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GROUPS IN WHICH EVERY FINITELY GENERATED SUBGROUP IS ALMOST A FREE FACTOR

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1. Introduction. In [**5**] M. Hall Jr. proved, without stating it explicitly, that every finitely generated subgroup of a free group is a free factor of a subgroup of finite index. This result was made explicit, and used to give simpler proofs of known results, in [**1**] and [**7**]. The standard generalization to free products was given in [**2**]: If, following [**13**], we call a group in which every finitely generated subgroup is a free factor of a subgroup of finite index an *M. Hall group*, then a free product of M. Hall groups is again an M. Hall group. The recent appearance of [**13**], in which this result is reproved, and the rather restrictive nature of the property of being an M. Hall group, led us to attempt to determine the structure of such groups. In this paper we go a considerable way towards achieving this for those M. Hall groups which are both finitely generated and accessible. (*Accessibility* means roughly that the process of decomposing the group as the fundamental group of a graph of groups with finite edge groups, terminates, possibly transfinitely. It is conjectured in [**4**] that all groups are accessible.) We first show, rather easily, that such an M. Hall group is free-by-finite. This leads to the conjecture that, conversely, a free-by-finite group is M. Hall if it satisfies a fairly obvious property of M. Hall groups, namely that every nontrivial finitely generated subgroup have finite index in its normalizer. In this direction we show that the amalgamated product $(A*B;\, U)$ where A, B are finite and $U \neq \{1\}$, is M. Hall if and only if at least one of A, B is a Frobenius group with Frobenius complement U.

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2. Background and statement of results. Since finite groups are trivially M. Hall groups, and, as mentioned above, the class of M. Hall groups is closed under the formation of free products, we have that free products of cyclic or finite groups are M. Hall. The stringency of the defining property led us to conjecture that these were all. This turns out not to be the case, as we shall show; however our first result, an easy consequence of Stallings' theory of ends, shows that M. Hall groups are nonetheless rather special. The proof occupies §4.

THEOREM 1. *A freely indecomposable, finitely generated M. Hall group is cyclic, or finite, or a proper amalgamated product $(A*B;\, U)$ where U is finite and nontrivial, or a proper* HNN *extension $\langle t, A \mid tUt^{-1} = V\rangle$ where U and V are finite and nontrivial.*

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COROLLARY 1. *A finitely generated, torsion-free M. Hall group is free.*

COROLLARY 2. *A finitely generated, accessible M. Hall group G is a finite extension of a free group.*

(Corollary 1 is immediate. Corollary 2 can be established as follows. By the accessibility, Theorem 1, and the proposition below (§3), G is the fundamental group of a graph of groups with all of its vertex groups either finite or infinite cyclic and with all edge groups finite. The fact that G is finitely generated then implies that the underlying graph may be assumed to be finite. It follows that G is a free product of a finite rank free group and the fundamental group of a finite graph of groups with finite vertex and edge groups. By [**10**, Theorem 1] such fundamental groups are precisely the finitely generated free-by-finite groups.)

To resume the discussion, there do exist finitely generated, freely indecomposable, accessible M. Hall groups (with torsion, naturally), which are neither cyclic nor finite; this is clear from our next theorem. To state it we need the following well-known concept, seemingly tailor-made for the purpose: a subgroup H of a group G is *malnormal* in G if $H \cap gHg^{-1} = \{1\}$ for all $g \in G\backslash H$. (If G is finite and $G > H > \{1\}$, then G is a *Frobenius* group.)

THEOREM 2. *The amalgamated product $G = (A*B; U)$ of finite groups A and B is M. Hall if and only if U is malnormal in at least one of A, B.*

Thus perhaps the simplest example of a freely indecomposable, non-cyclic infinite M. Hall group is the group $(S_3*C_4; C_2)$, where S_3 is the symmetric group on 3 letters, and C_4, C_2 are cycles of orders 4, 2.

From Theorem 2 it follows fairly easily, via the proposition in §3, that for the HNN extension $\langle t, A \mid tUt^{-1} = V\rangle$ (where U, V are finite) to be M. Hall, at least one of U, V must be malnormal in A, and also (using properties of Frobenius groups) that if A is finite, then $U = V = \{1\}$.

Our proof of Theorem 2 (see in particular Lemma 3), together with [**10**, Theorem 1], make it seem plausible that the finitely generated, accessible M. Hall groups are just those finite extensions of finite rank free groups in which the normalizer of every finite, nontrivial subgroup is finite.

Since the proof of the "if" part of Theorem 2 is long and technical, to ease its digestion, or to provide the impatient reader with the means for avoiding it, we shall prove in §5 the following much easier weakened version of Theorem 2, and relegate the proof of Theorem 2 itself to the final section (§6). The "programme" of proof is similar for the two versions.

THEOREM 2′. *If in $G = (A*B; U)$, A, B are finite and U is malnormal in both A and B, then G is M. Hall.*

3. Preliminary remarks. In [**13**], Tretkoff defines a group G to be M. Hall if, given a finitely generated, proper subgroup A of G, and $g_1, \ldots, g_n \in G\backslash A$, there exists a subgroup of finite index in G which contains A as a free factor and still avoids $g_1, \ldots, g_n$. However this property of avoidance of finitely many prescribed elements (which, incidentally, was the one originally of interest to M. Hall Jr., and which was also first shown to be preserved under free products by N. S. Romanovskii [**12**]) is redundant, so that our apparently weaker definition of M. Hall groups is equivalent to Tretkoff's. To see this we first observe that by using an argument similar to that in the proof of Part (i) of the proposition below, we may take $n = 1$. Thus let $g \in G \backslash A$ where G is M. Hall (in our sense), and A is a finitely generated subgroup. Then there exists $B \leqq G$ such that $\langle A, B \rangle = A*B$, and $A*B$ has finite index in G. If $g \notin A*B$ there is nothing to prove; suppose on the other hand that $g \in A*B$, and as a first case that g has finite order k. Then by the Kuroš subgroup theorem, g is in a conjugate of A different from A itself, or in a conjugate of B, so that $\langle A, g \rangle = A*\langle g \rangle = A_1$ say. Let B_1 be such that $\langle A_1, B_1 \rangle = A_1*B_1$, and has finite index in G. Then the normal closure in A_1*B_1 of $A*B_1$ has the desired properties. If g has infinite order then an elementary argument using the normal form of $g \in A*B$ shows that $g \notin \langle A, g^2 \rangle = A_2$ say. We also have, by the Kuroš subgroup theorem, that A is a free factor of A_2. Then, if A_2*B_2 has finite index in G, we have $g \notin A_2*B_2$.

We include in this section a proposition which has been used above, and whose second part will be of use later.

PROPOSITION. (i) *The subgroups of an M. Hall group are again M. Hall.* (ii) *Every nontrivial finitely generated subgroup of an M. Hall group has finite index in its normalizer.*

Proof. (i) Note first the following well-known consequence of the Kuroš subgroup theorem [**11**, p. 243]: If $H \leqq A*B$, then $H \cap A$ is a free factor of H. Now suppose $K \leqq G$ where G is M. Hall, and let A be a finitely generated subgroup of K. Then there is a $B \leqq G$ such that $\langle A, B \rangle = A*B$, and has finite index in G. Hence $H = K \cap (A*B)$ has finite index in K, and by the first observation has $H \cap A = K \cap A = A$ as a free factor.

(ii) The proof is easy: we leave it to the reader. Alternatively see [**1**, proof of Corollary 2].

4. Proof of theorem 1. If G is a finitely generated, freely indecomposable M. Hall group, then either all of its nontrivial subgroups are of finite index, in which case it is cyclic or finite, or else it has a proper free product as a subgroup of finite index. If this free product is not D_∞, the infinite dihedral group, then by [**3**, Theorem 3.1, p. 43] it has infinitely many ends, and hence by [**3**, Proposition 2.1, p. 20] so has G. Then, again by [**3**, Theorem 3.1, p. 43], G is an amalgamated product or HNN extension of the required sort. If the free product is D_∞ then by the same theorem and proposition, G has 2 ends, and is therefore by [**3**, Exercise, p. 31] an amalgamated product of the required sort or else has a finite nontrivial normal subgroup of infinite index. However by Part (ii) of the proposition of §3, the latter alternative cannot occur.

5. Proof of theorem 2′. We need two lemmas. The first one is due to Karrass and Solitar.

LEMMA 1. ([**9**, Theorem 5]). *Let $G = (A*B;\ U)$ where A, B are finite and U is malnormal in both A and B. Every subgroup of G is a free product of a free group, conjugates of subgroups of A or B, and conjugates of subgroups of the form $(A_1*B_1;\ U_1)$ where $A_1 \leqq A$, $B_1 \leqq B$ (and $U_1 = A_1 \cap U = B_1 \cap U$).*

For the second lemma we need a definition: we shall say that a group is *tractable* if every proper, finitely generated subgroup is contained in a proper subgroup of finite index.

LEMMA 2. *If a group G has a normal tractable subgroup of finite index then G is tractable.*

Proof. Denote by N the normal tractable subgroup of finite index in G, and let H be a proper finitely generated subgroup of G. We may suppose that $|G:H| = \infty$, since otherwise there is nothing to prove. We have that $H \cap N \lhd H$ and $|H:H \cap N| < \infty$. Let $\{g_1, \ldots, g_r\}$ be a complete set of representatives for H modulo $H \cap N$. Since N is tractable and $H \cap N$ is finitely generated and proper in N, there exists a subgroup K such that $H \cap N < K < N$ with $|N:K| < \infty$. Define K_1 to be the subgroup of G generated by $g_1, \ldots, g_r$ together with $M = \bigcap_{i=1}^{r} g_i K g_i^{-1}$. Then K_1 is the required proper subgroup of finite index in G containing H. The facts that $K_1 > H$ and $|G:K_1| < \infty$ are easily verified. That K_1 is proper in G follows from $K_1 \cap N = M < N$, also easily verified.

We are now ready for the proof of Theorem 2′. Suppose the theorem false and that $G = (A*B;\ U)$ with U malnormal in the finite groups A, B, is a counterexample with $|A| + |B|$ least. From Lemma 1 and the result of [**2**, 12] that free products of M. Hall groups are M. Hall, it follows that every proper subgroup of G is M. Hall. Let H be a finitely generated subgroup of G of infinite index. By [**10**, Theorem 1] G is free-by-finite, and therefore by Lemma 2 G is tractable. Hence there is a proper subgroup K, of finite index in G, containing H. Since K is M. Hall, H is a free factor of a subgroup of finite index in K, and therefore in G. Thus G is M. Hall, giving a contradiction.

6. Proof of theorem 2. The necessity of the condition that U be malnormal in A or B is not difficult; it is immediate from the following elementary lemma and Part (ii) of the proposition.

LEMMA 3. *Let $M = (C*D;\ V)$ where C, D are arbitrary groups and V is finite. If V is malnormal in neither C nor D, then V contains a nontrivial subgroup whose normalizer in M is infinite.*

Proof. Let $c \in C\backslash V$, $d \in D\backslash V$, be such that

$$cVc^{-1} \cap V = P > \{1\}, \text{ and } dVd^{-1} \cap V = Q > \{1\}.$$

Suppose that M is a counterexample to the lemma with $|V|$ least.

Case 1. Suppose $P < V$ and $Q < V$. We concentrate first on P. Since the normalizer in M of every nontrivial subgroup of V is finite, the same is true of P. Now it follows from [**8**, Theorem 5] (or an elementary normal-form argument) that

$$\langle cDc^{-1}, D\rangle \cong (cDc^{-1}*D; P) = L \text{ say.}$$

Since the nontrivial subgroups of P certainly have finite normalizers in L, we must have, by the minimality property of V, that P is malnormal in cDc^{-1} or D, and so certainly also in cVc^{-1} or V. By [**6**, Satz 8.3] P is malnormal in both cVc^{-1} and V, and has order coprime to its index in these groups. Hence any Sylow subgroup S of P is a Sylow subgroup of both cVc^{-1} and V; in other words $c^{-1}Sc$ and S are Sylow subgroups of V. Thus $v^{-1}c^{-1}Scv = S$ for some $v \in V$; i.e. S is normalized by an element $c_1(= cv)$ from $C\backslash V$.

The same argument gives that Q is malnormal in V, so that by [**6**, Sätze 8.3, 8.17], P and Q are conjugate in V, and $v_1Sv_1^{-1}$ is a Sylow subgroup of Q for some $v_1 \in V$. Again by the same argument as for P, $v_1Sv_1^{-1}$ is normalized by some element $\hat{d} \in D\backslash V$, so that S is normalized by some element $d_1(= v_1^{-1}\hat{d}v_1)$ from $D\backslash V$. We have therefore that the infinite subgroup $\langle c_1, d_1\rangle$ normalizes S, a contradiction.

Case 2. Suppose $P = V$. If $d \in D\backslash V$ normalized V, then the infinite subgroup $\langle c, d\rangle$ would normalize V, an impossibility. Hence $V > Q > \{1\}$ (with Q as in the beginning of the proof). As before any Sylow subgroup S_1 of Q is then a Sylow subgroup of V, and is normalized by some element $d_1 \in D\backslash V$. Since $cVc^{-1} = V$, we have that cS_1c^{-1} is also a Sylow subgroup of V. Since V is finite, it follows that $v_2cS_1c^{-1}v_2^{-1} = S_1$, for some $v_2 \in V$, so that S_1 is normalized by $v_2c = c_2 \in C\backslash V$, and therefore by the infinite subgroup $\langle c_2, d_1\rangle$, yielding the final contradiction.

The "if" part of the proof of Theorem 2 is relatively complicated, involving some of the technicalities of the Karrass–Solitar subgroup theorem for free products with amalgamation. Our immediate aim is to show that the proper subgroups of $G = (A*B; U)$ (with A, B finite and U malnormal in A) are in some sense simpler than G. To describe the proper subgroups of G we need the following construction. Define a *shrub* to be a finite tree of diameter $\leqq 2$, and a *Frobenius shrub product* to be a tree product of finite groups where the tree is a shrub and each edge group is malnormal in its extremal vertex group (if the shrub has just two vertices then we demand only that the single edge group be malnormal in at least one of the vertex groups). The *root* of a shrub with $\geqq 3$ vertices is the unique non-extremal vertex; the *root group* of such a shrub product is, naturally, the group associated with the root. (The root group of a Frobenius shrub product with just two vertices is that vertex group, if any, in which the edge group is not malnormal; if the edge group is malnormal in both vertex groups, choose either vertex as the root. The root of the one-vertex shrub is its single vertex.)

LEMMA 4. (cf. Lemma 1). *Let $G = (S*R;\ U)$ where S and R are finite and U is malnormal in S. Every subgroup H of G is a free product of a free group and Frobenius shrub products P, each of which has, for some $d \in G$, root group $H \cap dRd^{-1}$, and extremal vertices $H \cap (dr_i)S(dr_i)^{-1}$, $i = 1, 2, \ldots, n$, where the r_i belong to R and lie in different (H, S)-double cosets.*

Proof. By the Karrass–Solitar subgroup theorem [**8**, Theorem 5], H is a treed HNN group (or, in other terminology, the fundamental group of a graph of groups) where, in the notation of [**8**], the edge groups are of the form $U_H^{\ d}(= H \cap dUd^{-1})$, and the vertex groups have the form $S_H^{\ d}$, $R_H^{\ \delta}$, the d coming from different (H, R)-double cosets, and the δ from different (H, S)-double cosets. By arguing as in [**9**, end of proof of Theorem 4, p. 943, and proof of Theorem 5, p. 944] the malnormality of U in S can be seen to imply that in this standard presentation of H, at most one of the edge groups (including the associated subgroups) around each vertex group of the form $S_H^{\ d}$, is nontrivial. It follows immediately that the tree product base T say, of H (i.e. the subgroup generated by the vertex groups of H) is a free product of Frobenius shrub products P as described in the lemma. It remains to be shown that any stable letters occurring in the standard presentation of H can be successively split off as free factors of H. Thus let t be any such stable letter with nontrivial associated subgroups $U_H^{\ d}$, $U_H^{\ \delta}$ contained in vertex groups $S_H^{\ d}$, $R_H^{\ \delta}$ respectively, where $t = \delta u d^{-1}$ for some $u \in U$ (such will be the situation according to the Karrass–Solitar subgroup theorem); then all the other associated and amalgamated edge groups with $S_H^{\ d}$ as a vertex group are trivial. Hence $S_H^{\ d}$ is a free factor of the tree product base T; more specifically if T_1 is the subgroup of H generated by the vertex groups of H other than $S_H^{\ d}$, then $T = S_H^{\ d}*T_1$. But then $\langle t\rangle$ can be split off as a free factor of H; for $\langle t, T\rangle$ has presentation

$$\langle t, T\rangle = \langle t, S_H^{\ d}*T_1 |\ \text{rel}\ S_H^{\ d},\ \text{rel}\ T_1,\ U_H^{\ td} = U_H^{\ \delta}\rangle,$$

and since

$$S_H^{\ d} < \langle t, S_H^{\ \delta}\rangle \leqq \langle t, T_1\rangle \ (\text{using } d = t^{-1}\delta u),$$

it follows that $\langle t, T\rangle$ has presentation

$$\langle t, T\rangle = \langle t, T_1 |\ \text{rel}\ T_1\rangle = \langle t\rangle * T_1.$$

The effect of this re-presentation of $\langle t, T\rangle$ so that it is evident that $\langle t\rangle$ is a free factor, is to delete from the tree product base T the vertex and edge corresponding to $S_H^{\ d}$ and $U_H^{\ d}$ (together with these groups), thereby obtaining a new tree product base T_1.

The lemma follows by carrying out this process for each stable letter: if t_1 is another stable letter, represent $\langle t\rangle * \langle t_1, T_1\rangle$ as $\langle t\rangle * \langle t_1\rangle * T_2$, and so on.

In the next lemma we group together three relatively simple facts. Call a shrub product *proper* if each edge group is a nontrivial, proper subgroup of its (unique) extremal vertex group.

LEMMA 5. (i) *A proper shrub product of finite groups is freely indecomposable.*

(ii) *A finite subgroup of a tree product is contained in a conjugate of some vertex group.*

(iii) *Let G be a Frobenius shrub product with root group R and suppose that we have an element $g \in G$ and a vertex group A of G such that $R^g \cap A \neq \{1\}$. Then $g = ar$ for some $a \in A$, $r \in R$.*

Proof. (i) Suppose there exists a counterexample $K = C*D$ where C, D are nontrivial. Let X be the root group of K. Then by the Kuroš subgroup theorem, since X is finite it is contained in a conjugate of C or D; we may suppose without loss of generality that X is in a conjugate of C. Hence $X \cap kDk^{-1} = \{1\}$ for all $k \in K$, so that by the propriety of K the vertex groups adjacent to X are also in conjugates of C, rather than of D. Hence K is the normal closure of C, an impossibility.

(ii) This follows from the 2-vertex case ([**9**, Lemma 1]) by an easy induction.

(iii) Suppose $g \notin R$; then there is an extremal vertex group D say, such that g does not belong to the subgroup S of G generated by all vertex groups other than D. It is clear that $G = (D*S;\ U)$ where U is the edge group of the unique edge having D as one of its vertex groups. Write $g = d_1s_1 \ldots d_ns_n$, in the form of an alternating product, where $d_i \in D$, $s_i \in S$, and d_1 or s_n may be 1, but all other factors are outside U. Let $r \in R\backslash\{1\}$ be such that $grg^{-1} \in A$; then

$$grg^{-1} = d_1s_1 \ldots d_ns_nrs_n^{-1}d_n^{-1} \ldots s_1^{-1}d_1^{-1}.$$

If $s_nrs_n^{-1} \notin U$ then grg^{-1} has length >1 so it cannot lie in A (which is contained in S or equal to D). Hence we must have $s_nrs_n^{-1} \in U$; write $u = s_nrs_n^{-1}$. Then

$$grg^{-1} = d_1s_1 \ldots d_nud_n^{-1} \ldots s_1^{-1}d_1^{-1},$$

and by the malnormality of U in D, we have $d_nud_n^{-1} \in D\backslash U$. Again by length considerations, we deduce that $n = 1$, i.e. $g = d_1s_1 = d_ns_n$, and $grg^{-1} \in D$. Hence $D = A$. Now since $s_nrs_n^{-1} \in U$ we have that in S, a Frobenius shrub product with one fewer vertex than G, $s_nRs_n^{-1} \cap A \neq \{1\}$. If we suppose inductively that this implies that $s_n = ar_1$ for some $a \in A$, $r_1 \in R$ (the statement being trivial for shrub products with a single vertex), then we have $g = d_ns_n = d_nar_1$, which has the required form.

We are now in a position to prove the following result, the key to the proof of Theorem 2. We define the *size* of a proper Frobenius shrub product to be the sum of the orders of its extremal vertex groups.

THEOREM 3. *Let $G = (S*R;\ U)$ where R, S are finite and U is malnormal in S. Let P be an infinite, non-cyclic, freely indecomposable subgroup of G, so that (by Lemma 4) P can be presented as a proper Frobenius shrub product with root group R_P^d and extremal vertices $S_P^{dr_i}$, $i = 1, \ldots, n$, where $d \in G$, and the r_i belong to R and lie in different (P, S)-double cosets. Every proper subgroup K of P is a free product of a free group, finite groups, and infinite proper Frobenius shrub products each of which either has root group of smaller order than that of P, or else has root group of the same order as that of P, but has smaller size than P.*

Proof. If the theorem is true for $d^{-1}Pd$ then it is true for P. We may therefore assume a slightly simpler dsscription of P than that given in the theorem, namely that P has root group R_P and extremal vertex groups $S_P^{r_i}$, $i = 1, \ldots, n$ with the r_i as in the theorem. By Lemma 4 the group K, being a subgroup of G, is a free product of a free group, finite groups and infinite proper Frobenius shrub products, a typical one Q of which has, for some $d_1 \in G$, root group $R_K^{d_1} = R_Q^{d_1}$ and extremal vertex groups $S_K^{d_1\rho_i} = S_Q^{d_1\rho_i}$, $i = 1, \ldots, m$, where the $\rho_i \in R$ and the $d_1\rho_i$ lie in different (K, S)-double cosets. Our aim is to show that Q has smaller root group than P or else the same order root group but smaller "size". By Lemma 5(ii) the finite group $R_P^{d_1} \leqq B^p$ for some vertex group B of P and some $p \in P$.

Suppose $B = R_P$, the root group of P. Then $R_P^{d_1} \leqq R_P^p$, whence by Lemma 5 (iii)

(1) $d_1 = pr$ for some $r \in R$.

Suppose that on the other hand B is some extremal vertex group $S_P^{r_i}$ of P. We shall show by a more complicated argument that (1) holds in this case also. Note first that since $R_P^{d_1} \leqq S_P^{pr_i}$, we have by Lemma 5 (iii) that $d_1 = pr_is\hat{r}$ for some $s \in S$, $\hat{r} \in R$, so that $R_P^{d_1} = U_P^{pr_is}$. Since

$$U_P^{pr_is} = S_P^{pr_is} \cap U^{pr_is},$$

and since U is malnormal in S, we have that $U_P^{pr_is}$ is malnormal in $S_P^{pr_is} = S_P^{pr_i}$, i.e. that $R_P^{d_1}$ is malnormal in $S_P^{pr_i}$. Now $R_P \cap S_P^{r_i}$ (an edge group of P) is malnormal in $S_P^{r_i}$ (the corresponding extremal vertex group of P); the upshot is that $R_P^{p^{-1}d_1}$ and $R_P \cap S_P^{r_i}$, being both malnormal in $S_P^{r_i}$, are conjugate in $S_P^{r_i}$ [**6**, Sätze 8.3, 8.17]. Hence $R_P^{d_1}$ is conjugate in P to a subgroup of R_P; but this is just the first case above, from which we deduced (1).

We have so far shown that (1) holds for some $p \in P$, $r \in R$. We may assume that $p = 1$ since the theorem, if true for $p^{-1}Qp$, is true for Q; thus now Q has root group R_Q and vertex groups $S_Q^{r\rho_j}$, $j = 1, \ldots, m$. If $R_Q < R_P$ then we are finished, since then the root group of Q has smaller order than that of P; thus we may suppose $R_Q = R_P$. By Lemma 5 (ii) the extremal vertex groups of Q, being finite, are contained in conjugates of vertex groups of P. Suppose

$$S_Q^{r\rho_j} \leqq R_P^q \text{ for some } q \in P.$$

Since Q is proper we know that $S_Q{}^{r\rho_j} \cap R_Q \neq \{1\}$; therefore $R_P \cap R_P{}^q \neq \{1\}$, whence by Lemma 5 (iii), $q \in R$. But then

$$S_Q{}^{r\rho_j} \leqq R_P = R_Q,$$

which is impossible again by the propriety of Q. Thus each extremal vertex group of Q is contained in a conjugate in P of some extremal vertex group of P i.e. for each $j = 1, \ldots, m$, there exist $p_j \in P$, $k \in \{1, \ldots, n\}$, such that

(2) $S_Q{}^{r\rho_j} \leqq S_P{}^{p_j r_k}$.

We shall now show that we may suppose that the p_j all lie in R_P. We have already noted that $S_Q{}^{r\rho_j} \cap R_P \neq \{1\}$. This and (2) give that

$$S^{p_j r_k} \cap R_P \neq \{1\},$$

whence by Lemma 5 (iii) applied to the Frobenius shrub product P we have

$$p_j = \hat{\rho}_j s_j{}^{r_k}, \text{ where } \hat{\rho}_j \in R_P \text{ and } s_j{}^{r_k} \in S_P{}^{r_k}.$$

Hence $S^{p_j r_k} = S^{\hat{\rho}_j r_k}$; we may therefore suppose from now on that the p_j lie in R_P. As noted before, $S_Q{}^{r\rho_j} \nleqq R_P$. Hence there exists $s \in S \backslash U$ such that

$$(r\rho_j)s(r\rho_j)^{-1} \in S_Q{}^{r\rho_j}.$$

It follows from (2) that

$$[(p_j r_k)^{-1} r\rho_j] s [(p_j r_k)^{-1} r\rho_j]^{-1} \in S\backslash U;$$

thus this element has length 1 (in its normal form as an element of $(S^*R;\, U)$), so that since $p_j \in R$, we must have $(p_j r_k)^{-1} r\rho_j \in U$. Since $p_j \in R_P$, we obtain from this that $R_P r_k U = R_P r\rho_j U$, so that since $R_Q = R_P$ we certainly have

(3) $Qr_k S = Qr\rho_j S$.

It follows that we cannot have another extremal vertex group $S_Q{}^{r\rho_l}$ of Q contained in a conjugate $S_P{}^{p_l r_k}$ of the same vertex group of P, for we should then by the same argument arrive at

$$Qr_k S = Qr\rho_l S;$$

which, with (3), contradicts the hypothesis that $r\rho_j = d_1\rho_j$ and $r\rho_l = d_1\rho_l$ lie in different (K, S)-double cosets, and therefore certainly in different (Q, S)-double cosets.

To summarize: each extremal vertex group of Q lies in a conjugate by an element of R_P of an extremal vertex group of P, and no two extremal vertex groups of Q lie in conjugates of the same extremal vertex group of P. Hence $m \leqq n$, and by re-indexing the ρ_j we may suppose that

$$S_Q{}^{r\rho_i} \leqq S_P{}^{p_i r_i},\ p_i \in R_P,\ i = 1, \ldots, m.$$

It follows that Q has smaller size than P unless $m = n$ and $S_Q{}^{r\rho_i} = S_P{}^{p_i r_i}$

for all i. However since the $p_i \in R_P = R_Q$, it then follows that $P = Q$ contrary to hypothesis. This concludes the proof.

Theorem 2 can now be proved as follows. Let $\mathfrak{F}$ be the class of all infinite, proper Frobenius shrub products contained in groups of the form $(S*R; U)$ with S, R finite and U malnormal in S. Let $\mathfrak{S}$ be a subset of $\mathfrak{F}$ containing exactly one group from each isomorphism class of groups in $\mathfrak{F}$. Well-order $\mathfrak{S}$ lexicographically, first with respect to the order of the root group and then with respect to size. Suppose that not every group in $\mathfrak{S}$ is M. Hall, and let K be a counterexample minimal with respect to this well-ordering. From this point on the proof imitates that of Theorem 2'.

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