A TAUBERIAN THEOREM FOR THE RIEMANN-LIOUVILLE INTEGRAL OF INTEGER ORDER

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1. Notation. Let $s(x)$ be a function integrable[1] in every finite interval of $x \geqslant 0$. Then the Riemann-Liouville integral of $s(x)$, of order $\alpha > 0$, is defined for $x \geqslant 0$ by

$$s_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} s(t)dt. \tag{1}$$

The object of this note is to prove a Tauberian theorem for $s_\alpha(x)$ in the case in which α is a positive integer p, employing certain difference formulae due to Karamata (**4**, Lemma 2) and Bosanquet (**1**, Theorem 1) used already for a broadly similar purpose in an earlier paper (**12**) where α is any positive number.

Adopting a familiar notation, we shall write

$$C_\alpha(x) = \frac{\Gamma(\alpha + 1)\, s_\alpha(x)}{x^\alpha}, \qquad \alpha > 0, \tag{2}$$
$$C_0(x) = s_0(x) \equiv s(x),$$

and say that $s(x)$ is summable by the Cesàro mean of order α, or briefly, summable (C, α), to sum l, when

$$\lim_{x\to\infty} C_\alpha(x) = l,$$

l denoting a finite number as everywhere in this note. When $\lim C_\alpha(x)$ does not exist, as in the principal results of this note, it is convenient to write

$$\liminf_{x\to\infty} C_\alpha(x) = \underline{C}_\alpha, \qquad \limsup_{x\to\infty} C_\alpha(x) = \bar{C}_\alpha. \tag{3}$$

2. Scope of the main result. The following theorems, stated in the notation explained above, are known, at least in some part or form; and all of them turn out to be easy consequences or modifications of the single main result of this note featured as Theorem I.

THEOREM A. *If $s(x)$ is an integral,*

$$s'(x) = O_R(x^{q-p-1}) \quad \textit{as } x \to \infty \tag{4}$$

for almost all $x \geqslant 0$, p and q being real numbers of which the former is a positive integer, then

$$\frac{s_p(x)}{x^q} \to l \qquad (x \to \infty) \tag{5}$$

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[1]In this note integrability and integrals are always in the sense of Lebesgue.

implies

(6) $$\frac{s(x)}{x^{q-p}} \to lq(q-1)\ldots(q-p+1).$$

Theorem A was first proved by Doetsch (**2**, p. 174, Theorem II) with the restriction $q \geqslant p+1$[2] which was subsequently removed by Obrechkoff (**5**). A special case of Theorem A with $p = 1$ had been proved earlier by Hardy and Littlewood (**13**, p. 194, Corollary 4.4*a*), while a more general form of the theorem, with the positive integer p replaced by any positive number α was obtained later by Parthasarathy and Rajagopal (**6**, Theorem B, Case (2)).

A generalization of Theorem A is the following theorem wherein (4) is replaced by (4′), a condition which evidently holds whenever (4) holds.

THEOREM A′. *If* $s(x)$ *is such that*

(4′) $$\lim_{\lambda\to 1+0}\limsup_{t\to\infty}\sup_{t\leqslant t'\leqslant\lambda t}\frac{s(t')-s(t)}{t^{q-p}} \leqslant 0,$$

then (5) *implies* (6).

Theorem A′ and, in fact, its extension when the limit in (5) does not exist, are both included in the main result of this note whose proof is by the method used by Parthasarathy and Rajagopal (**6**) to obtain the extension of Theorem A in which p is replaced by any $\alpha > 0$.

The case $q = p$ of Theorem A′ is the classical result stated next.

THEOREM B. *If* $s(x)$ *is slowly increasing, that is,*

$$\lim_{\lambda\to 1+0}\limsup_{t\to\infty}\sup_{t\leqslant t'\leqslant\lambda t}\{s(t')-s(t)\} \leqslant 0,$$

and summable (C, p) *to* l,[3] *then* $s(x)$ *converges to* l *as* $x \to \infty$.

The following theorem is a companion to Theorem B; its case $p = 1$ has been proved in a somewhat different form by Pitt (**7**).

THEOREM C. *In Theorem* B *the condition of slow increase of* $s(x)$ *can be replaced by the following condition, without any other change:*

(7) $$\lim_{\lambda\to 1+0}\limsup_{t\to\infty}\sup_{t\leqslant t'\leqslant\lambda t}\left|\frac{1}{(\lambda-1)t}\int_t^{t'}\{s(u)-s(t)\}du\right| = 0.$$

A classical particularization of Theorem C is that in which (7) is replaced by the condition of slow oscillation of $s(x)$ which clearly implies (7).[4] A

[2]The case $q = p+1$ of Theorem A, with $s'(x)$ replaced by $s(x)$, gives the well-known theorem: *if* $s(x)$ *is bounded on one side and summable* (C, $p+1$) *to* l, *then it is summable* (C, 1) *to* l.

[3]In virtue of the first theorem of consistency for Cesàro summability, p in such cases may be replaced by any $\alpha > 0$.

[4]A condition which is effectively the same as that of slow oscillation is the "high-indices" condition, $\liminf \lambda_{n+1}/\lambda_n > 1$, when $s(x)$ is the λ_n − step function defined in the concluding remarks.

simple modification of the case $p = 1$ of Theorem C is like Pitt's theorem (**7**, Theorem 1) and unlike any of the classical Tauberian theorems for Cesàro summability in having no exact counterpart for Abel summability, that is, in not being always true when Cesàro summability is replaced by Abel summability (without any other change).

The last theorem to be now given includes Theorem B in the case $p = 1$.

THEOREM D. *If*

$$\limsup_{t\to\infty} \sup_{t\leqslant t'\leqslant\lambda t} \{s(t') - s(t)\} = w_1(\lambda), \tag{8}$$

then, for $0 < \theta < 1 < \lambda$,

$$-(\lambda - 1)\,\underline{C}_0 \leqslant -\lambda\underline{C}_1 + \bar{C}_1 + \int_1^\lambda w_1(t)dt,$$

$$(1 - \theta)\,\bar{C}_0 \leqslant -\theta\,\underline{C}_1 + \bar{C}_1 + \int_\theta^1 w_1(t)dt.$$

Theorem D is Karamata's[5] (**3**, Satz 1, first part), and its significance lies in the fact that it includes certain best-possible inequalities connecting C_0 and $\bar{C}_0$ with $\underline{C}_1$ and $\bar{C}_1$ first obtained by Fekete and Winn under the condition (8) with $w_1(\lambda) \leqslant K \log \lambda$. A generalization of Theorem D proved by me elsewhere (**8**, Lemma 3) is in the form of inequalities connecting $\underline{C}_\alpha$ and $\bar{C}_\alpha$ with $\underline{C}_{\alpha+1}$ and $\bar{C}_{\alpha+1}$ under condition (8) again. On the other hand, the generalization of Theorem D in this note, viz. Theorem I, takes the form of inequalities connecting lim inf $s(x)/x^{q-p}$ and lim sup $s(x)/x^{q-p}$ with lim inf $s_p(x)/x^q$ and lim sup $s_p(x)/x^q$ under a Tauberian condition which reduces to (8) when $q = p$. When

$$\liminf s_p(x)/x^q = \limsup s_p(x)/x^q,$$

the inequalities of Theorem I lead, in a special case, to Corollary I(1) which is Theorem A′ in the notation of Theorem I. When $q = p$, the inequalities of Theorem I become the inequalities of Corollary I(2) connecting $\underline{C}_0$ and $\bar{C}_0$ with $\underline{C}_p$ and $\bar{C}_p$, the case $p = 1$ of the latter inequalities constituting Theorem D. Modifications of the aforesaid inequalities connecting $\underline{C}_0$ and $\bar{C}_0$ with $\underline{C}_p$ and $\bar{C}_p$ are obtained in Corollary I(3) when (8) is replaced by the following condition implicit in (7):

$$\limsup_{t\to\infty} \sup_{t\leqslant t'\leqslant\lambda t} \left| \frac{1}{t(\lambda - 1)} \int_t^{t'} \{s(u) - s(t)\}du \right| = \Omega(\lambda).$$

In brief, Corollary I(2) and Corollary I(3) extend Theorem B and Theorem C respectively on the lines of Theorem D. Corollary I(4) following them refashions the case $p = 1$ of Corollary I(3) so as to produce in particular the (C, 1) summability theorem mentioned earlier as having no counterpart for Abel summability.

[5]Karamata's theorem has been restated here to match Theorem I, with $-s(x)$ in place of his $s(x)$.

3. The main result. The statement of this result, appearing as Theorem I, is necessarily elaborate by reason of the comprehensive character of the theorem. But the proof of the theorem is in essentials as simple as that of Theorem D, requiring nothing more than the Karamata-Bosanquet difference formulae referred to at the outset and embodied in the following lemmas easily verifiable by induction.

LEMMA 1. *If $h > 0$, $p = 1,2,3, \ldots$, then*

$$\Delta_h^p s_p(x) \equiv \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} s_p(x + \overline{p - \nu h})$$
$$= \int_x^{x+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \ldots \int_{t_{p-1}}^{t_{p-1}+h} s(t)dt.$$

LEMMA 2. *If $k > 0$, $p = 1,2,3, \ldots$, then*

$$\Delta_{-k}^p s_p(x) \equiv \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} s_p(x - \nu h)$$
$$= \int_{x-k}^{x} dt_1 \int_{t_1-k}^{t_1} dt_2 \ldots \int_{t_{p-1}-k}^{t_{p-1}} s(t)dt.$$

THEOREM I. *Let $s(x)$, integrable in every finite interval of $x \geqslant 0$, be such that, for $\lambda > 1$, one of the following two conditions holds and consequently the other also:*

$$\limsup_{t\to\infty} \sup_{t \leqslant t' \leqslant \lambda t} \frac{s(t') - s(t)}{t^{q-p}} = W_1(\lambda), \tag{9}$$

$$\limsup_{t\to\infty} \sup_{t \leqslant t' \leqslant \lambda t} \frac{s(t') - s(t)}{t'^{q-p}} = W_1^*(\lambda). \tag{9*}$$

Let $s_p(x)$ be defined for a positive integer p as in (1), *and let*

$$\liminf_{x\to\infty} \frac{s_p(x)}{x^q} = \underline{\sigma}_{p,q}, \quad \limsup_{x\to\infty} \frac{s_p(x)}{x^q} = \bar{\sigma}_{p,q}. \tag{10}$$

Then

$$-\left(\frac{\lambda-1}{p}\right)^p \liminf_{x\to\infty} \frac{s(x)}{x^{q-p}} \leqslant \mathfrak{A}_q(\lambda, p)\, \underline{\sigma}_{p,q} + \mathfrak{B}_q(\lambda, p)\, \bar{\sigma}_{p,q} + \left(\frac{\lambda-1}{p}\right)^{p-1} \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} W_1(t)dt, \tag{11}$$

where

$$\mathfrak{A}_q(\lambda, p) + \mathfrak{B}_q(\lambda, p) = -\sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \left\{1 + (p-\nu)\frac{\lambda-1}{p}\right\}^q, \tag{12}$$

$\mathfrak{A}_q(\lambda,p)$ is the part of the above sum consisting of the negative terms only and $\mathfrak{B}_q(\lambda,p)$ is the part of the same sum consisting of the positive terms only.

Further, for $0 < \theta < 1$, *we have*

$$(13)\qquad \left(\frac{1-\theta}{p}\right)^p \limsup_{x\to\infty} \frac{s(x)}{x^{q-p}} \leqslant \mathfrak{C}_q(\theta, p)\, \underline{\sigma}_{p,q} + \mathfrak{D}_q(\theta, p)\, \bar{\sigma}_{p,q} + \left(\frac{1-\theta}{p}\right)^{p-1} \int_\theta^{1-(1-p^{-1})(1-\theta)} W_1{}^*(t)dt,$$

where

$$(14)\qquad \mathfrak{C}_q(\theta, p) + \mathfrak{D}_q(\theta, p) = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \left\{1 - \nu\frac{1-\theta}{p}\right\}^q,$$

$\mathfrak{C}_q(\theta,p)$ *is the part of the above sum containing the negative terms alone and* $\mathfrak{D}_q(\theta,p)$ *is the part of the same sum containing the positive terms alone.*

(A condition such as (9) is to be read: "The left-hand member exists as a finite number and equals $W_1(\lambda)$."

(9*) follows from (9) since

$$\frac{s(t') - s(t)}{t'^{q-p}} = \left(\frac{t}{t'}\right)^{q-p} \frac{s(t') - s(t)}{t^{q-p}}$$

Similarly (9) follows from (9*).)

Proof. From Lemma 1 we have at once

$$- h^p \frac{s(x)}{x^q} = - \frac{\Delta_h^p s_p(x)}{x^q} + \int_x^{x+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \ldots \int_{t_{p-1}}^{t_{p-1}+h} \frac{s(t) - s(x)}{x^q} dt.$$

Denoting by I and J the first and the second terms respectively on the right, we can write the above relation as

$$(15)\qquad - \left(\frac{h}{x}\right)^p \frac{s(x)}{x^{q-p}} = I + J.$$

In J, t is such that $x \leqslant t_1 \leqslant t \leqslant t_1 + (p-1)h$, and so

$$J \leqslant \int_x^{x+h} \sup_{x \leqslant t \leqslant t_1 + (p-1)h} \left\{\frac{s(t) - s(x)}{x^q}\right\} h^{p-1} dt_1.$$

If $h = (\lambda - 1)x/p$ and $xt' = t_1 + (p-1)h$, this gives us

$$J \leqslant \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} \sup_{x \leqslant t \leqslant xt'} \left\{\frac{s(t) - s(x)}{x^{q-p}}\right\} \left(\frac{h}{x}\right)^{p-1} dt',$$

or, on account of (9),

$$(16)\qquad J \leqslant \left(\frac{h}{x}\right)^{p-1} \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} W_1(t')dt' + \left(\frac{h}{x}\right)^{p-1} \cdot o(1) \qquad (x \to \infty).$$

Next

$$(17)\qquad I = - \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \frac{s_p(x + \overline{p - \nu}h)}{(x + \overline{p-\nu}h)^q} \left(1 + \overline{p-\nu}\frac{h}{x}\right)^q$$

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where the factor multiplying $s_p(x + \overline{p - \nu h})/(x + \overline{p - \nu h})^q$, $\nu = 0,1,2, \ldots p$, is

$$- (-1)^\nu \binom{p}{\nu} \left\{1 + (p - \nu)\frac{\lambda - 1}{p}\right\}^q$$

which is independent of x. Consequently we get, letting $x \to \infty$ in (17) and recalling (10),

$$\limsup_{x\to\infty} I \leqslant \mathfrak{A}_q(\lambda, p)\, \underline{\sigma}_{p,q} + \mathfrak{B}_q(\lambda, p)\, \bar{\sigma}_{p,q} \tag{18}$$

by the definitions of $\mathfrak{A}_q$ and $\mathfrak{B}_q$ which follow (12). Taking upper limits of both sides of (15) as $x \to \infty$ and using (16) and (18), we establish the first conclusion (11).

To prove the second conclusion (13), we get from Lemma 2 the relation

$$k\frac{s(x)}{x^q} = \frac{\Delta^p_{-k}\, s_p(x)}{x^q} + \int_{x-k}^{x} dt_1 \int_{t_1-k}^{t_1} dt_2 \ldots \int_{t_{p-1}-k}^{t_{p-1}} \frac{s(x) - s(t)}{x^q} dt,$$

and rewrite it, denoting the first and the second terms on its right side by I^* and J^* respectively:

$$\left(\frac{k}{x}\right)^p \frac{s(x)}{x^{q-p}} = I^* + J^*. \tag{19}$$

In J^*, t lies in the interval $t_1 - (p-1)k \leqslant t \leqslant t_1 \leqslant x$, so that

$$J^* \leqslant \int_{x-k}^{x} \sup_{t_1-(p-1)k\leqslant t\leqslant x} \left\{\frac{s(x) - s(t)}{x^q}\right\} k^{p-1} dt_1.$$

If $k = (1-\theta)x/p$ and $xt' = t_1 - (p-1)k$, we can write the above inequality successively in the forms

$$J^* \leqslant \int_{\theta}^{1-(1-p^{-1})(1-\theta)} \sup_{xt'\leqslant t\leqslant x} \left\{\frac{s(x) - s(t)}{x^{q-p}}\right\} \left(\frac{k}{x}\right)^{p-1} dt',$$

$$J^* \leqslant \left(\frac{k}{x}\right)^{p-1} \int_{\theta}^{1-(1-p^{-1})(1-\theta)} W_1^*\left(\frac{1}{t'}\right) dt' + \left(\frac{k}{x}\right)^{p-1} \cdot o(1) \qquad (x \to \infty), \tag{20}$$

using (9*). Next

$$I^* = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \frac{s_p(x - \nu k)}{(x - \nu k)^q} \left(1 - \nu\frac{k}{x}\right)^q \tag{21}$$

where the factor multiplying $s_p(x - \nu k)/(x - \nu k)^q$, $\nu = 0,1,2, \ldots p$, is

$$(-1)^\nu \binom{p}{\nu} \left\{1 - \nu\frac{1-\theta}{p}\right\}^q$$

which is free from x. Therefore we obtain, letting $x \to \infty$ in (21),

$$\limsup_{x\to\infty} I^* \leqslant \mathfrak{C}_q(\theta, p)\, \underline{\sigma}_{p,q} + \mathfrak{D}_q(\theta, p)\, \bar{\sigma}_{p,q} \tag{22}$$

on account of (10) and our definitions of $\mathfrak{C}_q$, $\mathfrak{D}_q$ following (14). By taking upper limits of both sides of (19) as $x \to \infty$, and using (20) and (22), we immediately get (13).

4. Deductions. The deductions from Theorem I which have been outlined in an earlier section are effected by means of the two simple observations noted below as lemmas.

LEMMA 3. *If* $W_1(\lambda)$ *defined by* (9) *is such that*

$$\lim_{\lambda \to 1+0} W_1(\lambda) \leqslant 0,$$

then $W_1^*(\lambda)$ *defined by* (9*) *is also such that*

$$\lim_{\lambda \to 1+0} W_1^*(\lambda) \leqslant 0,$$

and conversely; further, the integrals in (11) *and* (13) *satisfy the conditions:*

$$\limsup_{\lambda \to 1+0} \frac{p}{\lambda - 1} \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} W_1(t)dt \leqslant 0,$$

$$\limsup_{\theta \to 1-0} \frac{p}{1-\theta} \int_{\theta}^{1-(1-p^{-1})(1-\theta)} W_1^*(t)dt \leqslant 0.$$

The proof is obvious.

LEMMA 4. *The function of* λ *defined in* (12) *and that of* θ *defined in* (14) *satisfy the conditions:*

$$(23)\quad \mathfrak{A}_q(\lambda,p) + \mathfrak{B}_q(\lambda,p) \sim -\left(\frac{\lambda-1}{p}\right)^p q(q-1)\ldots(q-p+1) \text{ as } \lambda \to 1+0,$$

$$(24)\quad \mathfrak{C}_q(\theta,p) + \mathfrak{D}_q(\theta,p) \sim \left(\frac{1-\theta}{p}\right)^p q(q-1)\ldots(q-p+1) \text{ as } \theta \to 1-0.$$

Proof. The proof of (23) is given below; that of (24) is similar.

By (12),

$$\begin{aligned}
\mathfrak{A}_q(\lambda,p) + \mathfrak{B}_q(\lambda,p) &= -\sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \left\{1 + (p-\nu)\frac{h}{x}\right\}^q \qquad \left(\frac{h}{x} = \frac{\lambda-1}{p}\right) \\
&= -x^{-q}\,\Delta_h^p\, x^q \\
&= -x^{-q} h^p \left(\frac{d^p x^q}{dx^p}\right)_{x=\xi} \qquad (x < \xi < x + ph = \lambda x) \\
&= -x^{p-q} \left(\frac{h}{x}\right)^p q(q-1)\ldots(q-p+1)\xi^{q-p} \\
&\sim -\left(\frac{\lambda-1}{p}\right)^p q(q-1)\ldots(q-p+1) \qquad (\lambda \to 1+0)
\end{aligned}$$

It is clear that, *in the particular case* $q = p$, (23) *and* (24) *reduce to*

$$(23')\qquad \mathfrak{A}_p(\lambda, p) + \mathfrak{B}_p(\lambda, p) = -\left(\frac{\lambda-1}{p}\right)^p p!\,,$$

$$(24')\qquad \mathfrak{C}_p(\theta, p) + \mathfrak{D}_p(\theta, p) = \left(\frac{1-\theta}{p}\right)^p p!\,.$$

To explain the derivation of Theorem A′ from Theorem I, we have only to rewrite the former as follows in the notation of the latter.

COROLLARY I(1). *If, in Theorem* I,

$$\lim_{\lambda\to 1+0} W_1(\lambda) \leqslant 0 \quad \textit{and hence} \quad \lim_{\lambda\to 1+0} W_1^*(\lambda) \leqslant 0,$$

and also

$$\underline{\sigma}_{p,q} = \bar{\sigma}_{p,q} = l,$$

then

$$\lim_{x\to\infty} \frac{s(x)}{x^{q-p}} = q(q-1)\ldots(q-p+1)l.$$

For, dividing (11) and (13) throughout by $(\lambda - 1)^p/p^p$ and $(1-\theta)^p/p^p$ respectively, and then letting $\lambda \to 1 + 1 - 0, \theta \to 0$, we get as a result of Lemmas 3 and 4,

$$-\liminf_{x\to\infty} \frac{s(x)}{x^{q-p}} \leqslant -q(q-1)\ldots(q-p+1)\iota,$$

$$\limsup_{x\to\infty} \frac{s(x)}{x^{q-p}} \leqslant q(q-1)\ldots(q-p+1)l,$$

which together imply the conclusion of Corollary I(1).

If $q = p$ in Theorem I, we find from (9), (9*) and (8) that

$$W_1(\lambda) = W_1^*(\lambda) = w_1(\lambda),$$

and from (3) and (10) that

$$\underline{\sigma}_{p,p} = \underline{C}_p/p! \quad , \quad \bar{\sigma}_{p,p} = \bar{C}_p/p! \quad .$$

Hence, when $q = p$ in Theorem I, the result is the following extension of Theorem D obtained by me some time ago (**9**, Theorem A).

COROLLARY I(2). *If $s(x)$ is integrable in every finite interval of $x \geqslant 0$ and such that, for $\lambda > 1$,*

$$\text{(8)} \qquad \limsup_{t\to\infty} \sup_{t\leqslant t'\leqslant\lambda t} \{s(t') - s(t)\} = w_1(\lambda),$$

then, for $0 < \theta < 1 < \lambda$,

$$\text{(11}'\text{)} \qquad -\left(\frac{\lambda-1}{p}\right)^p \underline{C}_0 \leqslant \frac{\mathfrak{A}_p(\lambda,p)\underline{C}_p + \mathfrak{B}_p(\lambda,p)\bar{C}_p}{p!} + \left(\frac{\lambda-1}{p}\right)^{p-1} \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} w_1(t)dt,$$

$$\text{(13}'\text{)} \qquad \left(\frac{1-\theta}{p}\right)^p \bar{C}_0 \leqslant \frac{\mathfrak{C}_p(\theta,p)\underline{C}_p + \mathfrak{D}_p(\theta,p)\bar{C}_p}{p!} + \left(\frac{1-\theta}{p}\right)^{p-1} \int_{\theta}^{1-(1-p^{-1})(1-\theta)} w_1(t)dt,$$

where $\mathfrak{A}_p, \mathfrak{B}_p, \mathfrak{C}_p, \mathfrak{D}_p$ are obtained with $q = p$ in $\mathfrak{A}_q, \mathfrak{B}_q, \mathfrak{C}_q, \mathfrak{D}_q$ respectively as defined immediately after (12) *and* (14).

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In the particular case in which the hypothesis is

$$\lim_{\lambda\to 1+0} w_1(\lambda) \leqslant 0 \quad , \quad \underline{C}_p = \bar{C}_p = l \quad ,$$

inequalities (11′) *and* (13′) *together reduce to the conclusion:*

$$\underline{C}_0 = \bar{C}_0 = l, \quad i.e. \quad \lim_{x\to\infty} s(x) = l,$$

on account of (23′), (24′) and Lemma 3 with $q = p$.

Theorem B is thus a particular case of Corollary I(2). Theorem C is a similar particular case of the next corollary got by making a small change in the proof of (11′) of Corollary I(2).

COROLLARY I(3). *If* $\lambda > 1$ *and*

$$\limsup_{t\to\infty} \sup_{t\leqslant t'\leqslant\lambda t} \left| \frac{1}{(\lambda-1)t} \int_t^{t'} \{s(u) - s(t)\} du \right| = \Omega(\lambda), \tag{25}$$

then

$$-\left(\frac{\lambda-1}{p}\right)^p \underline{C}_0 \leqslant \frac{\mathfrak{A}_p(\lambda,p)\underline{C}_p + \mathfrak{B}_p(\lambda,p)\bar{C}_p}{p!} + 2\left(\frac{\lambda-1}{p}\right)^p p\Omega(\lambda), \tag{26}$$

and there is a similar inequality with $\bar{C}_0$, $-\bar{C}_p$, $-\underline{C}_p$ *taking the places of* $-\underline{C}_0$, $\underline{C}_p$, $\bar{C}_p$ *respectively.*

In the particular case in which the hypothesis is that of Theorem C, *that is,*

$$\lim_{\lambda\to 1+0} \Omega(\lambda) = 0 \quad , \quad \underline{C}_p = \bar{C}_p = l,$$

inequality (26) *and its companion specified after it together yield the conclusion of Theorem* C, *viz.*

$$\underline{C}_0 = \bar{C}_0 = l,$$

as a result of (23′).

To prove Corollary I(3) in all its generality, we write down (15) with $q = p$ and find an upper estimate for J, using the following consequence of (25):

$$\begin{aligned}\int_{t_{p-1}}^{t_{p-1}+h} \{s(t) - s(x)\}dt &\leqslant \left|\int_x^{t_{p-1}} \{s(t) - s(x)\}dt\right| + \left|\int_x^{t_{p-1}+h} \{s(t) - s(x)\}dt\right| \\ &< 2\{\Omega(\lambda) + o(1)\}(\lambda-1)x \qquad (x\to\infty).\end{aligned}$$

From this we obtain in succession

$$J < x^{-p} \int_x^{x+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \ldots \int_{t_{p-2}}^{t_{p-2}+h} 2\{\Omega(\lambda) + o(1)\}(\lambda-1)x dt_1 \qquad (x\to\infty),$$

$$\limsup_{x\to\infty} J \leqslant 2\left(\frac{\lambda-1}{p}\right)^{p-1} (\lambda-1)\Omega(\lambda),$$

finally reaching (26) by a repetition of the rest of the argument used to prove (11). (26) has a companion as stated, resulting from the replacement of $s(x)$

by $-s(x)$ which is obviously permissible in our hypothesis (25) and all arguments therefrom.

In the case $p = 1$, Corollary I(3) can be modified to become a slight extension and simplification of Pitt's theorem already referred to (7, Theorem 1). This modification of Corollary I(3), analogous to Theorem D, is stated below.

Corollary I(4). *If, given some $\lambda > 1$, we can find, corresponding to every sufficiently large t, $R = R(t)$ tending to λ as $t \to \infty$ and such that*

$$\limsup_{t\to\infty} \left| \frac{1}{(R-1)t} \int_t^{Rt} \{s(u) - s(t)\} du \right| = \omega(\lambda), \tag{27}$$

then

$$-(\lambda - 1)\, \underline{C}_0 \leqslant -\lambda \underline{C}_1 + \bar{C}_1 + (\lambda - 1)\omega(\lambda), \tag{28}$$

$$(\lambda - 1)\, \bar{C}_0 \leqslant -\underline{C}_1 + \lambda \bar{C}_1 + (\lambda - 1)\omega(\lambda). \tag{29}$$

In particular, when (27) *is simply*

$$\lim_{t\to\infty} \left| \frac{1}{(R-1)t} \int_t^{Rt} \{s(u) - s(t)\} du \right| = 0, \tag{27'}$$

and $\underline{C}_1 = \bar{C}_1$, (28) *and* (29) *together reduce to the equality*

$$\underline{C}_0 = \bar{C}_0.$$

Corollary I(4) is proved from the following relation which is the case $q = p = 1$ of (15) with $h = (R-1)x$ and $R = R(x)$:

$$-(R-1)s(x) = -RC_1(Rx) + C_1(x) + \frac{1}{x}\int_x^{Rx} \{s(t) - s(x)\} dt.$$

Taking upper limits of both sides as $x \to \infty$ and using (27), we get at once (28) and deduce (29) from it by changing $s(x)$ to $-s(x)$, such a change being permissible in (27) and arguments based thereon.

Remark on Condition (27′). *This Tauberian condition, like Pitt's more complicated form of it* (7), *though sufficient to make the convergence of $s(x)$ follow from the* (C, 1) *summability of $s(x)$, is not always sufficient to make the convergence of $s(x)$ follow from the Abel summability of $s(x)$.*

(Pitt has, instead of (27′), the more complicated condition: *given $\epsilon > 0$, we can find $\eta(\epsilon) > 0$, $R = R(t, \epsilon)$ corresponding to every sufficiently large t, so that*

$$R \geqslant 1 + \eta, \quad \left| \int_T^{RT} \{s(u) - s(t)\} du \right| \leqslant (R-1)T\epsilon$$

for some $T = T(\epsilon, t)$ satisfying $tR^{-1} \leqslant T \leqslant t$.)

Pitt's example itself (7, Theorem 2) serves to establish this fact. The example is of a non-convergent $s(x)$ which is Abel summable and defined as follows:

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$s(x) = (-2)^m$ for $\lambda_m \leqslant x < \lambda_{m+1}$, $\lambda_m = (2m+1)\log(2m+1)$, $m = 0, 1, 2, \ldots$. Pitt's discussion shows that, for this $s(x)$ there is an $R = R(t)$ corresponding to every sufficiently large t, such that $R(t) \to \lambda$ as $t \to \infty$ and (27′) is fulfilled in the form

$$\frac{1}{(R-1)t}\int_t^{Rt}\{s(u) - s(t)\}du = 0.$$

(What Pitt has actually proved is that, corresponding to every sufficiently large t, $\lambda_M \leqslant t < \lambda_{M+1}$, we can find $R = R(M)$ tending to $\lambda = 2$ as $t \to \infty$, so that

$$\frac{1}{(R-1)\lambda_M}\int_{\lambda_M}^{R\lambda_M}\{s(u) - s(\lambda_M)\}du = 0.$$

However, it is easy to show that Pitt's result remains true when $\lambda = 2$ is replaced by any $\lambda > 1$, λ_M by t and $R = R(M)$ by another $R = R(t)$.)

5. A supplementary result. To make this study complete, a complement to Theorem I under a two-sided Tauberian condition is proved below. This complement, in the special case $q = p$, reduces to a result previously obtained by me (**9**, Theorem B), and, in the further special case $q = p = 1$, to Karamata's complement to Theorem D (**3**, Satz 1, second part) under the condition (8) together with a similar condition on $-s(x)$ instead of $s(x)$.

THEOREM II. *If, in Theorem* I, *we are given, in addition to either* (9) *or* (9*), *one of the following conditions which necessarily involves the other:*

$$\liminf_{t\to\infty}\ \inf_{t\leqslant t'\leqslant\lambda t}\frac{s(t') - s(t)}{t^{q-p}} = -W_2(\lambda), \tag{30}$$

$$\liminf_{t\to\infty}\ \inf_{t\leqslant t'\leqslant\lambda t}\frac{s(t') - s(t)}{t'^{q-p}} = -W_2^*(\lambda), \tag{30*}$$

we shall have, in addition to (11) *and* (13),

$$-\left\{\left(\frac{\lambda-1}{p}\right)^p + \left(\frac{1-\theta}{p}\right)^p\right\}\liminf_{x\to\infty}\frac{s(x)}{x^{q-p}}$$
$$\leqslant\begin{cases}\left\{\mathfrak{A}_q(\lambda,p) - \mathfrak{D}_q(\theta,p)\right\}\underline{\sigma}_{p,q} + \left\{\mathfrak{B}_q(\lambda,p) - \mathfrak{C}_q(\theta,p)\right\}\bar{\sigma}_{p,q} \\ \qquad + \left\{\dfrac{1+(-1)^{p-1}}{2}\right\}(\underline{\sigma}_{p,q} - \bar{\sigma}_{p,q}) \\ + \left(\dfrac{\lambda-1}{p}\right)^{p-1}\displaystyle\int_{1+(1-p^{-1})(\lambda-1)}^{\lambda}W_1(t)dt + \left(\dfrac{1-\theta}{p}\right)^{p-1}\int_\theta^{1-(1-p^{-1})(1-\theta)}W_2^*(1/t)dt,\end{cases} \tag{31}$$

and a similar inequality with lim sup $s(x)/x^{q-p}$ *in place of* $-$ lim inf $s(x)/x^{q-p}$ *deduced from* (31) *by taking* $-s(x)$ *in place of* $s(x)$.

Proof. By combining (15) and (19), we get

$$-\left(\frac{h}{x}\right)^p \frac{s(x)}{x^{q-p}} - \left(\frac{k}{x}\right)^p \frac{s(x)}{x^{q-p}} = I + J - I^* - J^*. \tag{32}$$

Taking $h = (\lambda - 1)x/p$, $k = (1 - \theta)x/p$, and arguing as in the derivation of (16) and (20), we obtain

$$\begin{aligned}\limsup_{x\to\infty} (J - J^*) &\leqslant \limsup_{x\to\infty} J + \limsup_{x\to\infty} (-J^*) \\ &\leqslant \left(\frac{\lambda - 1}{p}\right)^{p-1} \int_{1+(1-p^{-1})(\lambda-1)}^{\lambda} W_1(t)dt + \\ &\qquad \left(\frac{1-\theta}{p}\right)^{p-1} \int_{\theta}^{1-(1-p^{-1})(1-\theta)} W_2{}^*(1/t)dt.\end{aligned} \tag{33}$$

We have also, from the expression for I in (17) and that for I^* in (21),

$$\limsup_{x\to\infty} (I - I^*) \leqslant \begin{cases}\mathfrak{A}_q \underline{\sigma}_{p,q} + \mathfrak{B}_q \bar{\sigma}_{p,q} - \mathfrak{C}_q \bar{\sigma}_{p,q} - \mathfrak{D}_q \underline{\sigma}_{p,q} & \text{if } p \text{ is even,} \\ \mathfrak{A}_q \underline{\sigma}_{p,q} + (\mathfrak{B}_q - 1)\bar{\sigma}_{p,q} - \mathfrak{C}_q \bar{\sigma}_{p,q} - (\mathfrak{D}_q - 1)\underline{\sigma}_{p,q} & \text{if } p \text{ is odd,}\end{cases} \tag{34}$$

where the distinction between the cases of odd p and even p arises thus. If p is odd and only then, the last term in I is $s_p(x)/x^q$ and this cancels out the first term in $- I^*$ which is in any case $- s_p(x)/x^q$; the result is that the contribution (arising from I) to the positive terms which make up $\mathfrak{B}_q$ is less than what the form of I suggests, by 1, and the contribution (arising from $- I^*$) to the negative terms which make up $- \mathfrak{D}_q$ is more than what the form of $- I^*$ suggests, by 1. (31) follows from (32), (33) and (34).

6. Concluding remarks. There is a special case of interest in the results of this note, when $s(x)$ is, as in Pitt's example, a λ_n — step function with steps at points of any sequence $\{\lambda_n\}$ such that

$$0 < \lambda_0 < \lambda_1 < \ldots, \lambda_n \to \infty\ ,$$

that is,

$$s(x) = \begin{cases} a_0 + a_1 + \ldots + a_n & \text{for } \lambda_n \leqslant x < \lambda_{n+1}, \quad n \geqslant 0, \\ \qquad 0 & \text{for } 0 \leqslant x < \lambda_0. \end{cases}$$

In this case, the (C, α) summability of $s(x)$ becomes the summability of $\Sigma\, a_n$ by Riesz means of order α and type (λ_n), usually called (R, λ_n, α) summability; and Corollary I(2) can be used, as elsewhere **(10; 11)**, to extend certain Tauberian theorems of G. Ricci's for $\Sigma\, a_n$ summable to l by the method of Dirichlet's series or the (A, λ_n) method, that is, $\Sigma\, a_n$ such that

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \text{ converges for } s > 0 \text{ and tends to } l \text{ as } s \to +0.$$

An open question **(10**, §1.1**)** which may be recalled in this context is *whether the following theorem for* (R, λ_n, α) *summability is one possessing no precise analogue for* (A, λ_n) *summability*, i.e. one belonging possibly to a class of Tauberian theorems peculiar to Cesàro summability like the particular case of Corollary I(4).

THEOREM X. *If*

(i) $$\sum_{n=0}^{\infty} a_n$$

is (R, λ_n, α) *summable to l for some* $\alpha > 0$,

(ii) $$\lim_{\lambda \to 1+0} \limsup_{n \to \infty} \max_{\lambda_n < \lambda_m < \lambda\lambda_n} (a_{n+1} + a_{n+2} + \ldots + a_m) \leqslant 0,$$

then

$$\liminf_{n \to \infty} (a_0 + a_1 + \ldots + a_n) = l.$$

Theorem X (**10**, Theorem f) is a simple consequence of Corollary I(2), and it has the imperfect analogue for (A, λ_n) summability, stated below, whose special case $\alpha = 0$ follows from a reformulation of one of Ricci's theorems (**10**, Theorem G) and every case $\alpha \geqslant 0$ follows from Theorem X and my generalization (**10**, Lemma 2) of a theorem due to O. Szász.

THEOREM Y. *Theorem* X *can be restated with* (i) *replaced by the* (A, λ_n) *summability of* $\Sigma\, a_n$ *to* l *and* (ii) *augmented by the condition that, for some* $\alpha \geqslant 0$,

$$\sum_{\lambda_\nu \leqslant x} (x - \lambda_\nu)^\alpha a_\nu \lambda_\nu = O_R(x^{\alpha+1}) \qquad (x \to \infty).$$

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