Stellar Rotation
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Rotational Transport Processes

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Abstract. In this review, I discuss physical mechanisms leading to momentum and chemical transport in stars. Various instabilities leading to turbulence are discussed. I then present a self-consistent description of rotational mixing under the action of turbulence and meridional circulation in 1D models. Limitations of the model are discussed, both in terms of an extra mechanism for momentum transport in the Sun and solar-type stars (magnetic field and/or gravity waves) and in terms of our understanding of turbulent properties.

1. Introduction

Descriptions of rotational mixing have evolved a lot in recent years. The need for extra mixing in stars is widely recognized, and different rotational histories could well explain the variety of stellar behaviors (another key ingredient would be the star's initial magnetic field - see Spruit and Charbonneau, these proc.).

Early descriptions have relied on very simple power laws relating the amount of mixing to the star's rotation rate (see *e.g.* Zahn 1983). In the case of lithium destruction in low mass stars for example, this would lead to larger dispersion in Li than observed (Schatzman & Baglin 1991).

Very early on, it was recognized that such scaling laws fail to produce a proper depth dependent diffusion profile and that consistent models must be built that take into account the angular momentum evolution (Endal & Sofia 1978). To do so, one must consider all possible transport processes for angular momentum and then analyze their effect on the transport of chemicals.

2. Rotation driven instabilities

In this first section, we will be looking into hydrodynamical instabilities. They are presented by considering what happens if one displaces a fluid parcel from its original position.

2.1. Brunt-Väisälä frequency

Let us displace upwards a parcel of fluid from its equilibrium position in a radiative zone (Fig. 1A). If the displacement is adiabatic and if pressure equilibrium is maintained, the density inside the parcel will be larger than the local density

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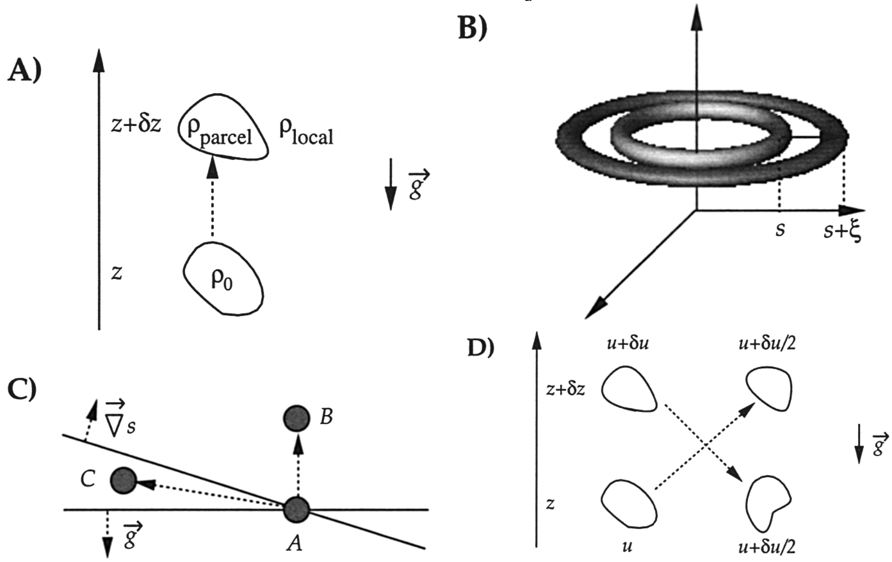

Figure 1. **A)** Brunt-Väisälä frequency **B)** Solberg-Høiland instability **C)** GSF instability **D)** Shear instability

by an amount

$$\frac{\rho_{\rm parcel}-\rho_{\rm local}}{\rho_0}=\delta z\left(\left.\frac{{\rm d}\ln T}{{\rm d}z}\right|_{\rm ad}-\frac{{\rm d}\ln T}{{\rm d}z}\right)+\delta z\frac{{\rm d}\ln\mu}{{\rm d}z}=\frac{\delta z}{H_P}\left(\nabla-\nabla_{\rm ad}-\nabla_\mu\right). \tag{1}$$

The parcel then experiences a restoring force $-g\left(\rho_{\rm parcel}-\rho_{\rm local}\right)$, giving rise to an oscillation characterized by the Brunt-Väisälä frequency

$$N^2=N_T^2+N_\mu^2=\frac{g}{H_P}\left(\nabla_{\rm ad}-\nabla+\nabla_\mu\right). \tag{2}$$

2.2. Solberg-Høiland instability

The first instability is related to the Coriolis force (Rayleigh 1916, Taylor 1923). We consider a fluid in *cylindrical rotation.* Neglecting gravity, pressure gradients balance centrifugal acceleration

$$-\frac{1}{\rho}\frac{{\rm d}P}{{\rm d}s}+s\Omega^2=0. \tag{3}$$

Let us displace a fluid torus over some distance ξ (*c.f.* Fig. 1B). The net force acting on the fluid depends on the difference between the torus value $(s+\xi)\Omega^2_{\rm torus}$ and the equilibrium value $(s+\xi)\Omega^2_{s+\xi}$. Since the Coriolis force ensures (specific) angular momentum ($j=s^2\Omega$) conservation, it may be written as

$$\left(s+\xi\right)\Omega^2_{\rm torus}-\left(s+\xi\right)\Omega^2_{s+\xi}=\frac{1}{\left(s+\xi\right)^3}\left(j_s^2-j_{s+\xi}^2\right)\simeq\frac{1}{s^3}\frac{{\rm d}\left(j^2\right)}{{\rm d}s}\xi \tag{4}$$

valid to first order in ξ. We use this condition to define the Rayleigh frequency

$$N_\Omega^2=\frac{1}{s^3}\frac{\rm d}{{\rm d}s}\left(s^2\Omega\right)^2 \tag{5}$$

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which corresponds to a restoring force if positive.

If gravity is taken into account, there are now two restoring forces. For radial displacements, the stability condition then reads

$$N^2 + N_\Omega^2 \geq 0. \tag{6}$$

For general axisymmetric perturbations, the stability criterion is twofold:
• Condition (6) must be satisfied;
• Specific angular momentum must increase from pole to equator.
The second condition corresponds to applying the Rayleigh criterion ($N_\Omega^2 > 0$) to an isentropic surface; in the limit of no rotation, the first condition corresponds to the Schwarzschild criterion. These conditions have been derived by Solberg (1936) and Høiland (1941).

2.3. Baroclinic instabilities

If the star's rotation law is not cylindrical, it can be shown that surfaces of constant pressure, density, entropy and gravity do not coincide. This is called a baroclinic state (in opposition to a barotropic state). Let us now imagine displacing a fluid element vertically[1] in such a star, going from A to B in Fig. 1C. Assuming adiabaticity, the fluid will then be denser than its environment and buoyancy will restore it to its original position. However, if the same parcel is moved to C, it will now have a lower density, and gravity will make it rise, leading to an instability, which is called a *baroclinic instability.* However, angular momentum conservation makes axisymmetric movements towards C impossible, and only non-axisymmetric perturbations may thus grow.

GSF instability: In stellar interiors, thermal diffusivity K_T is much larger than viscosity ν which controls angular momentum diffusion. This weakens the stabilizing effect of the density stratification (Goldreich & Schubert 1967). This leads to a modified version of the Solberg-Høiland criterion

$$\frac{\nu}{K_T} N_T^2 + N_\Omega^2 \geq 0 \tag{7}$$

for stability of axisymmetric perturbations. In the absence of viscosity, baroclinic instabilities may grow as soon as rotation is not cylindrical, that is if $|\partial\Omega/\partial z| \neq 0$; however, Fricke (1968) showed that viscosity modifies this condition and that the stability condition rather reads

$$\left| s \frac{\partial \Omega}{\partial z} \right|^2 < \frac{\nu}{K_T} N_T^2. \tag{8}$$

ABCD instability: If the star is not homogeneous, mean molecular weight gradients contribute to the stratification; this contribution is also weakened, but this time by molecular diffusivity K_μ. The Solberg-Høiland criterion is further modified and becomes

$$\frac{\nu}{K_T} N_T^2 + \frac{\nu}{K_\mu} N_\mu^2 + N_\Omega^2 \geq 0. \tag{9}$$

[1]The vertical is defined by the direction of gravity.

Since K_μ is rather small (of order ν), this instability likely does not set in. There is however another possibility. When displacing a parcel towards C, it will come back to its equilibrium position A due to momentum conservation. However, if thermal diffusion is large enough, it will then be cooler than its surroundings, and the parcel will sink further in. This will lead to growing oscillations, a situation called *overstability*. The condition for stability is then (Knobloch & Spruit 1983)

$$\frac{\nu}{K_T}\left(N_T^2 + N_\mu^2\right) + N_\Omega^2 \geq 0. \tag{10}$$

All criteria described so far are linear, and little work has been done on the non-linear behavior of the corresponding instabilities, implying that we do not know how efficient they are for mixing.

2.4. Shear instability

It can be shown that the minimum energy state of a rotating fluid is solid body rotation. Thus, if the star is rotating differentially, by homogenizing the velocities, it is possible to extract energy. However, to do so, one must be able to overcome the density stratification. By comparing both and assuming adiabaticity, one obtains the Richardson stability criterion (Fig. 1D)

$$Ri = \frac{N^2}{(\mathrm{d}u/\mathrm{d}z)^2} > Ri_{\mathrm{crit}} = \frac{1}{4} \tag{11}$$

(see Maeder 1995 for details).

Dynamical shear instability: Along isobars, there is no restoring force, and thus, shear is unstable as soon as horizontal differential rotation is present. Horizontal shear is thus a dynamical instability, acting on a dynamical hence quite short time-scale and leading to a large horizontal turbulent viscosity.

Secular shear instability: In the direction of entropy stratification, both the thermal and the mean molecular weight stratifications will hinder the growth of the instability. However, thermal diffusion and horizontal shear will act as to diminish those effects, leading to the *instability* criterion

$$\left(\frac{\Gamma}{\Gamma+1}\right) N_T^2 + \left(\frac{\Gamma_\mu}{\Gamma_\mu+1}\right) N_\mu^2 < Ri_{\mathrm{crit}} \left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 \tag{12}$$

where $\Gamma = v\ell/K_T$ and $\Gamma_\mu = v\ell/K_\mu$ (Talon & Zahn 1997). This criterion implies that, the smaller the eddy, the more efficient thermal diffusivity. In fact, there always exists an eddy that is small enough so that the instability criterion (12) will be satisfied. Turbulent diffusivity will be given by the largest eddy satisfying (12); this is a non-linear description, based on energy considerations. It is the formulation we shall adopt to describe the vertical shear instability.

3. Meridional circulation

Meridional circulation, related to the thermal imbalance present in rotating stars has been studied for a long time. However, since it is actually at least a 2D effect, not much progress has been done in incorporating it into stellar evolution codes until quite recently. The key ingredient to allow the modeling of meridional circulation as a 1D process is the assumption of highly anisotropic turbulence, with the vertical turbulent viscosity ν_v being much smaller than the horizontal turbulent viscosity ν_h. This property is related to the vertical stratification (Tassoul & Tassoul 1983, Zahn 1992). This leads to the assumption of shellular rotation *i.e.* $\Omega = \Omega(P)$. Conditions for this approximation to be valid are discussed by Tassoul & Tassoul (1983), Zahn (1992) and Charbonneau (1992 - this discussion is based on numerical simulations).

3.1. Heat flux

Under that assumption, and to first order, it is possible to derive an equation for the thermal imbalance in a star with a given (shellular) rotation profile. Then, meridional circulation $\vec{u}$ is obtained by stating that advection of entropy S must counterbalance this thermal imbalance

$$\rho T\vec{u}\cdot\vec{\nabla}S = \vec{\nabla}\cdot\left(\chi\vec{\nabla}T\right) + \rho\varepsilon - \vec{\nabla}\cdot\vec{F}_h \tag{13}$$

(this form of the equation also contains a turbulent horizontal heat flux $\vec{F}_h$, Maeder & Zahn 1998). The vertical component of the asymptotic circulation velocity is then

$$u = \frac{P}{\rho g C_P T}\frac{1}{(\nabla_{\rm ad} - \nabla + \nabla_\mu)}\left[\frac{L}{M}\left(E_\Omega + E_\Theta + E_\mu\right) + \frac{TC_P}{\delta}\frac{\partial\Theta}{\partial t}\right] P_2(\cos\theta) \tag{14}$$

(see Maeder & Zahn 1998 for the complete expression). Let us note here that the velocity is related to a term E_Ω which depends on the rotation rate, a term E_Θ which depends on the radial differential rotation ($\Theta \propto {\rm d}\Omega^2/{\rm d}r$) and a term E_μ which depends on the horizontal variations of the mean molecular weight ($\tilde{\mu}/\mu$). In a homogeneous solid body rotating star, only the first term survives, and one gets Sweet's (1950) classical solution characterized by a large circulation cell rising along the pole, with a reversed cell at the surface (Gratton 1945, Öpik 1951).

Let us note that if overshooting is not present, the solution diverges at the convective boundary and viscous boundary layers are then required (Tassoul & Tassoul 1982).

3.2. Momentum transport

If the only transport processes present are meridional circulation and hydrodynamical instabilities, the distribution of angular momentum within the star will evolve according to an advection-diffusion equation

$$\rho\frac{\rm d}{{\rm d}t}\left[r^2\Omega\right] = \frac{1}{5r^2}\frac{\partial}{\partial r}\left[\rho r^4\Omega u\right] + \frac{1}{r^4}\frac{\partial}{\partial r}\left[\rho\nu_v r^4\frac{\partial\Omega}{\partial r}\right] \tag{15}$$

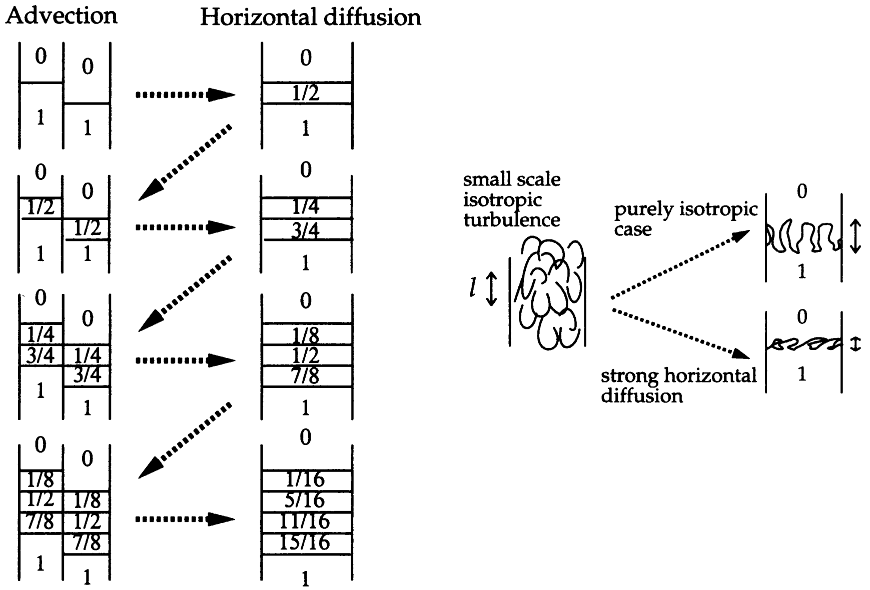

Figure 2. **left)** Combination of advection and horizontal transport leads to an effective vertical diffusion as illustrated in this cartoon **right)** Strongly anisotropic turbulence reduces the vertical extent of eddies, leading to a reduction of vertical transport.

(see Zahn 1992 for details). Let us note here the use of a vertical turbulent viscosity ν_v. This turbulence is attributed to relevant rotational instabilities detailed in § 2. Neglecting the star's evolution, this equation admits a stationary solution in which advective transport is counterbalanced by turbulent transport. If no turbulence is present and if the star is homogeneous, circulation will simply stop, resulting in a rotation profile with a core rotating about 30% faster than the surface (Urpin et al. 1996, Talon et al. 1997). The equilibrium profile in Θ thus depends on the nature and the vigor of rotational instabilities present in the star.

If the star is not chemically homogeneous, meridional circulation will produce horizontal variations of mean molecular weight out of the vertical chemical stratification. These contribute to the heat flux, via the equation of state. Equation (14) shows that, if horizontal variations of mean molecular weight are present, the equilibrium profile described in the previous paragraph exists; however, it now describes the function $\Theta - \widetilde{\mu}/\mu$. If rotation evolves solely under the action of meridional circulation and turbulence, this equilibrium requires an increase in differential rotation (compared to the homogeneous case) in order to maintain the asymptotic solution. This effect has been observed in fully consistent unidimensional calculations including differential rotation and mean molecular weight gradients (Talon et al. 1997, Palacios et al. 2003).

If the star is constrained to rotate as a solid body[2] and does not evolve, the build up of horizontal mean molecular weight fluctuations can thus also lead to a circulation free state, as originally suggested by Mestel (1953).

In an evolving stellar model, the required equilibrium profile also evolves (e.g. Talon et al. 1997). This leads to the appearance of a "re-adjustment" circulation, which is added to the asymptotic circulation. This circulation cannot be avoided, even in chemically inhomogeneous stars rotating as solid bodies.

3.3. Wind-driven circulation

If the star's surface is braked via a magnetic torque, as is the case of Pop I stars cooler than about 7000 K, the internal distribution of angular momentum is rapidly moved away form its equilibrium profile. This is the regime of "wind-driven" circulation, and the model presented up to now predicts important mixing related to such a state. This regime is discussed further in § 4, and in Balachandran (these proc.).

3.4. Chemical transport

While momentum transport remains a truly advective process when considering the combined effect of meridional circulation and horizontal diffusion, it is possible to understand heuristically that this combination leads to (vertical) diffusion in the case of chemicals (Fig. 2 left, see also Chaboyer & Zahn 1992); this has also been shown in numerical simulations (Charbonneau 1992). We assume that diffusion coefficients are equal to the turbulent viscosities *i.e.* $D_v = \nu_v$ and $D_h = \nu_h$. The advection + horizontal diffusion + vertical diffusion equation can thus be replaced by a purely diffusive equation, with vertical turbulent transport and an effective diffusivity depending on horizontal turbulent transport D_h and advective velocity u

$$\rho \frac{\mathrm{d}c}{\mathrm{d}t} = \rho \dot{c} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \rho u_{\mathrm{mic}} c \right] + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \rho \left(D_{\mathrm{eff}} + D_v \right) \frac{\partial c}{\partial r} \right]. \qquad (16)$$

The effective diffusivity is $D_{\mathrm{eff}} = r^2 u^2 / 30 D_h$ (Chaboyer & Zahn 1992).

4. Other sources of momentum transport

This fully consistent 1D description of the evolution of angular momentum under the action of various rotational instabilities and meridional circulation has been applied to stars in various mass ranges with great success. It explains among other things He enrichment in massive stars, the hot side of the Li dip, as well as Li underabondances of massive stars. For details, see Talon et al. (1997), Maeder & Meynet (2000), Palacios et al. (2003), Maeder & Meynet (these proc.). However, various simulations have lead to the conclusion that there must be another transport process for angular momentum in solar type stars (Chaboyer et al. 1995, Matias & Zahn 1998). Indeed, those models lead

[2]In solid body rotation, no hydrodynamical instabilities can develop.

to a large amount of differential rotation at the age of the Sun, in clear disagreement with helioseismology. Thus, any fully consistent study of low mass stars has to invoke another transport mechanism. Furthermore, surface lithium measurements imply that it must be far more efficient for momentum than it is for chemicals.

A fossil magnetic field (uncoupled to the surface convection zone) could explain the rotation of the Sun's radiative zone (Charbonneau & MacGregor 1993, Barnes et al. 1998, see also Charbonneau, these proc.).

Another suggestion for momentum transport is related to gravity waves, excited at the base of the surface convection zone (Schatzman 1993, Kumar & Quataert 1997, Zahn et al. 1997).

4.1. Gravity waves

Much like the pressure waves observed via helioseismology, gravity waves can be excited by convection, either via Reynolds stresses (Balmforth 1992, Goldreich et al. 1994) or via convective penetration (Hurlburt et al. 1986, García López & Spruit 1991). While high frequency waves can form standing waves (the elusive g modes of helioseismology) low frequency waves are damped rapidly. If prograde waves (that travel in the direction of rotation) and retrograde waves (that travel in the opposite direction) are damped differently, damping leads to the redistribution of angular momentum in the star.

On very short time-scales, this differential damping leads to the formation of a double peaked, oscillating shear layer just below the convection zone (Ringot 1998, Gough & McIntyre 1998, Kumar et al. 1999). However, if differential rotation is initially present below that shear layer (as in stars spun down by magnetic torquing), it does not oscillate symmetrically; this leads to differential filtering of the waves. Talon et al. (2002) recently showed that this extracts momentum from the radiative zone. This process could even be efficient enough to produce an inner core rotating slightly slower than the convection zone as suggested by certain helioseismic inversions (see Turck-Chièze, these proc.). If such a feature was confirmed in the data, it could allow to evaluate the relative importance of magnetic and gravity wave transport.

Another important feature of wave transport is its dependence on stellar mass. Indeed, as suggested by Talon & Charbonnel (1998), the hot side of the Li dip can be well understood in the mixing scenario described herein. The diminution of Li with decreasing temperature is then related to braking, which is known to begin at precisely the same temperature. Then, the rise of Li abundances on the cold side of the dip has to be related to the appearance of the extra mechanism for momentum transport. Momentum transport by gravity waves has the required dependence (Talon & Charbonnel, these proc.).

5. Open problems

The 1D modeling of meridional circulation is certainly a very important step in understanding and testing models of stellar evolution with rotation. However, one must not forget that it is actually an advective transport and that unidimensional modeling remains an approximation.

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The major limitation of the model comes from the invoked horizontal turbulent diffusion coefficient. While in actual models it is linked to the source of horizontal shear (that is, some function of u) its magnitude remains quite uncertain. Furthermore, the hypothesis is that its effect will be to homogenize horizontal fluctuations. In the earth atmosphere, this is known to happen, but only up to scales that are smaller than the Rossby radius[3], which is given by

$$L_{\rm Rossby} = \frac{H_P N}{2\Omega \sin\phi} \tag{17}$$

where ϕ is the latitude. With typical values for H_P and N, it is of order $L_{\rm Rossby} \simeq (50R)/(v_{\rm rot} \sin\phi)$, with $v_{\rm rot}$ in km.s^{-1}. This limit will thus become relevant for stars rotating faster than about 100 km.s^{-1}.

Another issue that has not been addressed yet concerns the vertical transport of chemicals in the presence of strongly anisotropic turbulence. As a turbulent eddy is displaced vertically in the fluid, it looses some of its coherence by being mixed by horizontal turbulence (see Fig. 2 right). This effect has been demonstrated in numerical simulations (Vincent et al. 1996). It is not clear however how general these results are, or what the effect of such anisotropic turbulence would be on a non-passive component (*e.g.* angular momentum). More work is required in that framework to understand how this will affect vertical diffusion.

A better understanding of the interaction between meridional circulation and turbulent properties could be obtained via the resolution of the full Navier-Stokes equations of hydrodynamics in 3D (see also Garaud, these proc., who uses a 2D code). This is a numerical challenge, as many time-scales are involved in the problem.

A first step was taken in that direction by Talon et al. (2003) who wrote a code to tackle that problem. Used in direct numerical simulations with unrealistically large viscosities, it does lead to stationary solutions. A comparison of these solutions with those of analytical models should permit a verification of their validity in the limit of large viscosities.

The goal is then to reduce the viscosities and introduce a sub-scale model in order to represent unresolved features and to examine how solutions are modified.

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