WIDTHS AND HEIGHTS OF (0,1)-MATRICES

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Introduction. A number of combinatorial problems may be regarded as particular instances of the following rather general situation. Given a set X composed of n elements $x_1, x_2, \ldots, x_n$, and m subsets $X_1, X_2, \ldots, X_m$ of X, find a minimal system of representatives for $X_1, X_2, \ldots, X_m$. That is, single out a subset X^* of X such that $X_i \cap X^*$ is non-empty for $i = 1, 2, \ldots, m$, and no subset of X containing fewer elements than X^* has this property. To illustrate, each of the following can be thought of in these terms.

(a) Find the fewest number of nodes that touch all arcs in a linear graph. Thus the sets $X_1, X_2, \ldots, X_m$ are the arcs of the graph, each set consisting of two elements, its end nodes. A famous example of this is the eight queens chessboard problem. Here one forms a graph by connecting two cells of the board if a queen can move from one cell to the other. Then the complement of a minimal system of cells that touch all arcs represents positions in which the maximal number of queens can be placed so that no two attack each other.

(b) Given two distinct nodes in a graph, find a set of arcs, minimal in number, that cut all chains leading from one node to the other. Here the elements $x_1, x_2, \ldots, x_n$ are the arcs of the graph, and the sets $X_1, X_2, \ldots, X_m$ are all chains that join the two given nodes. A similar problem is to find the fewest number of arcs that cut all directed cycles in a directed graph.

(c) Given the truth table for a proposition letter formula F in r proposition letters $p_1, p_2, \ldots, p_r$, find a disjunctive normal form of F that has the fewest number of terms. That this problem, which arises, for example, in the design of switching circuits, falls in the category of minimal set representative problems, can be seen as follows: As elements of the fundamental set X, admit all terms having one of the forms $q_i, q_iq_j, q_iq_jq_k, \ldots, q_1q_2 \ldots q_n$, where q_i is either p_i or its negation $\bar{p}_i$, and such that the term takes the value t (true) only if $F(p_1, p_2, \ldots, p_r)$ does also, for all values of the proposition letters $p_1, p_2, \ldots, p_r$. In other words, a t in the truth table for an admissible term implies a t in the same position for the F truth table. The subsets to be represented are formed by grouping together, for each assignment of values to $p_1, p_2, \ldots, p_r$ that makes $F(p_1, p_2, \ldots, p_r)$ true, all of the admissible terms that are also true for this assignment of values. For example, suppose that $F(p_1, p_2, p_3)$ is given by the truth table below (Table I).

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TABLE I

p_1	p_2	p_3	F
f	f	f	f
f	f	t	f
f	t	f	t
f	t	t	f
t	f	f	t
t	f	t	t
t	t	f	t
t	t	t	f

Then the elements of X are

$$p_1\bar{p}_2,\ p_1\bar{p}_3,\ p_2\bar{p}_3,\ \bar{p}_1p_2\bar{p}_3,\ p_1\bar{p}_2\bar{p}_3,\ p_1\bar{p}_2p_3,\ p_1p_2\bar{p}_3,$$

and the four subsets to be represented are

$$\begin{aligned} X_1 &= \{p_2\bar{p}_3,\ \bar{p}_1p_2\bar{p}_3\} \\ X_2 &= \{p_1\bar{p}_2,\ p_1\bar{p}_3,\ p_1\bar{p}_2\bar{p}_3\} \\ X_3 &= \{p_1\bar{p}_2,\ p_1\bar{p}_2p_3\} \\ X_4 &= \{p_1\bar{p}_3,\ p_2\bar{p}_3,\ p_1p_2\bar{p}_3\}. \end{aligned}$$

A minimal system of representatives is given by selecting the terms $p_1\bar{p}_2$, $p_2\bar{p}_3$, that is,

$$F(p_1, p_2, p_3) \equiv p_1\bar{p}_2 + p_2\bar{p}_3$$

and F cannot be represented by a disjunctive normal form having fewer terms.

Many other combinatorial problems can be viewed as minimal representative problems. (But doing so is unlikely to make the problem any easier.) Of the three listed above, only one, so far as we know, might properly be termed solved. This is the first problem mentioned under (b), for which the max flow min cut theorem provides a theoretical answer on the one hand, and on the other hand, an algorithm based on network flow considerations can be used to construct, in a highly efficient manner, a minimal cut set of arcs for any particular graph **(2)**. For undirected graphs, the second problem under (b) is easy, the answer being the cyclomatic number of the graph, but for directed graphs, very little seems to be known. The problem in this latter form has been proposed by Moore (cf. **14**). Berge **(1)** has obtained some results on problem (a), and Roth **(9)** has studied problems of type (c) using combinatorial topological methods.

From the computational standpoint, any minimal set representative problem can be put in the form of an integer linear programme, for which Gomory **(6)** has devised promising algorithms. Thus, for example, we may take the constraint matrix $A = [a_{ij}]$ for the programme to be the incidence matrix of sets vs. elements, that is, $a_{ij} = 1$ if x_j is in X_i, $a_{ij} = 0$ otherwise. Then the minimal set representative problem asks for non-negative integers

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$w_1, w_2, \ldots, w_n$ that minimize the linear form $\sum_{j=1}^{n} w_j$ over all selections of non-negative integers satisfying the constraints

$$\sum_{j=1}^{n} a_{ij} w_j \geqslant 1, \qquad i = 1, 2, \ldots, m.$$

In general, however, the incidence matrix A is much too large to make such a computation feasible. Sometimes one can obtain other linear programmes that are not so formidable in size, and in certain cases, the programme may even be formulated so that optimal solutions are always integral. This is the situation, for example, in the first problem listed under (b), for which an appropriate formulation (not in terms of the incidence matrix A of chains vs. arcs) can be described that is both reasonable in size and automatically yields integer answers.

The results of this paper are not aimed at a solution of the minimal set representative problem per se, but may be viewed as providing some information on this problem. Specifically, we are interested in obtaining bounds on the minimal number of representatives by allowing the incidence matrix to vary over all matrices of zeros and ones having the same row and column sums as the given A, that is, the class $\mathfrak{A}$ generated by A **(10)**. From this standpoint, the present paper may be regarded as a continuation of **(4**; **7**; **11**; **12)**, in which other combinatorially significant quantities associated with an incidence matrix A have been so studied.

In order to avoid repeating the cumbersome phrase "the number of elements in a minimal set of representatives for A," we call this simply the "width" of A, or more precisely, the "1-width" of A, since we generalize the problem to α-widths, that is, we insist that each subset be represented at least α times. Throughout we let $\epsilon(\alpha)$ denote the α-width of a specified A; $\tilde{\epsilon}(\alpha)$ and $\bar{\epsilon}(\alpha)$ then denote, respectively, the minimum and maximum α-widths taken over all A in $\mathfrak{A}$. The problem of determining $\tilde{\epsilon}(\alpha)$ in terms of the given row and column sums that characterize $\mathfrak{A}$ is completely solved in the sequel, but our efforts to pin down $\bar{\epsilon}(\alpha)$ have so far been unsuccessful.* In solving the $\tilde{\epsilon}(\alpha)$ problem, an auxiliary notion, the "α-height" of A, turns out to be important. This, and the other notions introduced informally above, will be defined more precisely in §1.

Throughout the paper we use purely combinatorial methods in establishing results. It should be mentioned, however, that the formula obtained for $\tilde{\epsilon}(\alpha)$ can also be derived using network flows, and was in fact first obtained in this way. From the viewpoint of flow theory, the function $N(\epsilon, e, f)$ introduced in § 4 can be interpreted as representing possible minimal cut capacities in an appropriate flow network.

*Since the results of this paper were obtained, it has been shown by one of the authors that a solution to the $\bar{\epsilon}(1)$ problem would settle the existence question for finite projective planes. See **(13)**. Thus the maximal width problem appears to be considerably deeper and more important than the minimal width problem.

1. Concepts and notation. Let A be a matrix of m rows and n columns and let each entry of A be 0 or 1. We call A a (0, 1)-matrix of size m by n. Let the sum of row i of A be denoted by r_i and let the sum of column j of A be denoted by s_j. We call $R = (r_1, r_2, \ldots, r_m)$ the *row sum vector* and $S = (s_1, s_2, \ldots, s_n)$ the *column sum vector* of A. The vectors R and S determine a class

$$\mathfrak{A} = \mathfrak{A}(R, S) \tag{1.1}$$

consisting of all (0, 1)-matrices A of size m by n, with row sum vector R and column sum vector S. Simple necessary and sufficient conditions on R and S are available in order that the class $\mathfrak{A}$ be non-empty **(5; 10)**. Let A be in $\mathfrak{A}$ and consider the 2 by 2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a minor of type A_1 into A_2, or *vice versa*, and leaves all other elements of A unaltered. The interchange theorem **(10)** asserts that if A and A' belong to $\mathfrak{A}$, then A is transformable into A' by interchanges. In our study we may suppose without loss of generality that $\mathfrak{A}$ is non-empty and that

$$r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m > 0, \tag{1.2}$$

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n > 0. \tag{1.3}$$

Such a class is called *normalized*. Henceforth we take $\mathfrak{A}$ normalized.

Let α be an integer in the interval

$$0 \leqslant \alpha \leqslant r_m, \tag{1.4}$$

and let ϵ be an integer in the interval

$$1 \leqslant \epsilon \leqslant n. \tag{1.5}$$

Let A be a matrix in the normalized class $\mathfrak{A}(R, S)$ and suppose A has an m by ϵ submatrix E^* each of whose row sums is at least α. An integer α fulfilling these requirements is said to be *compatible with ϵ in A*.

Suppose now that α is positive and compatible with ϵ in A. If this is the case, then we say that the ϵ columns of our m by ϵ submatrix E^* of A form an *α-set of representatives* for the matrix A. Let $\epsilon(\alpha)$ be the minimal number of columns of A that form an α-set of representatives for A. Such a column set is called a *minimal α-set of representatives* for A and $\epsilon(\alpha)$ is called the *α-width* of A. The integer α and the matrix A uniquely determine $\epsilon(\alpha)$. We note at the outset that the α-width $\epsilon(\alpha)$ of A is invariant under arbitrary permutations of the rows and columns of A. However, the α-width of A^T, the transpose of A, may differ drastically from that of A.

Let E^* be a submatrix of A of size m by $\epsilon(\alpha)$ that yields a minimal α-set of representatives for A. Let E be the submatrix of E^* composed of all of the rows of E^* that contain α 1's and $\epsilon(\alpha) - \alpha$ 0's. The matrix E is called a *critical α-submatrix* of A. Note that E cannot be empty since if all row sums of E^* exceed α, then deletion of any column of E^* yields an α-set of representatives for A, contradicting the minimality of $\epsilon(\alpha)$.

THEOREM 1.1. *The matrix A has an α-width $\epsilon(\alpha)$ for each α in the interval $1 \leqslant \alpha \leqslant r_m$. A critical α-submatrix E of A associated with an α-width $\epsilon(\alpha)$ contains no zero columns.*

Proof. Suppose that a critical α-submatrix E of A associated with an α-width $\epsilon(\alpha)$ contains a zero column. Let E^* be the m by $\epsilon(\alpha)$ submatrix of A containing E. The column of E^* containing the 0 column of E may be deleted and this yields an m by $\epsilon(\alpha) - 1$ matrix with minimal row sum α. But this contradicts the minimality of $\epsilon(\alpha)$.

Each of the critical α-submatrices E of A must contain $\epsilon(\alpha)$ columns. But the number of rows in the various critical α-submatrices need not be fixed. Let E be a critical α-submatrix containing the minimal number of rows $\delta(\alpha)$. The positive integer $\delta(\alpha)$ is called the *α-height* of A. Both $\epsilon(\alpha)$ and $\delta(\alpha)$ are basic invariants of the matrix A. Evidently

$$\epsilon(1) < \epsilon(2) < \ldots < \epsilon(r_m) \tag{1.6}$$

and by Theorem 1.1,

$$\delta(1) \geqslant \epsilon(1). \tag{1.7}$$

Thus far we have discussed for the most part a specified matrix A in the normalized class $\mathfrak{A}(R, S)$. We now turn our attention to properties of the class $\mathfrak{A}(R, S)$. Let α and ϵ be fixed and let α be *compatible with* ϵ. This means that α and ϵ are restricted by (1.4) and (1.5). Moreover, there exists an A in $\mathfrak{A}(R, S)$ with an m by ϵ submatrix E^* whose minimal row sum is at least α. Now consider the class of all m by ϵ submatrices E'' of the matrices A in $\mathfrak{A}(R, S)$ with the row sums of E'' greater than or equal to α. Let δ'' denote the number of row sums in E'' equal to α. The non-negative integer δ equal to the minimum of the integers δ'' is called the *multiplicity* of α with respect to ϵ. An α compatible with ϵ may be of multiplicity 0 with respect to ϵ. This will be the case whenever there exists an m by ϵ E'' with all of its row sums greater than α.

Let $1 \leqslant \alpha \leqslant r_m$. Then each A in $\mathfrak{A}(R, S)$ determines an α-width $\epsilon(\alpha)$ and α is compatible with $\epsilon(\alpha)$. For each α let the minimum of these $\epsilon(\alpha)$'s over all A in $\mathfrak{A}(R, S)$ be denoted by

$$\tilde{\epsilon} = \tilde{\epsilon}(\alpha). \tag{1.8}$$

Then α is compatible with $\tilde{\epsilon}(\alpha)$ and, by the minimality of $\tilde{\epsilon}(\alpha)$, if $\beta > \alpha$, then β is not compatible with $\tilde{\epsilon}(\alpha)$. We call $\tilde{\epsilon} = \tilde{\epsilon}(\alpha)$ the *minimal α-width* of the class $\mathfrak{A}(R, S)$. Let

$$\tilde{\delta} = \tilde{\delta}(\alpha) \tag{1.9}$$

denote the multiplicity of α with respect to $\tilde{\epsilon}(\alpha)$. The integer $\tilde{\delta}(\alpha)$ is positive and is equal to the minimum of the $\delta(\alpha)$'s for all matrices $A_{\tilde{\epsilon}}$ in $\mathfrak{A}(R, S)$ of α-width $\tilde{\epsilon}(\alpha)$. It is clear that

$$\tilde{\epsilon}(1) < \tilde{\epsilon}(2) < \ldots < \tilde{\epsilon}(r_m) \tag{1.10}$$

and

$$\tilde{\delta}(1) \geqslant \tilde{\epsilon}(1). \tag{1.11}$$

Similarly for each α let the maximum of the $\epsilon(\alpha)$'s over all A in $\mathfrak{A}(R, S)$ be denoted by

$$\bar{\epsilon} = \bar{\epsilon}(\alpha). \tag{1.12}$$

We call $\bar{\epsilon} = \bar{\epsilon}(\alpha)$ the *maximal α-width* of the class $\mathfrak{A}(R, S)$. A direct application of the interchange theorem allows us to prove that if ϵ is an integer in the interval

$$\tilde{\epsilon} \leqslant \epsilon \leqslant \bar{\epsilon}, \tag{1.13}$$

then there exists a matrix A_ϵ in $\mathfrak{A}(R, S)$ of α-width ϵ (see § 3).

In § 2 we take an α compatible with ϵ and of multiplicity δ with respect to ϵ. Under these conditions we establish the existence of a (0, 1)-matrix in $\mathfrak{A}(R, S)$ with an unusually simple block decomposition. An application of this theorem yields matrices of α-width $\tilde{\epsilon}$ and α-height $\tilde{\delta}$ in $\mathfrak{A}(R, S)$ called canonical matrices. Their study in §§ 3 and 4 leads to simple and explicit formulas for both $\tilde{\epsilon}$ and $\tilde{\delta}$. A straightforward construction for a canonical matrix is given in § 5. Section 6 concludes with applications to the special classes of (0, 1)-matrices containing k 1's in each row or k 1's in each column.

2. A block decomposition theorem. Let $0 \leqslant \alpha \leqslant r_m$ and let $1 \leqslant \epsilon \leqslant n$. Let α be compatible with ϵ and of multiplicity δ with respect to ϵ. We now prove the block decomposition theorem that plays a fundamental role in our subsequent investigations involving $\tilde{\epsilon}$ and $\tilde{\delta}$.

THEOREM 2.1. *Let α be compatible with ϵ and of multiplicity δ with respect to ϵ. Then there exists a matrix A in the normalized class $\mathfrak{A}(R, S)$ of the form*

$$A = \left[\begin{array}{c|c|c} M & J & * \\ \hline F & * & 0 \\ \cline{1-1} E & & \end{array}\right]. \tag{2.1}$$

Here E is of size δ by ϵ with exactly α 1's in each row. M is a matrix of size e by ϵ with $\alpha + 1$ or more 1's in each row. F is a matrix of size $m - (e + \delta)$ by ϵ with exactly $\alpha + 1$ 1's in each row. J is a matrix of 1's of size e by $f - \epsilon$ and 0 is a zero matrix. The degenerate cases $e = 0$, $e + \delta = m$, $\delta = 0$, $f = \epsilon$, and $f = n$ are not excluded.

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Proof. Let A be a matrix in the normalized class $\mathfrak{A}(R, S)$ and let A contain a submatrix E^* of size m by ϵ with δ row sums equal to α and the remaining $m - \delta$ row sums $> \alpha$. Let $\eta_1, \eta_2, \ldots, \eta_\epsilon$ be the column vectors of E^*. The matrix A is selected so that the vectors $\eta_1, \eta_2, \ldots, \eta_\epsilon$ are to the left as far as possible among all matrices A in $\mathfrak{A}$ containing an m by ϵ submatrix E^* of the type described. Let η be a column vector of A and suppose that η appears to the left of η_t, where η_t is one of $\eta_1, \eta_2, \ldots, \eta_\epsilon$. Now the class $\mathfrak{A}$ is normalized, so the column sums of A are non-increasing. We apply interchanges involving only the two columns η and η_t, and replace η by η', and η_t by η_t'. The column η' is to have 1's in all of the positions in which η_t has 1's. These interchanges yield a new matrix A' in $\mathfrak{A}$. Now columns $\eta', \eta_1, \ldots, \eta_{t-1}, \eta_{t+1}, \ldots, \eta_\epsilon$ of A' form an m by ϵ submatrix of A' with row sums $\geqslant \alpha$. Moreover, the number of row sums in this submatrix equal to α is $\leqslant \delta$. Hence the matrix A may be selected so that the m by ϵ submatrix E^* is confined to the first ϵ columns.

If $\delta = 0$, then A is of form (2.1) with $e = m, f = \epsilon$. Let δ be positive and suppose that in the first ϵ columns of A a row vector of E^* of sum α occurs above a row vector of E^* of sum $> \alpha$. Since the row sums of A are non-increasing, we may apply interchanges to A and lower the row vector of E^* of sum α. Hence we may obtain a matrix A in the normalized $\mathfrak{A}$ with the submatrix E of (2.1) in the lower left corner.

We now take this matrix and by interchanges obtain a matrix of the following form

$$
\left[\begin{array}{c|ccc}
M_1 & J & & C_0 \\
\hline
F_1 & W & & X \\
\hline
E & Y & Z & 0
\end{array}\right]. \tag{2.2}
$$

Here E is the matrix of (2.1). F_1 has exactly $\alpha + 1$ 1's in each row and M_1 has $\alpha + 2$ or more 1's in each row. J is a matrix of 1's and C_0 has at least one 0 in each column. The matrix 0 in the lower right corner must be a zero matrix, since otherwise an interchange involving the blocks M_1, C_0, E, and 0 contradicts the minimality of δ. (The tacit assumption that M_1 and C_0 both appear is unimportant. For if this were not the case, (2.2) is already a degenerate case of (2.1).)

Now let Z be the zero matrix of t columns that appears in all matrices of the form (2.2) in the normalized class $\mathfrak{A}$. The integer t is to be maximal, but the case $t = 0$ is not excluded. Then there exists a matrix of the form (2.2) with a 1 in the last column of Y. (Again if Y does not appear, then (2.2) is a degenerate case of (2.1).) Suppose that a 1 appears in row j of X and that a 0 appears in row j of W. We may apply an interchange if necessary and assume that a 0 appears in row j and the last column of W. Now an interchange involving the 1 in row j of X and the 1 in the last column of Y places a 1 in 0 or in Z. This contradicts either the minimality of δ or the presence of Z in all matrices of the form (2.2) in $\mathfrak{A}$. Thus if X contains a 1 in row j, then row j of W is a row of 1's. This means that there exists a matrix A in the normalized $\mathfrak{A}$ of the form (2.1).

3. The minimal α-width $\tilde{\epsilon}(\alpha)$.

THEOREM 3.1. *Let $\tilde{\epsilon} = \tilde{\epsilon}(\alpha)$ be the minimal α-width of the normalized class $\mathfrak{A}(R, S)$. Let $\tilde{\delta} = \tilde{\delta}(\alpha)$ be the multiplicity of α with respect to $\tilde{\epsilon}(\alpha)$. Then there exists a matrix $A_{\tilde{\epsilon}}$ of α-width $\tilde{\epsilon}$ in $\mathfrak{A}(R, S)$ of the form*

$$A_{\tilde{\epsilon}} = \left[\begin{array}{c|c|c} M & J & * \\ \hline F & * & 0 \\ \cline{1-1} E & & \end{array}\right]. \tag{3.1}$$

Here E is a critical submatrix of $A_{\tilde{\epsilon}}$ of size $\tilde{\delta}$ by $\tilde{\epsilon}$. M is a matrix of size e by $\tilde{\epsilon}$ with $\alpha + 1$ or more 1's *in each row. F is a matrix of size $m - (e + \tilde{\delta})$ by $\tilde{\epsilon}$ with exactly $\alpha + 1$* 1's *in each row. J is a matrix of size e by $f - \tilde{\epsilon}$ consisting entirely of* 1's, *and* 0 *is a zero matrix. Each of the first $\tilde{\epsilon}$ columns of $A_{\tilde{\epsilon}}$ contains more than $m - \tilde{\delta}$* 1's. *The degenerate cases $e = 0$, $e + \tilde{\delta} = m$, $f = \tilde{\epsilon}$, and $f = n$ are not excluded.*

Proof. In Theorem 2.1 let $\epsilon = \tilde{\epsilon}(\alpha)$ and $\delta = \tilde{\delta}(\alpha)$. Then (2.1) establishes the existence of a matrix $A_{\tilde{\epsilon}}$ of the form (3.1). Note that in Theorem 3.1 the integers α, $\tilde{\epsilon}$, and $\tilde{\delta}$ are positive and the degenerate case $\delta = 0$ of Theorem 2.1 is excluded. The matrix $A_{\tilde{\epsilon}}$ is of α-width $\tilde{\epsilon}(\alpha)$. Each of the first $\tilde{\epsilon}$ columns of $A_{\tilde{\epsilon}}$ contains more than $m - \tilde{\delta}$ 1's. For if this were not the case we could apply interchanges confined to the first $\tilde{\epsilon}$ columns of $A_{\tilde{\epsilon}}$ and replace a column of E by 0's. But this contradicts the minimality of $\tilde{\epsilon}$.

The special case $\alpha = 1$ of Theorem 3.1 deserves mention. A (0, 1)-matrix M is maximal **(10)** provided that in each row of M no 0 occurs to the left of a 1. We prove that for $\alpha = 1$ the matrices M and F of (3.1) may be selected as maximal matrices. Let E^* be the m by $\tilde{\epsilon}(1)$ matrix composed of the first $\tilde{\epsilon}(1)$ columns of (3.1). Let the sum of column 1 of E be e_1. We minimize e_1 by applying interchanges to E^*. This means that column 1 of M and column 1 of F must be columns of 1's. We cannot have $e_1 = 0$, for this contradicts the minimality of $\tilde{\epsilon}(1)$. Let the sum of column 2 of the transformed $\tilde{\delta}(1)$ by $\tilde{\epsilon}(1)$ E matrix be e_2. We minimize e_2 by applying interchanges to the last $\tilde{\epsilon}(1) - 1$ columns of the transformed E^*. Thus column 2 of M and column 2 of F must be columns of 1's, and again $e_2 > 0$. But F contains only two 1's in each row and hence F is the maximal matrix with exactly two 1's in each row. Let the sum of column 3 of the transformed $\tilde{\delta}(1)$ by $\tilde{\epsilon}(1)$ E matrix be e_3. We minimize e_3 by applying interchanges to the last $\tilde{\epsilon}(1) - 2$ columns of the transformed E^*, and continue this minimizing process until the matrix M is maximal.

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Theorem 3.1 is the basis for the simple formula for $\tilde{\epsilon}(\alpha)$ derived in § 4. Unfortunately the decomposition (3.1) does not have an apparent analogue for a matrix $A_{\bar{\epsilon}}$ of maximal α-width $\bar{\epsilon}(\alpha)$. Indeed, the class generated by the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

has $\bar{\epsilon}(1) = 3$. Columns 1, 2, and 4 intersect a critical submatrix of A. But it is not possible to replace A by a matrix in its class with a critical submatrix in the first three columns.

The following information on intermediate α-widths follows without difficulty.

THEOREM 3.2. *If ϵ is an integer in the interval*

$$\tilde{\epsilon} \leqslant \epsilon \leqslant \bar{\epsilon}, \tag{3.3}$$

then there exists an A_ϵ of α-width ϵ in the normalized class $\mathfrak{A}$.

Proof. We show that a single interchange applied to a matrix A_ϵ of α-width ϵ in $\mathfrak{A}$ cannot raise the α-width by two or more. For consider the case in which a matrix A_ϵ of α-width ϵ is transformed by one interchange into a matrix A' of α-width $\epsilon + 2$ or more. The matrix A_ϵ must have a critical submatrix E of size δ by ϵ. It is essential that the single interchange remove a 1 from the critical submatrix E, for otherwise we would have a matrix of α-width ϵ or less. Let the column vectors $\eta_1, \eta_2, \ldots, \eta_\epsilon$ of A_ϵ intersect the critical submatrix E. The interchange affects two column vectors η_t and η of A_ϵ. Here η_t is one of the vectors $\eta_1, \eta_2, \ldots, \eta_\epsilon$ and η is some other column vector of A_ϵ. Let the interchange transform η_t into η_t' and η into η'. But now in A' the $\epsilon + 1$ columns $\eta_1, \eta_2, \ldots, \eta_t', \ldots, \eta_\epsilon, \eta'$ are an α-set of representatives for A'. Hence one interchange can raise the α-width of A_ϵ by at most 1. But by the interchange theorem we may transform by interchanges a matrix $A_{\tilde{\epsilon}}$ of α-width $\tilde{\epsilon}$ into a matrix $A_{\bar{\epsilon}}$ of α-width $\bar{\epsilon}$. This establishes the existence of the matrix A_ϵ of α-width ϵ.

4. Canonical matrices.

For the normalized class $\mathfrak{A}(R, S)$ let

$$t_{ef} = r_{e+1} + r_{e+2} + \ldots + r_m - (s_1 + s_2 + \ldots + s_f) + ef. \tag{4.1}$$

Here e and f are integer parameters such that

$$0 \leqslant e \leqslant m, \tag{4.2}$$

$$0 \leqslant f \leqslant n. \tag{4.3}$$

Let A be in $\mathfrak{A}(R, S)$ and suppose that

$$A = \begin{bmatrix} W & * \\ * & Z \end{bmatrix} \tag{4.4}$$

with W of size e by f. For a (0, 1)-matrix Q let $N_0(Q)$ denote the number of 0's in Q and let $N_1(Q)$ denote the number of 1's in Q. Then (4.1) can be rewritten in the form

$$t_{ef} = N_0(W) + N_1(Z). \tag{4.5}$$

The invariants t_{ef} of $\mathfrak{A}(R, S)$ are useful in determining the maximal and minimal trace **(12)** and the maximal term rank **(11)** of the matrices in $\mathfrak{A}(R, S)$.

We now define invariants $N(\epsilon, e, f)$ of $\mathfrak{A}(R, S)$ which are generalizations of (4.1). These invariants turn out to be effective in determining the minimal α-width of the matrices in $\mathfrak{A}(R, S)$. Let

$$N(\epsilon, e, f) = r_{e+1} + r_{e+2} + \ldots + r_m - (s_{\epsilon+1} + s_{\epsilon+2} + \ldots + s_f) + e(f - \epsilon). \tag{4.6}$$

Here ϵ, e, f are integer parameters such that

$$0 \leqslant \epsilon \leqslant n, \tag{4.7}$$

$$0 \leqslant e \leqslant m, \tag{4.8}$$

$$\epsilon \leqslant f \leqslant n. \tag{4.9}$$

Note that

$$N(0, e, f) = t_{ef}, \tag{4.10}$$

and for $\epsilon = 0$, (4.9) reduces to (4.3). Moreover, (4.1) and (4.6) imply

$$N(\epsilon, e, f) = t_{ef} + (s_1 + s_2 + \ldots + s_\epsilon) - \epsilon e. \tag{4.11}$$

Let A be in $\mathfrak{A}(R, S)$ and suppose that

$$A = \begin{bmatrix} * & Y & * \\ X & * & Z \end{bmatrix}, \tag{4.12}$$

with X of size $m - e$ by ϵ and Y of size e by $f - \epsilon$. Then by (4.6),

$$N(\epsilon, e, f) = N_1(X) + N_0(Y) + N_1(Z). \tag{4.13}$$

Let A' be a matrix in the normalized class of the form

$$A' = \left[\begin{array}{c|c|c} M' & J & * \\ \hline F' & & \\ \hline E' & * & 0 \end{array}\right]. \tag{4.14}$$

Here E' is a matrix of size δ' by ϵ' with exactly α 1's in each row. M' is a matrix of size e' by ϵ' with $\alpha + 1$ or more 1's in each row. F' is a matrix of size $m - (e' + \delta')$ by ϵ' with exactly $\alpha + 1$ 1's in each row. J is a matrix of 1's of size e' by $f' - \epsilon'$ and 0 is a zero matrix. Each of the first ϵ' columns of A' contains more than $m - \delta'$ 1's. We require δ' and $\epsilon' > 0$ but the degenerate cases $e' = 0$, $e' + \delta' = m$, $f' = \epsilon'$, and $f' = n$ are not excluded. A matrix fulfilling these requirements is called *canonical*, and e' and f' are said to be *decomposition numbers* for A'. The decomposition numbers for a specified A' need not be unique.

It is clear that the matrix $A_{\tilde{\epsilon}}$ of Theorem 3.1 is canonical with $\epsilon' = \tilde{\epsilon}$, $\delta' = \tilde{\delta}$. The e and f of Theorem 3.1 are decomposition numbers for $A_{\tilde{\epsilon}}$.

THEOREM 4.1. *The ϵ' of the canonical matrix A' of* (4.14) *equals the first non-negative integer ϵ such that*

$$N(\epsilon, e, f) \geqslant \alpha(m - e) \tag{4.15}$$

for all integer parameters e and f restricted by (4.8) *and* (4.9).

Proof. Let ϵ be fixed and restricted by (4.7) and suppose that for some e and f restricted by (4.8) and (4.9)

$$N(\epsilon, e, f) < \alpha(m - e). \tag{4.16}$$

Then

$$\epsilon < \epsilon'. \tag{4.17}$$

For suppose that (4.16) holds and that $\epsilon \geqslant \epsilon'$. Then the first ϵ columns of A' contain at least α 1's in each row. But then by (4.13) the ϵ, e, and f of (4.16) satisfy $N(\epsilon, e, f) \geqslant \alpha(m - e)$ and this contradicts (4.16). Hence (4.16) implies (4.17).

Let ϵ be fixed and restricted by (4.7) and suppose that for each e and f restricted by (4.8) and (4.9)

$$N(\epsilon, e, f) \geqslant \alpha(m - e). \tag{4.18}$$

Then

$$\epsilon' \leqslant \epsilon. \tag{4.19}$$

For suppose that (4.18) holds and that $\epsilon < \epsilon'$. Then for the decomposition numbers e' and f' of (4.14)

$$0 \leqslant e' < m, \tag{4.20}$$

and

$$\epsilon < \epsilon' \leqslant f' \leqslant n. \tag{4.21}$$

By (4.13),

$$N(\epsilon, e', f') = N(\epsilon', e', f') + N_0(T) - N_1(U), \tag{4.22}$$

where T denotes the submatrix formed by the intersection of rows $1, 2, \ldots, e'$ and columns $\epsilon + 1, \epsilon + 2, \ldots, \epsilon'$ of A', and U the intersection of rows $e' + 1, e' + 2, \ldots, m$ and columns $\epsilon + 1, \epsilon + 2, \ldots, \epsilon'$ of A'. Now each of the first ϵ' columns of A' contains more than $m - \delta'$ 1's. Hence

$$N_1(U) - N_0(T) + e'(\epsilon' - \epsilon) = s_{\epsilon+1} + s_{\epsilon+2} + \ldots + s_{\epsilon'} > (m - \delta')(\epsilon' - \epsilon), \tag{4.23}$$

and

$$N_1(U) - N_0(T) > m - (e' + \delta'). \tag{4.24}$$

Moreover,

$$N(\epsilon', e', f') = (\alpha + 1)(m - e') - \delta'. \tag{4.25}$$

Hence by (4.22), (4.25), and (4.24),

$$\begin{aligned} N(\epsilon, e', f') &= (\alpha + 1)(m - e') - \delta' + N_0(T) - N_1(U) \\ &< (\alpha + 1)(m - e') - \delta' - m + e' + \delta' = \alpha(m - e'). \end{aligned} \tag{4.26}$$

But this contradicts (4.18). Hence (4.18) implies (4.19) and this proves Theorem 4.1.

THEOREM 4.2. *Let $\tilde{\epsilon}$ be the minimal α-width of the normalized class $\mathfrak{A}(R, S)$. The ϵ' of the canonical matrix A' of* (4.14) *equals $\tilde{\epsilon}$ and $\tilde{\epsilon}$ is the first non-negative integer ϵ such that*

$$N(\epsilon, e, f) \geqslant \alpha(m - e) \tag{4.27}$$

for all integer parameters e and f restricted by (4.8) *and* (4.9).

Proof. This follows from Theorem 4.1 and the fact that the matrix $A_{\tilde{\epsilon}}$ of Theorem 3.1 is canonical.

Theorem 4.2 provides a simple computation for $\tilde{\epsilon}$. One can successively calculate the arrays $N(\hat{\epsilon}, e, f) + \alpha e$, $N(\hat{\epsilon} + 1, e, f) + \alpha e, \ldots,$ each for appropriate e and f, where $\hat{\epsilon}$ is the first ϵ such that $s_1 + s_2 + \ldots + s_\epsilon \geqslant \alpha m$, stopping when all entries of the array are at least equal to αm. The starting value $\hat{\epsilon}$ in the calculation is clearly a lower bound for $\tilde{\epsilon}$.

The next theorem shows that all pairs of decomposition numbers e', f' can be singled out from the array $N(\tilde{\epsilon} - 1, e, f) + \alpha e$.

THEOREM 4.3. *Let A' be the canonical matrix of* (4.14) *with $\epsilon' = \tilde{\epsilon}$. Let*

$$\tilde{\gamma} = \min_{e,f} [N(\tilde{\epsilon} - 1, e, f) + \alpha e], \tag{4.28}$$

where $0 \leqslant e \leqslant m$ and $\tilde{\epsilon} - 1 \leqslant f \leqslant n$. Then

$$\tilde{\gamma} = N(\tilde{\epsilon} - 1, e', f') + \alpha e' \tag{4.29}$$

if and only if e' and f' are decomposition numbers for A'.

Proof. Let e' and f' be decomposition numbers for A'. Then $0 \leqslant e' \leqslant m$ and $\tilde{\epsilon} \leqslant f' \leqslant n$. We consider first the case in which $e \leqslant e'$ and $\tilde{\epsilon} \leqslant f \leqslant n$. Then

$$N(\tilde{\epsilon} - 1, e', f') = N(\tilde{\epsilon} - 1, e, f) + N_0(T) - N_1(U) - N_0(V) - N_1(W). \tag{4.30}$$

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Here T is the intersection of rows $e+1, e+2, \ldots, e'$ and column $\tilde{\epsilon}$ of A', U is the intersection of rows $e+1, e+2, \ldots, e'$ and columns $1, 2, \ldots, \tilde{\epsilon}-1$ of A', V is the intersection of rows $1, 2, \ldots, e$ and columns $\tilde{\epsilon}+1, \tilde{\epsilon}+2, \ldots, f$ of A', and W is the intersection of rows $e+1, e+2, \ldots, m$ and columns $f+1, f+2, \ldots, n$ of A'. Now since e' and f' are decomposition numbers for A',

$$N_1(U) + [e' - e - N_0(T)] = (\alpha + 1)(e' - e) + p, \tag{4.31}$$

where p is a non-negative integer. Hence

$$N_0(T) - N_1(U) = \alpha(e - e') - p \tag{4.32}$$

and

$$N(\tilde{\epsilon} - 1, e', f') + \alpha e' = N(\tilde{\epsilon} - 1, e, f) + \alpha e - p - N_0(V) - N_1(W). \tag{4.33}$$

Thus

$$N(\tilde{\epsilon} - 1, e', f') + \alpha e' \leqslant N(\tilde{\epsilon} - 1, e, f) + \alpha e \tag{4.34}$$

and equality holds if and only if $p = 0$, $N_0(V) = 0$, $N_1(W) = 0$. But $p = 0$, $N_0(V) = 0$, $N_1(W) = 0$ if and only if e and f are decomposition numbers for A'.

Next consider the case in which $e' < e$ and $\tilde{\epsilon} \leqslant f \leqslant n$. Then

$$N(\tilde{\epsilon} - 1, e', f') = N(\tilde{\epsilon} - 1, e, f) + N_1(U) - N_0(T) - N_0(V) - N_1(W). \tag{4.35}$$

Here T is the intersection of rows $e'+1, e'+2, \ldots, e$ and column $\tilde{\epsilon}$ of A', U is the intersection of rows $e'+1, e'+2, \ldots, e$ and columns $1, 2, \ldots, \tilde{\epsilon}-1$ of A', V is the intersection of rows $1, 2, \ldots, e$ and columns $\tilde{\epsilon}+1, \tilde{\epsilon}+2, \ldots, f$ of A', and W is the intersection of rows $e+1, e+2, \ldots, m$ and columns $f+1, f+2, \ldots, n$ of A'. Now

$$N_1(U) + [e - e' - N_0(T)] = \alpha(e - e') + q, \tag{4.36}$$

where q is a non-negative integer satisfying

$$q + e' - e \leqslant 0. \tag{4.37}$$

Hence

$$N(\tilde{\epsilon} - 1, e', f') + \alpha e' = N(\tilde{\epsilon} - 1, e, f) + \alpha e + e' - e + q - N_0(V) - N_1(W). \tag{4.38}$$

Thus

$$N(\tilde{\epsilon} - 1, e', f') + \alpha e' \leqslant N(\tilde{\epsilon} - 1, e, f) + \alpha e \tag{4.39}$$

and equality holds if and only if $q = e - e'$ and $N_0(V) = N_1(W) = 0$, that is, if and only if e and f are decomposition numbers for A'.

We now extend the range of f to $\tilde{\epsilon} - 1 \leqslant f \leqslant n$. It suffices to show that if $f = \tilde{\epsilon} - 1$, then

$$N(\tilde{\epsilon} - 1, e, f) + \alpha e > N(\tilde{\epsilon} - 1, e', f') + \alpha e'.$$

But this follows without difficulty from the equations

$$N(\tilde{\epsilon} - 1, e, \tilde{\epsilon} - 1) = r_{e+1} + r_{e+2} + \ldots + r_m$$

and

$$N(\tilde{\epsilon} - 1, e', f') = (\alpha + 1)(m - e') - \delta' + e' - s_{\tilde{\epsilon}}.$$

This completes the proof of Theorem 4.3.

THEOREM 4.4. *Let $\tilde{\delta}$ be the multiplicity of α with respect to $\tilde{\epsilon}$. The δ' of the canonical matrix A' of* (4.14) *equals $\tilde{\delta}$ and*

$$\tilde{\delta} = (\alpha + 1)m - \tilde{\gamma} - s_{\tilde{\epsilon}}. \tag{4.40}$$

Proof. Let A' be the canonical matrix of (4.14). Then

$$N(\tilde{\epsilon}, e', f') = N(\tilde{\epsilon} - 1, e', f') + s_{\tilde{\epsilon}} - e'. \tag{4.41}$$

But

$$N(\tilde{\epsilon}, e', f') = (\alpha + 1)(m - e') - \delta' \tag{4.42}$$

and hence

$$\delta' = (\alpha + 1)m - \tilde{\gamma} - s_{\tilde{\epsilon}}. \tag{4.43}$$

Moreover, the matrix $A_{\tilde{\epsilon}}$ of Theorem 3.1 is canonical so that $\delta' = \tilde{\delta}$.

We conclude this section with a numerical example illustrating the computation of $\tilde{\epsilon}(1)$, $\tilde{\delta}(1)$, and the decomposition numbers e', f' for a normalized class. Let $\mathfrak{A}(R, S)$ be determined by

$$R = (6, 5, 3, 2, 2, 2, 1, 1),$$

$$S = (4, 4, 4, 4, 4, 1, 1).$$

The arrays $N(2, e, f) + e$, for $0 \leqslant e \leqslant 8$, $2 \leqslant f \leqslant 7$, and $N(3, e, f) + e$, for $0 \leqslant e \leqslant 8$, $3 \leqslant f \leqslant 7$, yield all pertinent information. They are shown in Table II.

TABLE II

	$\epsilon = 2$							$\epsilon = 3$				
$e \backslash f$	2	3	4	5	6	7	$e \backslash f$	3	4	5	6	7
0	22	18	14	10	9	8	0	22	18	14	13	12
1	17	14	11	8	8	8	1	17	14	11	11	11
2	13	11	9	7	8	9	2	13	11	9	10	11
3	11	10	9	8	10	12	3	11	10	9	11	13
4	10	10	10	10	13	16	4	10	10	10	13	16
5	9	10	11	12	16	20	5	9	10	11	15	19
6	8	10	12	14	19	24	6	8	10	12	17	22
7	8	11	14	17	23	29	7	8	11	14	20	26
8	8	12	16	20	27	34	8	8	12	16	23	30

The recursions

$$N(\epsilon, e+1, f) = N(\epsilon, e, f) - r_{e+1} + f - \epsilon, \tag{4.44}$$

$$N(\epsilon, e, f+1) = N(\epsilon, e, f) + e - s_{f+1}, \tag{4.45}$$

$$N(\epsilon+1, e, f) = N(\epsilon, e, f) + s_{\epsilon+1} - e \tag{4.46}$$

are useful in constructing such arrays.

Since

$$N(2, 2, 5) + 2 = 7 < 8,$$
$$N(3, e, f) + e \geqslant 8, \qquad (0 \leqslant e \leqslant 8,\ 3 \leqslant f \leqslant 7),$$

we have $\tilde{\epsilon}(1) = 3$. Also $\tilde{\gamma} = 7$ corresponding to the unique decomposition numbers $e' = 2, f' = 5$, and hence $\tilde{\delta}(1) = 5$. A canonical matrix in the class is given by

$$A = \left[\begin{array}{ccc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

5. Construction of canonical matrices. We are now in a position to give a simple procedure for the construction of a canonical matrix A'. Before doing so we recall some facts about the construction of a (0, 1)-matrix of size m by n having a specified row sum vector $R = (r_1, r_2, \ldots, r_m)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$ (**5**; **7**; **10**). Let R_1 be a row vector of r_1 1's and $n - r_1$ 0's. Let the 1's be inserted in the positions in which S has its r_1 largest components. Let R_2 be a row vector of r_2 1's and $n - r_2$ 0's. Let the 1's be inserted in the positions in which $S - R_1$ has its r_2 largest components. R_3 is a row vector whose 1's are in the positions in which $S - R_1 - R_2$ has its r_3 largest components, and so on. Now let A be a matrix with row sum vector R and column sum vector S. We may apply interchanges to A and replace row 1 of A by R_1. Then we apply interchanges to the transformed matrix and replace row 2 by R_2. These interchanges do not involve R_1. In this way we transform A by interchanges into a matrix A^* composed of the row vectors $R_1, R_2, \ldots, R_m$. But this tells us that A^* has row sum vector R and column sum vector S, and hence we have a procedure for constructing a matrix in the class $\mathfrak{A}(R, S)$.

We now construct a canonical matrix A' of the form (4.14) in the normalized class $\mathfrak{A}(R, S)$. The theorems of § 4 give formulas for the integers

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$\epsilon' = \tilde{\epsilon}$, $\delta' = \tilde{\delta}$, e' and f' in terms of R and S. The submatrix of A' formed by the intersection of rows $e' + 1, e' + 2, \ldots, m$ and columns $\epsilon' + 1, \epsilon' + 2, \ldots, f'$ has its row and column sum vectors determined. Hence this submatrix may be inserted.

Let

$$B = \left[\begin{array}{c|c} M' & G' \\ \hline F' & \\ \cline{1-1} E' & 0 \end{array}\right] \tag{5.1}$$

be the m by $n - (f' - \epsilon')$ submatrix of A' formed from A' by the deletion of columns $\epsilon' + 1, \epsilon' + 2, \ldots, f'$. The matrix B comprises the undetermined portion of A'. We know the row sums of B, F', and E' and the column sums of B and G'. Let

$$B' = [M' \; G'] \tag{5.2}$$

denote the first e' rows of B and let

$$S' = (s_{f'+1}, s_{f'+2}, \ldots, s_n) \tag{5.3}$$

denote the column sum vector of G'. We apply interchanges to B' and place the 1's in column 1 of G' in those rows of B' that possess the $s_{f'+1}$ largest row sums. Now we ignore column 1 of G' of the transformed B' matrix and apply interchanges to column 2 of G'. These interchanges do not disturb column 1 of G' and they place the 1's in column 2 of G' in those rows of B' that possess, with column 1 of G' excluded, the $s_{f'+2}$ largest row sums. This gives a construction for G'. But then this determines a row sum vector for G' and hence a row sum vector for M'. The construction for G' is such that each of the components of the row sum vector of M' exceed α. In fact 1's are inserted in the columns of G' by a procedure that keeps the size of the row sums of M' as uniform as possible. The undetermined portion of B now consists of the first ϵ' columns of B. But we know the row sum vector and column sum vector of this m by ϵ' matrix, and hence we have a construction for a canonical matrix A'.

6. Special classes. Let $\mathfrak{A}(K, S)$ denote the normalized class of m by n (0, 1)-matrices having row sum vector $K = (k, k, \ldots, k)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$. Similarly, let $\mathfrak{A}(R, K)$ denote the normalized class of m by n (0, 1)-matrices having row sum vector $R = (r_1, r_2, \ldots, r_m)$ and column sum vector $K = (k, k, \ldots, k)$. For these special classes, the canonical form (4.14) is always degenerate.

THEOREM 6.1. *Every canonical matrix A' of form* (4.14) *in* $\mathfrak{A}(K, S)$ *has decomposition numbers $e' = 0, f' = n$. Every canonical matrix A' of form* (4.14) *in* $\mathfrak{A}(R, K)$ *has decomposition number $f' = n$.*

Proof. Let A' of form (4.14) be in the normalized class $\mathfrak{A}(K, S)$, and suppose $e' > 0$. Then, comparing first and last row sums of A', we have $\alpha + 1 + f' - \epsilon' \leqslant k \leqslant \alpha + f' - \epsilon'$. This contradiction shows that $e' = 0$, and hence $f' = n$.

Let A' of form (4.14) be in the normalized class $\mathfrak{A}(R, K)$, and suppose $f' < n$. Comparing first and last column sums of A' yields $e' \geqslant k > m - \delta'$, a contradiction. Thus $f' = n$.

For the class $\mathfrak{A}(K, S)$, the lower bound for $\tilde{\epsilon}$ mentioned following the proof of Theorem 4.2 is always achieved: $\tilde{\epsilon}$ is the first ϵ such that

$$s_1 + s_2 + \ldots + s_\epsilon \geqslant \alpha m. \tag{6.1}$$

For in A' of (4.14) with $e' = 0$,

$$s_1 + s_2 + \ldots + s_{\tilde{\epsilon}} = \alpha m + (m - \tilde{\delta}). \tag{6.2}$$

Hence $s_1 + s_2 + \ldots + s_{\tilde{\epsilon}-1} < \alpha_m$ and $s_1 + s_2 + \ldots + s_{\tilde{\epsilon}} \geqslant \alpha m$. Moreover, $\tilde{\delta}$ for the normalized class $\mathfrak{A}(K, S)$ is given by

$$\tilde{\delta} = (\alpha + 1)m - (s_1 + s_2 + \ldots + s_{\tilde{\epsilon}}). \tag{6.3}$$

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