)$ and the number c are unique. The functions f and g are unique up to a positive scalar multiple.*

Proof. First we demonstrate uniqueness. Suppose (f, g) and (f', g') each satisfy the theorem with positive constants c and c', respectively. We may suppose that $c \leqq c'$. Then

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$$\int_M f(x)h(x,y)g(y)d\mu(x) = 1, \quad \int_M f'(x)h(x,y)g'(y)d\mu(x) = 1, \quad y \in N,$$

and

$$\int_N f(x)k(x,y)g(y)d\nu(y) = c, \quad \int_N f'(x)k(x,y)g'(y)d\nu(y) = c', \quad x \in M.$$

Put $u = f/f'$ and $v = g/g'$. Then

$$\max u(x) = u(x_1) = c[\int_N f'(x_1)k(x_1,y)v(y)g'(y)\, d\nu(y)]^{-1}$$
$$\leqq (c/c')\, [\min v(y)]^{-1} \leqq [\min v(y)]^{-1}$$

and

$$\min v(y) = v(y_1) = [\int_M u(x)f'(x)h(x,y_1)g'(y_1)\, d\mu(x)]^{-1}$$
$$\geqq [\max u(x)]^{-1}.$$

Since non-void open sets have positive measure, equality is possible in both cases only if u and v are constant functions. Since equality is forced, this must be so. It is clear then that $uv = 1$ and $c = c'$. Thus $fg = f'g'$.

Define

$$\phi(f,g) = \max\int_N f(x)k(x,y)g(y)\, d\nu(y) - \min\int_N f(x)k(x,y)g(y)\, d\nu(y).$$

Let

$$f_0(x) = 1, \qquad g_0(y) = [\int_M h(x,y)\, d\mu(x)]^{-1}.$$

An iteration is now constructed in the following way. Begin with (f_0, g_0) above, and set $f_{n+1}(x) = a_n(x)f_n(x)$, $g_{n+1}(y) = b_n(y)g_n(y)$, $n = 0, 1, 2, \ldots,$ where

$$a_n(x) = [\int_N f_n(x)k(x,y)g_n(y)\, d\nu(y)]^{-1},$$
$$b_n(y) = [\int_M a_n(x)f_n(x)h(x,y)g_n(y)\, d\mu(x)]^{-1}.$$

Let $E_n = \{y \in N: b_n(y) < [\underline{b}_n + \bar{b}_n]/2\}$. (In this definition and hereafter we shall indicate the minimum and maximum values of a function on its domain of definition by bars under and over the function, respectively.) Then define

$$d_n(x) = \int_{E_n} a_n(x)f_n(x)k(x,y)g_n(y)\, d\nu(y)$$

and note that

$$\int_{E'_n} a_n(x)f_n(x)k(x,y)g_n(y)\, d\nu(y) = 1 - d_n(x),$$

where $E'_n = N - E_n$.

Then

$$\phi(f_{n+1}, g_{n+1}) = \max[a_{n+1}(x)]^{-1} - \min[a_{n+1}(x)]^{-1} = [a_{n+1}(x_2)]^{-1} - [a_{n+1}(x_1)]^{-1}$$
$$= [\int_{E_n} a_nf_nkb_ng_n\, d\nu(y) + \int_{E'_n} a_nf_nkb_ng_n\, d\nu(y)]_{x=x_2}$$
$$- [\int_{E_n} a_nf_nkb_ng_n\, d\nu(y) + \int_{E'_n} a_nf_nkb_ng_n\, d\nu(y)]_{x=x_1}$$
$$\leqq [d_n(x_2)[\underline{b}_n + \bar{b}_n]/2 + \bar{b}_n[1 - d_n(x_2)]]$$
$$- [\underline{b}_nd_n(x_1) + [1 - d_n(x_1)][\underline{b}_n + \bar{b}_n]/2]$$
$$= \tfrac{1}{2}[1 - d_n(x_2) + d_n(x_1)][\bar{b}_n - \underline{b}_n],$$

where x_1 and x_2 depend upon n. Since $0 \leqq d_n(x) \leqq 1$,

$$\tfrac{1}{2}[1 - d_n(x_2) + d_n(x_1)] \leqq 1.$$

We shall show that the quantity above is in fact bounded away from 1 for all n.

For each $n > 0$ define $\tilde{f}_n(x) = \underline{h}^{-\frac{1}{2}} M_{n-1}{}^{-1} f_n(x)$ and $\tilde{g}_n(y) = \underline{h}^{\frac{1}{2}} M_{n-1} g_n(y)$, where $M_n = ||a_n f_n||_1 = \int_M a_n(x) f_n(x)\, d\mu(x)$, $n \geqq 0$, and set $\tilde{f}_0(x) = \underline{h}^{-\frac{1}{2}}/\mu(M)$, $\tilde{g}_0(y) = \underline{h}^{\frac{1}{2}}[\int_M h(x, y)\, d\mu(x)]^{-1}\mu(M)$. It readily follows that $||\tilde{f}_n||_1 = \underline{h}^{-\frac{1}{2}}$ and $||\tilde{g}_n||_\infty \leqq \underline{h}^{-\frac{1}{2}}$ if one observes that

$$b_n(y) g_n(y) = [\int_M a_n(x) f_n(x) h(x, y)\, d\mu(x)]^{-1} \leqq \underline{h}^{-1} M_n{}^{-1}.$$

Consequently,

$$a_n(x)\tilde{f}_n(x) = [\int_N k(x, y)\tilde{g}_n(y)\, d\nu(y)]^{-1} \geqq \bar{k}^{-1}\underline{h}^{\frac{1}{2}}/\nu(N)$$

and

$$\tilde{g}_n(y) = [\int_M \tilde{f}_n(x) h(x, y)\, d\mu(x)]^{-1} \geqq \bar{h}^{-1}\underline{h}^{\frac{1}{2}}$$

and therefore

$$a_n(x) f_n(x) k(x, y) g_n(y) = a_n(x)\tilde{f}_n(x) k(x, y)\tilde{g}_n(y) \geqq \frac{\underline{h}\underline{k}}{\bar{h}\bar{k}} \frac{1}{\nu(N)} > 0$$

for all n and all x and y.

Let $\alpha = \underline{h}\underline{k}/4\bar{h}\bar{k}$ and suppose that for some n

$$1 - \alpha < \tfrac{1}{2}[1 - d_n(x_2) + d_n(x_1)] \leqq 1.$$

Then

$$1 - 2\alpha < d_n(x_1) - d_n(x_2) \leqq d_n(x_1) \leqq 1,$$

which gives

$$0 \leqq 1 - d_n(x_1) < 2\alpha, \qquad 0 \leqq d_n(x_2) < 2\alpha,$$

i.e.,

$$0 \leqq [\int_{E'_n} a_n f_n k g_n\, d\nu(y)]_{x=x_1} < 2\alpha,$$

$$0 \leqq [\int_{E_n} a_n f_n k g_n\, d\nu(y)]_{x=x_2} < 2\alpha.$$

Then we would have

$$0 \leqq [\underline{h}\underline{k}/\bar{h}\bar{k}][\nu(E_n) + \nu(E'_n)]/\nu(N) = \underline{h}\underline{k}/\bar{h}\bar{k} < 4\alpha,$$

a contradiction. Thus

$$\tfrac{1}{2}[1 - d_n(x_2) + d_n(x_1)] \leqq 1 - \alpha = \theta < 1,$$

for all n. Furthermore, from the definitions of a and b we see that

$$(\bar{a}_n)^{-1} \leqq \underline{b}_n \leqq (\bar{a}_{n+1})^{-1} \leqq (\underline{a}_{n+1})^{-1} \leqq \bar{b}_n \leqq (\underline{a}_n)^{-1};$$

hence,

$$[\bar{b}_n - \underline{b}_n] \leqq [(\underline{a}_n)^{-1} - (\bar{a}_n)^{-1}].$$

Then

$$\phi(f_{r+1}, g_{n+1}) \leqq \theta[(\underline{a}_n)^{-1} - (\bar{a}_n)^{-1}] = \theta\phi(f_n, g_n),$$

and

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$$\phi(f_n, g_n) \leqq \theta^n \phi(f_0, g_0) = K\theta^n \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that there is a number $c > 0$ such that $\underline{b}_n \uparrow c$, $\bar{b}_n \downarrow c$, $\underline{a}_n \uparrow 1/c$, $\bar{a}_n \downarrow 1/c$ so that $a_n(x)$ converges uniformly to $1/c$ and $b_n(y)$ uniformly to c. Next consider the infinite product $(c\bar{a}_0)\,(c\bar{a}_1)\ldots(c\bar{a}_n)\ldots$. Since

$$[(\underline{a}_n)^{-1} - (\bar{a}_n)^{-1}] \leqq K\theta^n,$$

we have

$$1 \leqq c\bar{a}_n \leqq (1 - c^{-1}K\theta^n)^{-1}$$

and thus

$$0 \leqq \log c\bar{a}_n \leqq \log(1 - c^{-1}K\theta^n)^{-1}, \tag{1}$$

if n is sufficiently large so that $1 - c^{-1}K\theta^n > 0$. The ratio test shows that $\sum \log(1 - c^{-1}K\theta^n)^{-1}$ converges. Thus the product $(c\bar{a}_0)\,(c\bar{a}_1)\ldots(c\bar{a}_n)\ldots$ converges. Consequently, there is an $R > 0$ such that $(c\bar{a}_0)\,(c\bar{a}_1)\ldots(c\bar{a}_n) \leqq R$ for all n. Note that

$$|c^{n+1}f_{n+1}(x) - c^n f_n(x)| = c^n|ca_n(x) - 1|f_n(x)$$
$$\leqq |ca_n(x) - 1|(c\bar{a}_0)\,(c\bar{a}_1)\ldots(c\bar{a}_{n-1})f_0(x) \leqq |ca_n(x) - 1|R$$

and therefore, if $m > p$,

$$|c^m f_m(x) - c^p f_p(x)| \leqq \sum_{n=p}^{m-1} |c^{n+1}f_{n+1}(x) - c^n f_n(x)| \leqq \sum_{n=p}^{m-1} |ca_n(x) - 1|R. \tag{2}$$

But from $[(\underline{a}_n)^{-1} - (\bar{a}_n)^{-1}] \leqq K\theta^n$ $(1 - c^{-1}K\theta^n > 0)$ we see that

$$\frac{-K\theta^n}{c + K\theta^n} \leqq ca_n(x) - 1 \leqq \frac{K\theta^n}{c - K\theta^n}$$

and, therefore,

$$|ca_n(x) - 1| \leqq \frac{K\theta^n}{c - K\theta^n}. \tag{3}$$

By the ratio test, $K\theta^n/(c - K\theta^n)$ is summable. Thus, $|ca_n(x) - 1|$ is summable, uniformly on M. It follows that $c^n f_n(x)$ converges uniformly on M to a continuous limit $f(x)$. Since $(\bar{b}_n - \underline{b}_n) \leqq K\theta^n$,

$$0 \leqq \log c^{-1}\bar{b}_n \leqq \log(1 + c^{-1}K\theta^n), \tag{4}$$

and since $\sum_{n=0}^{\infty} \log(1 + c^{-1}K\theta^n)$ converges, so must $\Pi_{n=0}^{\infty}\,(c^{-1}\bar{b}_n)$. Thus there is a number $R' > 0$ such that $\Pi_{k=0}^{n}\,(c^{-1}\bar{b}_k) \leqq R'$ for all n, whence

$$\begin{aligned} |c^{-n-1}g_{n+1}(y) - c^{-n}g_n(y)| &= c^{-n-1}|\,b_n(y) - c|g_n(y) \\ &\leqq c^{-1}(\bar{b}_n - \underline{b}_n)R'g_0(y) \leqq c^{-1}R'\,||g_0||_\infty K\theta^n, \end{aligned} \tag{5}$$

and therefore

$$\begin{aligned} |c^{-m}g_m(y) - c^{-p}g_p(y)| &\leqq \sum_{n=p}^{m-1} |c^{-n-1}g_{n+1}(y) - c^{-n}g_n(y)| \\ &\leqq \sum_{n=p}^{m-1} c^{-1}R'\,||g_0||_\infty K\theta^n. \end{aligned} \tag{6}$$

It follows that $c^{-n}g_n(y)$ converges uniformly on N to a continuous limit $g(y)$.

Since

$$\int_M f_n(x)h(x, y)g_n(y)\, d\mu(x) = 1$$

and

$$\int_N f_{n+1}(x)k(x, y)g_n(y)\, d\nu(y) = 1,$$

then

$$\int_M c^n f_n(x)h(x, y)c^{-n}g_n(y)\, d\mu(x) = 1$$

and

$$\int_N c^{n+1}f_{n+1}(x)k(x, y)c^{-n}g_n(y)\, d\nu(y) = c,$$

and, consequently,

$$(7) \qquad \int_M f(x)h(x, y)g(y)\, d\mu(x) = 1, \qquad \int_N f(x)k(x, y)g(y)\, d\nu(y) = c.$$

At this point it is clear that f and g are positive.

COROLLARY 1. *Under the hypothesis of the theorem, the functions f and g and the quantity c are continuously dependent upon h and k.*

Proof. Let h_0 and k_0 be positive and continuous on $M \times N$ and let ρ be such that $0 < \rho < \frac{1}{2}\min(\underline{h}_0, \underline{k}_0)$. Denote by D the collection of all ordered pairs (h, k) which are positive and continuous on $M \times N$ and such that $||h - h_0||_\infty < \rho$ and $||k - k_0||_\infty < \rho$. It readily follows that $\bar{h} - \bar{h}_0 < \rho$, $\underline{h}_0 - \underline{h} < \rho$, $\bar{k} - \bar{k}_0 < \rho$, $\underline{k}_0 - \underline{k} < \rho$ for any $(h, k) \in D$, whence, for any such (h, k),

$$\frac{\underline{h}\underline{k}}{4\bar{h}\bar{k}} > \frac{(\underline{h}_0 - \rho)(\underline{k}_0 - \rho)}{4(\bar{h}_0 + \rho)(\bar{k}_0 + \rho)} = \alpha_0 > 0.$$

Let $\theta_0 = 1 - \alpha_0$. Then $0 \leqq 1 - \underline{h}\underline{k}/4\bar{h}\bar{k} \leqq \theta_0$.

If $(h, k) \in D$ and if $g_0(y) = [\int_M h(x, y)\, d\mu(x)]^{-1}$, it follows that

$$[(\bar{h}_0 + \rho)\mu(M)]^{-1} \leqq ||g_0||_\infty \leqq [(\underline{h}_0 - \rho)\mu(M)]^{-1}.$$

Also, if $K = \phi(f_0, g_0)$,

$$0 \leqq K \leqq \frac{(\bar{k}_0 + \rho)\nu(N)}{(\underline{h}_0 - \rho)\mu(M)} - \frac{(\underline{k}_0 - \rho)\nu(N)}{(\bar{h}_0 + \rho)\mu(M)} = K_0.$$

If $a_0(x) = [\int_N k(x, y)g_0(y)\, d\nu(y)]^{-1}$,

$$0 < \bar{a}_0 \leqq \frac{(\bar{h}_0 + \rho)\mu(M)}{(\underline{k}_0 - \rho)\nu(N)} = c_0^{-1}.$$

For the given $(h, k) \in D$, $\underline{a}_n \leqq \underline{a}_{n+1} \leqq c^{-1} \leqq \bar{a}_{n+1} \leqq \bar{a}_n$ for all n, and therefore

$$0 < c^{-1} \leqq c_0^{-1}.$$

Equations (1) and (4) show that there exists an $R_0 > 0$ such that $R \leqq R_0$ and $R' \leqq R_0$ for all $(h, k) \in D$.

Thus (2) and (6) may be written, respectively, as

$$||c^m f_m - c^p f_p||_\infty \leqq \sum_{n=p}^{m-1} \left[\frac{K_0\theta_0^{\,n}}{c_0 - K_0\theta_0^{\,n}}\right] R_0$$

and

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$$||c^{-m}g_m - c^{-p}g_p||_\infty \leqq \sum_{n=p}^{m-1} c_0^{-1} R_0[(\underline{h}_0 - \rho)\mu(M)]^{-1}K_0\theta_0^{\ n}$$

for all $(h, k) \in D$. Thus $c^n f_n \to f$ and $c^{-n}g_n \to g$ uniformly on D. It follows that f and g depend continuously on h and k. The second equation in (7) shows that the quantity c likewise depends continuously on h and k.

COROLLARY 2. *Let A and B be positive $n \times n$ matrices. There exist diagonal matrices D_1 and D_2 with positive main diagonals and a positive number c such that D_1AD_2 and cD_2BD_1 are stochastic. The number c is unique and the D_i are unique up to a scalar multiple.*

3. Consequences.

THEOREM 2. *Let M and N be compact topological spaces and let μ and ν be non-negative regular Borel measures on M and N, respectively, such that $\mu(M) > 0$ and $\nu(N) > 0$. Let $H(x, y)$, $F(x)$, and $G(y)$ be positive and continuous on $M \times N$, M, and N, respectively, such that $\int_M F\,d\mu = \int_N G\,d\nu$. Then there exist functions $f(x)$ and $g(y)$ positive and continuous on M and N, respectively, such that $\int_N f(x)H(x, y)g(y)\,d\nu(y) = F(x)$ and*

$$\int_M f(x)H(x, y)g(y)\,d\mu(x) = G(y).$$

The form $f(x)H(x, y)g(y)$ can be obtained as a limit to the iteration of alternately scaling the function H to have the correct integrals over M and N. If, in addition, each non-void open set in M and N has positive measure, then $f(x)H(x, y)g(y)$ is unique and the functions f and g are unique up to a positive scalar multiple.

Proof. Use $h(x, y) = H(x, y)/G(y)$ and $k(x, y) = H(x, y)/F(x)$. It remains to show that $c = 1$.

Given $0 < \epsilon \leqq c$, pick n so that $c - \epsilon < [a_n(x)]^{-1} < c + \epsilon$. Then

$$\int_M (c - \epsilon)F(x)\,d\mu(x) < \int_M [F(x)/a_n(x)]\,d\mu(x) < \int_M (c + \epsilon)F(x)\,d\mu(x)$$

or

$$(c - \epsilon)\int_M F(x)\,d\mu(x) < \int_M d\mu(x)\int_N f_n(x)H(x, y)g_n(y)\,d\nu(y) < (c + \epsilon)\int_M F(x)\,d\mu(x).$$

Using Fubini's theorem and cancelling $\int_M F(x)\,d\mu(x) = \int_N G(y)\,d\nu(y)$, we get $c - \epsilon < 1 < c + \epsilon$. Thus $c = 1$.

Some interpretations of Theorem 2 are listed below.

COROLLARY 1 (Sinkhorn (**3**)). *Let A be a positive $n \times m$ matrix and let $r_1, \ldots, r_n, c_1, \ldots, c_m$ be given positive numbers with $\sum r_i = \sum c_j$. Then there exists a unique matrix of the form D_1AD_2 with row sums r_i and column sums c_j, where D_1 and D_2 are diagonal matrices with positive main diagonals. The D_i are themselves unique up to a scalar multiple.*

COROLLARY 2. *Let* $h_k(x)$, $k = 1, \ldots, n$, *be positive and continuous on* $[0, 1]$. *Let* $F(x)$ *be positive and continuous on* $[0, 1]$ *and let* $r_1, \ldots, r_n$ *be positive numbers such that*

$$\int_0^1 F(x)\,dx = r_1 + \ldots + r_n.$$

There exists a unique collection of functions $H_k(x)$, $k = 1, \ldots, n$, *on* $[0, 1]$ *of the form* $f(x)h_k(x)g_k$ *such that*

$$\int_0^1 H_k(x)\,dx = r_k$$

while $\sum H_k(x) = F(x)$. *The function* $f(x)$ *is positive and continuous on* $[0, 1]$ *and each* $g_k > 0$. *The function* $f(x)$ *and the numbers* g_k *are themselves unique up to a scalar multiple.*

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