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PERMUTATION CHARACTERS OF FINITE GROUPS OF LIE TYPE

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Abstract

Let $\mathscr{G}$ be a connected reductive linear algebraic group, and let $G = \mathscr{G}_\sigma$ be the finite subgroup of fixed points, where σ is the generalized Frobenius endomorphism of $\mathscr{G}$. Let x be a regular semisimple element of G and let w be a corresponding element of the Weyl group W. In this paper we give a formula for the number of right cosets of a parabolic subgroup of G left fixed by x, in terms of the corresponding action of w in W. In case G is untwisted, it turns out that x fixes exactly as many cosets as does w in the corresponding permutation representation.

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1. Introduction

Let k be the finite field of q elements and let K be an algebraic closure of k. Let $\mathscr{G}$ be a connected reductive linear algebraic group defined over k. Let σ denote the generalized Frobenius endomorphism of $\mathscr{G}$ into $\mathscr{G}$, and let $G = \mathscr{G}_\sigma$ be the finite group of fixed points in $\mathscr{G}$. If $\mathscr{G}$ is semisimple and has an irreducible root system, then, with a few exceptions, G is a central extension of a finite simple Chevalley group, of normal or twisted type; Steinberg (1968), Sections 11 and 12. Let P be a parabolic subgroup of G and let 1_P^G be the permutation character induced by the action of G on the right cosets of P in G. In this paper we show that for any regular semisimple element $x \in G$, $1_P^G(x)$ is given by a formula (Theorem 4.1, below) involving only the Weyl group W. Thus, the value of $1_P^G(x)$ is independent of q. The reader should note that this formula is obtainable via the Deligne–Lusztig theory, see Deligne and Lusztig (1976), Sections 7 and 8. However, since the methods considered herein make no reference to the 'virtual representations' or

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to the étale cohomology used to construct the virtual representations, we consider the present treatment considerably more elementary.

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2. Regular semisimple elements of G

Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$ and let $\mathcal{B}$ be a Borel subgroup of $\mathcal{G}$ containing $\mathcal{T}$, both fixed by σ. Let $\mathcal{N}$ be the normalizer of $\mathcal{T}$ in $\mathcal{G}$ and set $W = \mathcal{N}/\mathcal{T}$, the *Weyl group* of $\mathcal{G}$. An element $x \in \mathcal{G}$ is called *semisimple* if there exists an element $a \in \mathcal{G}$ such that $axa^{-1} \in \mathcal{T}$. If, in addition, $(axa^{-1})^w \neq axa^{-1}$, for all $w \in W$, we call x a *regular* semisimple element.

In order to describe regular semisimple elements of G more intrinsically, we consider the so-called *maximal tori* (Cartan subgroups) of G; Springer and Steinberg (1970), or Srinivasan (1971). If $a \in \mathcal{G}$ is such that the subgroup $a^{-1}\mathcal{T}a$ is σ-invariant, we set $H = (a^{-1}\mathcal{T}a)_\sigma$, and call H a *maximal torus* of G. The following is well known; see Springer and Steinberg (1970), p. 186, or Srinivasan (1971), p. 3.

THEOREM 2.1. *The nonconjugate tori in G are in one-to-one correspondence with the σ-conjugacy classes of W. More precisely, let $a, b \in \mathcal{G}$ with $aa^{-\sigma}$, $bb^{-\sigma} \in \mathcal{N}$. Then $a^{-1}\mathcal{T}a$ and $b^{-1}\mathcal{T}b$ are both σ-stable and $(a^{-1}\mathcal{T}a)_\sigma$ and $(b^{-1}\mathcal{T}b)_\sigma$ are conjugate in G if and only if $aa^{-\sigma}$ and $bb^{-\sigma}$ are σ-conjugate in $\mathcal{N}$ $(\operatorname{mod}\mathcal{T})$. Finally, every semisimple element of $\mathcal{G}$ is contained in some maximal torus of G.*

Now let $a \in \mathcal{G}$ where $a^{-1}\mathcal{T}a$ is σ-stable. If $aa^{-\sigma} \to w$ under the natural projection $\mathcal{N} \to W$, we set $H_w = (a^{-1}\mathcal{T}a)_\sigma$. The σ-centralizer of $w \in W$ is the subgroup $C'_W(w) = \{w_1 \in W \mid w_1 w w_1^{-\sigma} = w\}$ of W.

PROPOSITION 2.1. *Let $x \in G$ be semisimple and let H_w be a maximal torus in G containing x. Then*

(i) $N_G(H_w)/H_w \simeq C'_W(w)$,

(ii) *x is regular if and only if $x^n \neq x$ for all $n \in N_G(H_w) - H_w$.*

Statement (i) is proved in Srinivasan (1971), Lemma 4, and (ii) follows from (i) together with the definition of regularity.

In case $t \in \mathcal{T}$, there is an alternative description of regularity as follows; see Springer and Steinberg (1970), p. 217. Let Φ be the set of roots associated with $\mathcal{T}$, and let $\mathcal{X}_r$, $r \in \Phi$, be the one-parameter subgroups of $\mathcal{G}$. If Φ^+ denotes the set of positive roots of Φ, then t acts via conjugation on the unipotent subgroup

$\mathscr{U} = \langle \mathscr{X}_r \mid r \in \Phi^+ \rangle$. Moreover, t stabilizes each $\mathscr{X}_r$ with action

$$t x_r(c) t^{-1} = x_r(\chi(r) c), \quad c \in K,$$

with $\chi \in \mathrm{Hom}(P, K)$, $P = \mathbf{Z}\Phi$. Thus we have a homomorphism $\mathscr{T} \to \mathrm{Hom}(P, K)$ with kernel $Z(\mathscr{G})$. In addition, if $w \in W$ and $t \to \chi$ then $t^w \to \chi^w$ where

$$\chi^w(r) = \chi(w^{-1}(r)), \quad r \in \Phi,$$

where W acts on P as a reflection group.

In view of the factorization $\mathscr{U} = \Pi \mathscr{X}_r$ $(r \in \Phi^+)$, the following is immediate.

LEMMA 2.1. *Let $t \in \mathscr{T}$ be regular. Then t acts fixed point freely on each one-parameter subgroup $\mathscr{X}_r$ of $\mathscr{G}$. Therefore if $\mathscr{U}$ is the unipotent radical of $\mathscr{B}$, then t acts fixed point freely on $\mathscr{U}$.*

We remark that it is possible for an element $t \in \mathscr{T}$ to act fixed point freely on $\mathscr{U}$ without being regular. In fact, from Springer and Steinberg (1970) (4.1), p. 201, we see that this happens for any $t \in \mathscr{T}$ that fixes pointwise no $\mathscr{X}_r$, $r \in \Phi$, and has a disconnected centralizer $C_{\mathscr{G}}(t)$.

3. More fixed point free actions

Let R be a set of fundamental reflections in W, let $J \subseteq R$, and let $P_J = \mathscr{B} W_J \mathscr{B}$, where $W_J = \langle J \rangle$. The parabolic subgroups of G are of the form $P_J = (\mathscr{P}_J)_\sigma$ where $\mathscr{P}_J$ is σ-stable, and admit a Levi decomposition as follows. Let Π_J be the set of fundamental roots corresponding to the reflections in J, and let Φ_J be the root system generated by Π_J. Set $\mathscr{L}_J = \langle \mathscr{T}, \mathscr{X}_r | r \in \Phi_J \rangle$, $\mathscr{V}_J = \langle \mathscr{X}_r \mid r \in \Phi^+ - \Phi_J \rangle$, and let $L_J = (\mathscr{L}_J)_\sigma$, $V_J = (\mathscr{V}_J)_\sigma$. Then $\mathscr{V}_J$ is the unipotent radical of $\mathscr{P}_J$, and the mapping $\mathscr{L}_J \times \mathscr{V}_J \to \mathscr{P}_J$ given by multiplication of coordinates is a k-isomorphism of algebraic varieties; Borel and Tits (1965). Moreover $\mathscr{L}_J$ is a reductive connected linear algebraic group; Springer and Steinberg (1970) (4.1b), p. 201. Thus we have a semidirect product $P_J = L_J V_J$, called the *Levi decomposition* of P_J. The subgroup L_J of P_J is called the *Levi factor* of P_J. For a different approach to the Levi decomposition, see Curtis (1975).

Henceforth we consider only those subsets $J \subseteq R$ for which $\mathscr{P}_J$ is σ-stable. The next proposition generalizes Lemma 2.1 for regular semisimple elements of the finite group G.

PROPOSITION 3.1. *Let x be a regular semisimple element of G and assume that $x \in L_J$ for some $J \subseteq R$. Then x acts fixed point freely on V_J.*

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PROOF. Since x is semisimple, there is a maximal torus of L_J containing x. Thus there is an element $a \in \mathscr{L}_J$ such that $axa^{-1} \in \mathscr{T}$. But since $\mathscr{L}_J$ normalizes $\mathscr{V}_J$ we have $a\mathscr{V}_J a^{-1} \leqslant \mathscr{V}_J$. Let $1 \neq v \in V_J$ and let $v' = ava^{-1} \in \mathscr{V}_J$. Then $xv = vx$ implies

$$a^{-1} tv'a = a^{-1} v'ta.$$

Thus t commutes with v', contradicting the fact that t acts fixed point freely on $\mathscr{U}$.

4. The main result

Let $J \subseteq R$ and let D_J be the set of *distinguished* right W_J-coset representatives in W. Thus $w \in D_J$ if and only if w is the unique element of minimal length in $W_J w$. Moreover, $\Pi_J \subseteq w\Phi^+$, and $l(w_1 w_2) = l(w_1) + l(w_2)$, $w_1 \in W_J$, $w_2 \in D_J$, where l is the length function on W; see Carter (1972), Chapter 2.

Let $\mathscr{U}^- = \langle \mathscr{X}_r \mid r \in \Phi^- \rangle$, $U^- = (\mathscr{U}^-)_\sigma$, and, if $w \in W$, set $\mathscr{U}_w^- = \mathscr{U} \cap w^{-1}\mathscr{U}^- w$. Then from Bruhat's lemma, Borel and Tits (1965), p. 100, the 'strong form' of the Bruhat decomposition holds:

$$\mathscr{G} = \bigcup \mathscr{B} w \mathscr{U}_w^- \quad (w \in W),$$

where each $g \in \mathscr{G}$ can be expressed uniquely as $g = bwu$, $b \in \mathscr{B}$, $w \in W$ and $u \in \mathscr{U}_w^-$. More generally, let $J \subseteq R$. Then since $l(w_1 w_2) = l(w_1) + l(w_2)$, $w_1 \in W_J$, $w_2 \in D_J$, we have $Bw_1 w_2 B = Bw_1 Bw_2 B$ from the (B, N)-pair axioms satisfied by $\mathscr{G}$; see for example, Carter (1972), Chapter 8. This implies a strong Bruhat decomposition relative to the pair $(\mathscr{P}_J, \mathscr{B})$:

$$\mathscr{G} = \bigcup \mathscr{P}_J w \mathscr{U}_w^- \quad (w \in D_J),$$

with uniqueness of expression. Correspondingly, the finite group G admits a similar decomposition $G = \bigcup P_J w U_w^- \, (w \in (D_J)_\sigma)$, with uniqueness of expression, where $U_w^- = (\mathscr{U}_w^-)_\sigma$.

If $J, K \subseteq R$, let $D_{K,J}$ be the set of distinguished $(W_K - W_J)$-double coset representatives in W; Bourbaki (1968), p. 37. Then $W = \bigcup W_K w W_J \, (w \in D_{K,J})$, and each $w \in W_{K,J}$ is the unique element of minimal length in $W_J w W_K$.

PROPOSITION 4.1. *Let x be a semisimple element of L_J, where $J \subseteq R$ is minimal with respect to $x \in L_J^G$. Let $w \in W$ with $x \in P_K^w \, (= w^{-1} P_K w)$, where $J \subseteq K$. Then $x \in L_K^w$.*

PROOF. It suffices to assume that $w \in (D_{K,J})_\sigma$. Then $x \in L_J \cap P_K^w$ is a standard parabolic subgroup of L_J:

$$L_J \cap P_K^w = L_J \cap P_I,$$

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where $\Pi_I = \Pi_J \cap w^{-1}(\Pi_K)$: Curtis (1975), p. 672. But since x is a p'-element ($p = \operatorname{char} k$) acting on the p-group V_I, we have that x is conjugate to an element of L_I, by the Schur–Zaussenhaus theorem. This contradicts the minimality of J, unless $\Pi_J \subseteq w^{-1}(\Pi_K)$. But then $\mathscr{L}_J \leqslant \mathscr{L}_K^w$ and so $L_J \leqslant L_K^w$.

If $K \subseteq R$ we denote by $1_{P_K}^G$ the character of the action of G on the right cosets of P_K in G. Thus, if $x \in G$, $1_{P_K}^G(x)$ is the number of right cosets of P_K in G fixed by x. If $w \in W$, we set $\sigma 1_{W_K}^W(w)$ equal to the number of right cosets $W_K w_1$ $w_1 \in W$, where

$$W_K w_1 w = W_K w_1^\sigma.$$

Thus, if G is untwisted, $\sigma 1_{W_K}^W$ is the ordinary permutation character $1_{W_K}^W$. We are now in a position to prove the main result.

THEOREM 4.1. *Let $x \in G$ be regular semisimple, and let $w_x \in W$ be such that $x \in H_{w_x}$ Then, for all $K \subseteq R$ such that $\mathscr{P}_K$ is σ-stable,*

$$1_{P_K}^G(x) = \sigma 1_{W_K}^W(w_x).$$

PROOF. Without loss of generality we may assume that $x \in L_J$, where $J \subseteq R$ is minimal with respect to $x \in L_J^G$. Since $x = a^{-1}ta$ for some $t \in \mathscr{T}$, $a \in \mathscr{L}_J$, and since $aa^{-\sigma} \to w_x$ under the projection $\mathscr{N} \to W$ we see that J is also minimal with respect to

$$\{ww_x w^{-\sigma} \mid w \in W\} \cap W_J \neq \varnothing.$$

Therefore, if K is properly contained in J, we see that

$$1_{P_K}^G(x) = 0 = \sigma 1_{W_K}^W(w_x).$$

Therefore, assume that $J \subseteq K$ and let D_K be as usual. Assume that x fixes the coset $P_K w$, where $w \in (D_K)_\sigma$. If x also fixes the coset $P_K wu$ where $u \in U_w^-$, then $wuxu^{-1}w^{-1} \in P_K$, and so $vx'v^{-1} \in P_K$, where $v = wuw^{-1}$ and $x' = wxw^{-1}$. But since $\mathscr{U}_w^- = \mathscr{U} \cap w^{-1}\mathscr{U}^- w$, and since $w \in D_K$, we have

$$v \in \mathscr{U}^- \cap w\mathscr{U}w^{-1} = \langle \mathscr{X}_r \mid r \in \Phi^- \cap w\Phi^+ \rangle \leqslant \mathscr{V}_K^-,$$

where $\mathscr{V}_K^- = \langle \mathscr{X}_r \mid r \in \Phi^- - \Phi_K \rangle$. Therefore $v \in V_K^- = (\mathscr{V}_K)_\sigma$. But by Proposition 4.1, $x' = wxw^{-1} \in P_K$ implies that $x' \in L_K$. Since L_K normalizes V_K, and hence V_K^-, and since x' is regular, we conclude that x' acts fixed point freely on V_K^-. Therefore $vx' = x'v'$ where $v \neq v' \in V_K^-$. But then $vx'v^{-1} = x'v'v^{-1} \in P_K$, and since $x' \in P_K$, we have

$$v'v^{-1} \in P_K \cap V_K^- = 1,$$

a contradiction. Thus, each regular semisimple element fixes at most one right coset in $P_K wU_w^-$.

Finally, by arguing as in the proof of Proposition 4.1, we see that x fixes $P_K w$ precisely when w_x fixes $W_K w$, $w \in (D_K)_\sigma$. Therefore it suffices to prove that if $W_K w w_x = W_K w^\sigma$, $w \in D_K$, then $w = w^\sigma$. If $w \in D_{K,J}$ then $W_K w W_J = W_K w^\sigma W_J$, and so $w = w^\sigma$ by uniqueness. Otherwise write $w = w_1 w_2$ with $w_1 \in D_{K,J}$, $w_2 \in W_J$. Then $W_K w_1 w_2 w_x = W_K w_1^\sigma w_2^\sigma$. As above $w_1 = w_1^\sigma$, and so

$$w_2 w_x w_2^{-\sigma} \in W_J \cap W_K^{w_1}.$$

But by a result of Kilmoyer (1969), p. 66, $W_J \cap W_K^{w_1}$ is a (σ-stable) parabolic subgroup of W, which contains W_J by the minimality of J. Thus

$$W_K w_1 w_2 w_x = w_1 w_1^{-1} W_K w_1 w_2 w_x = W_K w_1 w_x.$$

Thus $w_2 = 1$ since $w = w_1 w_2 \in D_K$, and so $w = w^\sigma$.

5. Concluding remarks

In case $G = GL_n(q)$ (or $SL_n(q)$, $PSL_n(q)$), the results are well known; see Green (1955), Lemma 2.8, *p*. 413. Moreover, Theorem 4.1 has an important generalization for these groups. Namely, let G be one of these finite groups and let B be a Borel subgroup of G. The irreducible character constituents of 1_B^G and the irreducible characters of the Weyl group W are in a natural one-to-one correspondence; see Curtis and others (1971), Section 2. Let ϕ be an irreducible character of W and let ξ be the corresponding irreducible constituent of 1_B^G. Then in Steinberg (1951), Theorem 2.1, p. 275, one has that ϕ and ξ can be expressed as linear combinations of permutation characters $1_{P_J}^G$ and $1_{W_J}^W$, respectively, and *with the same coefficients*. Therefore it follows that if x is a regular semisimple element of G, then,

$$\xi(x) = \phi(w_x),$$

for each irreducible constituent ξ of 1_B^G.

For other groups of Lie type this situation does not seem to hold, as the examples $G = Sp_4(q)$ and $G = G_2(q)$ indicate. The author has learned, however, that George Lusztig has proved that $\xi(x)$ is always independent of q, and has given a formula for $\xi(x)$ in terms of w and certain generalized characters of W; see Lusztig (1976).

Finally, we should mention that the above equality is always valid in the case when x is a regular semisimple element already contained in the 'split' torus $T = (\mathscr{T})_\sigma$, and we replace W with the relative Weyl group W_σ. This is a result of Curtis (1975), Corollary 6.2, p. 683.

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