# VARIETIES OF METABELIAN $p$-GROUPS OF CLASS $p, p+1$

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## Introduction

A large number of results are available on the lattice of subvarieties of the variety of metabelian groups. When considering metabelian $p$-groups (for odd $p$), the immediate division is between groups of nilpotency class less than $p$, and those of class at least $p$. The first case was dealt with in some detail in [1], and this paper extends the results to the next interesting cases, classes $p$ and $p+1$. The main results are stated in Theorems 2 and 4, which give the basis laws for certain varieties, and 3 and 5, which assert the existence of specific generating groups for these varieties, and hence their non-trivial existence. The notation, and the essential parts of the logic, are as in [1]; for the purposes of this paper, the following modification of Lemma 1.1 of [1], and also the ring-of-integer operations used in its proof, are together dubbed the 'Stirling manipulation':

*'Let $\mathfrak{V}$ be a variety of p-groups, with the law $\prod_{k=0}^{n} W_k = 1$, $n < p$, where the $W_k$ are distinct commuting words. If there is a set of valid transformations $\{\phi_\lambda\}$, $\lambda = 1, 2, \cdots n+1$ such that $\phi_\lambda(W_k) = W_k^{\lambda^k}$, then $W_k = 1$ is a law in $\mathfrak{V}$ for each $k : k = 0, 1, 2, \cdots n$.'*

Certain other modifications can be made in the manner in which the properties of the Stirling numbers can be exploited, and they are 'obvious' when the appropriate arrays are written down. Consequently, these details will not be given in the proofs.

## 1. Laws in metabelian $p$-groups of "large class"

Results obtained by N. D. Gupta and M. F. Newman [2] (on laws with repeated entries in commutator words) can be extended to the more general results below. These include some of the results of H. Meier-Wunderli [3], but the demonstration of the inclusion is left.

THEOREM 1. *Let $G$ be a metabelian p-group of class $c = p+m$ where $m \geqq 0$, and let the exponent of $\gamma_c(G)$ be $p^{\alpha_c}$, where $\alpha_c$ is minimal. Then* no *law of the form*

$$(x_1, n_2 x_2, (n_1-1)x_1, n_3 x_3, \cdots, n_t x_t)^{p^\mu} = 1$$



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*will hold in G, with* $\mu < \alpha_c$, *the t variables being distinct,* $\sum_{i=1}^{t} n_i = c$, *and all* $n_i > 0$, *with* $t \geqq 3$ *when* $m = 0$, $t \geqq m+2$ *when* $m \neq 0$.

(The proof is almost the same as that for the more special Lemma 2 of [1], the restriction on $t$ being sufficient to allow use of the Stirling manipulation, so the details are not given.) By extending the methods of 2.1 of [1], it is possible to obtain:

LEMMA 1.1. *Let G be a metabelian p-group of class c; if G has the law in* $n \leqq s+2$ *variables*:

$$(x, my, v_1, v_2, \ldots, v_s)^{p^\mu} = 1, \quad m < p+1, \quad m+1+c = c, \quad s \geqq 0$$

(*where none of the* $v_i$ *is x or y, but the* $v_i$ *need not be distinct*) *then G also has the law in* $n+1$ *variables*: $(x, y, (m-1)z, v_1, \cdots, v_s)^{(m+1)p^\mu} = 1$, *and in particular, if p does not divide* $(m+1)$, *the law*: $(x, y, (m-1)z, v_1, \cdots, v_s)^{p^\mu} = 1$.

One further lemma is useful:

LEMMA 1.2. *Let G be a metabelian p-group of class* $c \leqq p+1$. *Then if G has the law*

$$(x, (c-1)y)^{p^\mu} = 1$$

*it follows that every two-generator group H in Var* $(G)$ *has* $\gamma_c(H)$ *of exponent at most* $p^\mu$.

PROOF. Substitute $xy$ for $y$ in the given law, and use the given law itself, to obtain

$$\prod_{k=1}^{c-2} (x, y, (k-1)y, (c-1-k)x)^{\binom{c-2}{k-1}p^\mu} = 1.$$

Apply the Stirling manipulation using the substitution of $y^\lambda$ for $y$ as the 'valid transformation'. Since $c \leqq p+1$, then immediately

$$(x, ky, (c-1-k)x)^{p^\mu} = 1, \qquad k = 1, 2, \cdots, c-1.$$

The immediate consequences of the above results for a metabelian $p$-group $G$ are then:

(i) *If G is of class p, then* $(x, (p-1)y)^{p^\mu} = 1$ *implies* $(x, (p-2)y, z)^{p^{\mu+1}} = 1$ *which implies* $\gamma_p(G)$ *is of exponent at most* $p^{\mu+1}$.

(ii) *If G is of class* $p+1$, *then* $(x, py)^{p^\mu} = 1$ *implies* $(x, (p-1)y, z)^{p^\mu} = 1$ *which implies* $\gamma_{p+1}(G)$ *is of exponent at most* $p^\mu$.

*Using* (i) *and* (ii) *together with Theorem* 1 *and Lemma* 1.1

(iii) *If* $G/\gamma_{p+1}(G)$ *has both the laws* $(x_1, x_2, \cdots x_p)^{p^{\alpha_p}} = 1$ *and* $(x, (p-1)y)^{p^{\alpha_p - 1}} = 1$, $\alpha_p$ *and* $\alpha_p - 1$ *being minimal, then* $G/\gamma_{r+1}(G)$ *has the law* $(x_1, x_2, \cdots x_r)^{p^{\alpha_p - 1}} = 1$ *for* $r \geqq p+1$.

Unfortunately, it appears that the minimal number of variables, $t$, needed to ensure that:
*'any c-weight commutator law*

$$(x_1, n_2 x_2, (n-1)x_1, n_3 x_3, \cdots, n_t x_t)^{p^\mu} = 1 \quad in \ G/\gamma_{c+1}$$

*implies that $\gamma_c(G)$ has exponent $p^\mu$'*
might well be given by Theorem 1 and some easy extensions on special word-forms. No attempt will be made to settle the question here.

## 2. The lattice of varieties of class $p$

The inductive steps to prove that certain laws form part of a 'basis', used to prove Theorem 3 of [1], are valid for class $p$, as long as we allow for the possibility of $(x, (p-1)y)^{p^{\alpha_p - 1}} = 1$ and $(x_1, x_2, \cdots, x_p)^{p^{\alpha_p}} = 1$ both being true, with $\alpha_p$ being minimal.

The exponent of the variety has a bearing on the existence of the two-variable law. Although results from another article [4] are used to establish the lemma below, the lemma itself properly occurs in this context.

LEMMA 2.1. *Let $G$ be a metabelian $p$-group of class $p$ and exponent $p^n$. Then every two-generator group $H$ in $Var(G)$ has $\gamma_p(H)$ of exponent at most $p^{n-1}$.*

PROOF. $H/H^p$ is of exponent $p$, and hence has $(x, (p-1)y) = 1$ as a law, ([3], [4]), and so $(x, (p-1)y) \in H^p$ for all $x, y \in H$. But $H^p$ has exponent $p^{n-1}$ at most, by the elementary result that $(a^{p^\alpha}, b^{p^\alpha}) = S^{p^\alpha}$, $S \in \gamma_2(\langle a, b\rangle)$, in any metabelian $p$-group of class less than $2p$, and elementary calculation.

The strategy to establish a basis for the laws of a variety of class $p$ is as follows; we note first that (from its proof) the inductive lemma 5 of [1] applies to the free group generating the variety, thus allowing certain laws in commutator words of weight $< p$ to be elements of a basis, so our concern here is with $p$-weight commutator laws. If $(x, (p-1)y)^{p^{\alpha_p}} = 1$ but $(x, (p-1)y)^{p^{\alpha_p - 1}} \neq 1$, we have no trouble, because exactly the same technique applies as for the small-class case. The real concern then lies with the situation where

$$(x, (p-1)y)^{p^\mu} = 1 \ \text{and} \ (x, (p-2)y, z)^{p^{\mu+1}} = 1,$$

both indices being minimal, and $\alpha_p = \mu+1$. Since the first law implies the second, we need to show that any $p$-weight commutator law follows from $(x, (p-1)y)^{p^\mu} = 1$, irrespective of the number of distinct variables. Lemma 4 of [1] (which was adequate for the small-class case) is clearly insufficient, since it is part of an induction assuming that $\mu \nless \alpha_c$. The result we need is provided in the following three lemmas:

LEMMA 2.2. *Let $G$ be a metabelian $p$-group of class $p$, in which the law $\mathscr{L}$ : $(x, (p-1)y)^{p^\mu} = 1$ holds. Then any law in $G$ of the form $\prod_i L_i = 1$, where the $L_i$*

*are commutator words of weight p, each of which contains no more than three distinct variables, is a consequence of the law $\mathscr{L}$.*

LEMMA 2.3. *Let $a_1, a_2, \cdots, a_n$ be a partition of p, such that each $a_i > 0$, and $a_1 + \cdots + a_n = p$. Let G be a metabelian p-group of class p in which the law $\mathscr{L} : (x, (p-1)y)^{p^\mu} = 1$ holds. Then one of the consequences of $\mathscr{L}$ in G is a law in n distinct variables $x_1, x_2, \cdots, x_n$, of the form:*

$$\prod_{i=1}^{n-1} (x_i, x_n, (a_n-1)x_n, a_{n-1}x_{n-1}, \cdots, (a_i-1)x_i, \cdots, a_1 x_1)^{\beta_i p^\mu} = 1$$

*in which no $\beta_i$ is divisible by p, and further, p does not divide $\Sigma_{i=1}^{n-1} \beta_i$.*

LEMMA 2.4. *Let G be a metabelian p-group of class p, in which*

$$\mathscr{L} : (x, (p-1)y)^{p^\mu} = 1$$

*is a law. Then any law in G in p-weight commutator words, in $n \geqq 3$ variables, can be derived from $\mathscr{L}$.*

The last Lemma follows from the previous two, which are proved as follows:

PROOF OF 2.2. We assume the non-trivial case $\alpha_p = \mu+1$. In the law $\prod_i L_i = 1$, select any three variables, say $x, y, z$, and set the rest to 1. The resulting law can be written in 'basic' commutator form, according to the ordering $x \geqq y \geqq z$, and then re-collected into the form $\prod_{k=0}^{p-1} W_k = 1$, where $W_k$ is a product each of whose factors has precisely $k$ entries of $z$. By applying the Stirling manipulation, we obtain each $W_k = 1$. $W_0 = 1$ and $W_{p-1} = 1$ are each consequences of $\mathscr{L}$, by 1.2 and by the proof of 1.2 of [1]. Taking any $W_m = 1$, $m \neq 0$, $m \neq p-1$, we can write it in the form $\prod_{k=0}^{p-c} V_{m,k} = 1$, where each $V_{m,k}$ is a product of commutators with $m$ entries of $z$ and $k$ entries of $x$; the Stirling manipulation then gives each $V_{m,k} = 1$ as a law, $V_{m,0} = 1$ and $V_{1,p-1} = 1$ being consequences of $\mathscr{L}$. We can rewrite each remaining case in the form

$$V_{m,a} = (x, z, y, (a-1)x, (b-1)y, (m-1)z)^{\beta_q}(y, z, x, (a-1)x, (b-1)y, (m-1)z)^{\gamma_q} = 1$$

with $a \neq 0$, $b \neq 0$, $m \neq 0$, and where $q$ is the partition $a$, $b$, $m$ of $p$. By putting $x$ for $z$, we obtain $p^\mu | \gamma_q$ by the minimality of $\mu$, and similarly $p^\mu | \beta_q$. If $p^{\mu+1} | \gamma_q$ then $p^{\mu+1} | \beta_q$ and $V_{m,a} = 1$ is an immediate consequence of $\mathscr{L}$. The only case left to consider is when $\gamma_q = \gamma p^\mu$, $\beta_q = \beta p^\mu$, $\gamma \not\equiv 0$, $\beta \not\equiv 0 \pmod p$. By substituting $yz$ for $y$, $xz$ for $x$, expanding, and using the Stirling manipulation twice we can arrive at the law:

$$V' : (x, z, y, (p-3)z)^{b\beta p^\mu}(y, z, x, (p-3)z)^{a\gamma p^\mu} = 1.$$

We can combine this with one of the consequences of $\mathscr{L}$ (see, for example, the proof of 2.1 of [1]) to show $(x, z, y, (p-3)z)^{(b\beta - a\gamma)p^\mu} = 1$ and hence $b\beta - a\gamma \equiv 0 \pmod p$ by Theorem 1. Now another of the consequences of $\mathscr{L}$ (see [1]) is the set of laws

$$(x, z, y, (p-2-k)y, (k-1)z)^{\binom{p-1}{k}p^\mu}(y, z, x, (p-2-k)y, (k-1)z)^{-\binom{p-1}{k}p^\mu} = 1,$$

for $k = 0, 1, \cdots p-2$. If we select $p-2-k = b-1$, and raise to the power $t$, where $t\binom{p-1}{b} \equiv 1 \bmod p$, substitute $xz$ for $z$, and apply the Stirling manipulation, we obtain, as a consequence of $\mathscr{L}$,

$$(x, z, y, (a-1)x, (b-1)y, (m-1)z)^{Mp^\mu}(y, z, x, (a-1)x, (b-1)y, (m-1)z)^{Np^\mu} = 1$$

where

$$M = \binom{p-b-2}{a-1} - b\binom{p-b-2}{a-2} = \binom{p-b-1}{a-1}\left(\frac{p-b-a-b(a-1)}{p-b-1}\right)$$
$$\equiv a\binom{p-b-1}{a-1} \bmod p$$

and

$$N = \binom{p-b-2}{a-1} b + \binom{p-b-2}{a-1} b = b\binom{p-b-1}{a-1}.$$

Since $\binom{p-b-1}{a-1} \not\equiv 0 \bmod p$, we have now

$$(x, z, y, (a-1)x, (b-1)y, (m-1)z)^{ap^\mu}(y, z, x, (a-1)x, (b-1)y, (m-1)z)^{bp^\mu} = 1$$

as a consequence of $\mathscr{L}$. Since, however, $\beta/\gamma \equiv a/b \pmod p$, the law $V_{m,a} = 1$ can be derived from this. We have now, in fact, shown that the original word $\prod L_i$ is a product of words each of which is 1 as a consequence of $\mathscr{L}$.

PROOF OF 2.3. ($\alpha_p = \mu+1$). The case $n = 3$ being true by the last part of the proof in the previous Lemma, take as inductive assumption the truth of 2.3 for $n-1$ variables, $n > 3$: we have that $\mathscr{L}$ implies a law

$$\prod_{i=1}^{n-2}(x_i, x_n, (a_{n-1}+a_n-1)x_n, a_{n-2}x_{n-2}, \cdots, a_1x_1)^{\gamma_i p^\mu} = 1$$

with no $\gamma_i$ divisible by $p$, and $\sum_i \gamma_i$ not divisible by $p$. By putting $x_{n-1}x_n$ for $x_n$, writing $r$ for $a_{n-1}+a_n-1$, expanding, and using the Stirling manipulation, we arrive at the law in the statement of 2.3, with $\beta_i = \binom{r+1}{a_n}\gamma_i$ for $1 \leqq i \leqq n-2$ and $\beta_{n-1} = -\binom{r}{a_n}\sum \gamma_i$.

The proof of Lemma 2.4 is by induction on $n$: considering any law of the form mentioned, we proceed in a manner similar to the proof of 2.2, using the Stirling manipulation and 2.3 at the appropriate places. The details are tedious but straightforward, and are therefore omitted.

We now have all the necessary results to establish a basis for the laws of any variety of metabelian $p$-groups nilpotent of class $p$. The proof is simply a minor re-wording of the corresponding proof in [1], using 2.4 when discussing commutator laws of weight $p$. Hence:

THEOREM 2. *Any variety of metabelian p-groups of class $p \geqq 3$ and finite exponent $p^{\alpha_1}$ has as a basis for its laws, either the set S:*

$$S \begin{cases} ((x, y), (u, v)) = 1, (x_1, x_2, \cdots, x_{p+1}) = 1, x^{p^{\alpha_1}} = 1, \\ (x, (i-1)y)^{p^{\alpha_i}} = 1, \ i = 2, 3, \cdots, p-1 \\ (x, (p-2)y, z)^{p^{\alpha_p}} = 1 \ (\textit{which is equivalent to } (x_1, x_2, \cdots, x_p)^{p^{\alpha_p}} = 1) \end{cases}$$

*or the set S together with the law L:*

$$L: \quad (x, (p-1)y)^{p^{\alpha_{p-1}}} = 1$$

*for some suitable set of indices $\alpha_i$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_p > 0$, all indices being minimal.*

*In case $\alpha_1 = \alpha_2 = \cdots = \alpha_p$, the set S implies the law L.*

The actual existence of such varieties is not certain until we have classes of groups which generate them. For the set $S$ with $\alpha_1 > \alpha_p$, and without the law $L$, a two-generator group which will generate the variety is easily constructed, as in [1], so each such variety is generated by its reduced-free group of rank two. For $\alpha_1 = \alpha_p$, or for $\alpha_1 > \alpha_p$ with $S$ and $L$, the existence of the variety is given by Theorem 3 below. With the existence settled, the lattice of such varieties can be described by the minimal indices in the basis laws, it is distributive, and the lattice of varieties of metabelian $p$-groups of class $< p$ can be included in it with the same description as in [1].

THEOREM 3. *Given a set of indices $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_p > 0$, $p \geqq 3$, there exists a metabelian p-group G of precise class p which has, as a basis for its laws, the set S of Theorem 2, together with the law*

$$(x, (p-1)y)^{p^{\alpha_{p-1}}} = 1,$$

*all indices being minimal. G is on 3 generators.*

PROOF. By construction; the recipe follows. Take $2(p-1)$ cyclic groups $\langle B_i \rangle$, $\langle C_i \rangle$, $1 \leqq i \leqq p-1$, of orders $p^{\alpha_{i+1}}$ for $1 \leqq i < p-1$, and $p^{\alpha_{p-1}}$ for $i = p-1$. Take $\binom{p}{2}$ cyclic groups $\langle Q_{ij} \rangle$ with $i \geqq 1$, $j \geqq 1$, $i+j \leqq p$, $\langle Q_{ij} \rangle$ of order $p^{\alpha_{i+j}}$ for $i+j < p$, $\langle Q_{ij} \rangle$ of order $p^{\alpha_{p-1}}$ for $i+j = p$. Take $((p-1)/2)^2$ cyclic groups $\langle M_{ij} \rangle$, $1 \leqq i \leqq (p-1)/2$, $1 \leqq j \leqq (p-1)/2$ of orders $p^{\alpha_{i+j+1}}$. Take the direct product of all these cyclic groups, and name it $G'$. Then extend $G'$ by the cyclic group $\langle a \rangle$ of order $p^{\alpha_3}$ by using the automorphism

$$B_i^a = B_i, \ C_i^a = C_i, \ M_{ij}^a = M_{ij}$$

$$Q_{ij}^a = Q_{ij} M_{ij}^{-2} \text{ unless either } i \text{ or } j \text{ is } > \frac{p-1}{2}, \text{ in which case } Q_{ij}^a = Q_{ij}.$$

Then extend the group $\langle a, G'\rangle$ by a cyclic group $\langle b\rangle$ of order $p^{\alpha_2}$ by using the automorphism

$$a^b = aB_1,\ B_i^b = B_iB_{i+1} \text{ for } 1 \leqq i < p-1,\ B_{p-1}^b = B_{p-1}$$

$$C_i^b = C_iM_i^{-1} \text{ for } i = 1, 2, \cdots \frac{p-1}{2},\ C_i^b = C_i \text{ for } i > \frac{p-1}{2}$$

$$M_{ij}^b = M_{ij}M_{i+1,j} \text{ for } i < \frac{p-1}{2},\ M_{ij}^b = M_{ij} \text{ for } i \geqq \frac{p-1}{2}$$

$$Q_{ij}^b = Q_{ij}Q_{i+1,j} \text{ for } i+j < p,\ Q_{ij}^b = Q_{ij} \text{ for } i+j = p.$$

Then extend the group $\langle a, b, G'\rangle$ by a cyclic group $\langle c\rangle$ of order $p^{\alpha_2}$ by using the automorphism

$$a^c = a\quad_1,\ b^c = bQ_{11},$$

$$B_i^c = B_iM_{i1} \text{ for } 1 \leqq i \leqq \frac{p-1}{2},\ B_i^c = B_i \text{ for } i > \frac{p-1}{2}$$

$$C_i^c = C_iC_{i+1} \text{ for } 1 \leqq i < p-1,\ C_{p-1}^c = C_{p-1}$$

$$M_{ij}^c = M_{ij}M_{i,j+1} \text{ for } 1 \leqq j < \frac{p-1}{2},\ M_{i,(p-1)/2}^c = M_{i,(p-1)/2}$$

$$Q_{ij}^c = Q_{ij}Q_{i,j+1} \text{ for } i+j < p,\ Q_{ij}^c = Q_{ij} \text{ for } i+j = p.$$

The resulting group is $\langle a, b, c, G'\rangle$, and in fact is $\langle a, b, c\rangle$, the derived group being the direct product named $G'$. If $\alpha_1 = \alpha_2$, this group will do for $G$; if $\alpha_1 > \alpha_2$, take the direct product of this group with a cyclic group $\langle z\rangle$ of order $p^{\alpha_1}$, and take $G$ as the subgroup $\langle az, b, c\rangle$. The verification that the group so constructed *does* have the required properties is straightforward.

## 3. The lattice of varieties of class $p+1$

Nearly all the inductive steps needed to prove that a certain set of laws forms a 'basis' are valid for class $(p+1)$ as well as for class $p$ or less, as can be seen from their proofs. We need, however, a supplement to Lemma 5 of [1], to allow the particular step from laws in class $\leqq p$ to laws in class $p+1$. This is provided by:

LEMMA 3.1. *Let the variety $\mathfrak{V}$ of metabelian p-groups contain the reduced-free group F. Then if $(x, (p-1)y)^{p^\lambda} = 1$ and $(x, (p-2)y, z)^{p^\mu} = 1$ are laws in $F/\gamma_{p+1}(F)$, so also are they laws in $F/\gamma_{p+2}(F)$.*

We note (by Theorem 1 and Lemma 1.1) that $(x, py)^{p^\mu} = 1$ implies $(x_1, x_2, \cdots x_{p+1})^{p^\mu} = 1$, so that the divisibility arguments of Lemma 5 of [1] will apply; Lemma 4 of [1] will allow induction on the number of variables in deriveable laws, as long as we provide the starting point by Lemma 3.2 below. (That Lemma 4 of

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[1] is valid for $c = p+1$ can be established by using the Stirling manipulation instead of the argument about the non-singularity of a certain matrix in its proof.) The starting point is:

LEMMA 3.2. *Let $\mathfrak{B}$ be a variety of metabelian p-groups of class $p+1$, with the law $(x, py)^{p^\mu} = 1$, $\mu$ minimal. If there is any law of the form*

$$\prod_{k=1}^{p} (x, ky, (p-k)x)^{\beta_k} = 1$$

*in $\mathfrak{B}$, then $(x, ky, (p-k)x)^{\beta_k} = 1$, $k = 1, 2, \cdots p$, are all laws in $\mathfrak{B}$, each $\beta_k$ is divisible by $p^\mu$, each of these laws is a consequence of $(x, py)^{p^\mu} = 1$, and further, any law of the form $(x, ky, (p-k)x)^n = 1$ implies $(x, py)^n = 1$.*

Lemma 4 of [1] then implies that all $(p+1)$-weight commutator laws in $\mathfrak{B}$ follow from $(x, py)^{p^\mu} = 1$.

Thus, by the same logic as in the proof of Theorem 3 of [1] and Theorem 2 of this, we have:

THEOREM 4. *Any variety of metabelian p-groups of class $p+1$ and finite exponent $p^{\alpha_1}$ has as a basis for its laws* either *the set $S$:*

$$S \begin{cases} ((x, y), (u, v)) = 1, \ (x_1, x_2, \cdots x_{p+2}) = 1, \ x^{p^{\alpha_1}} = 1 \\ (x, (i-1)y)^{p^{\alpha_i}} = 1, \ i = 2, 3, \cdots p-1, p+1 \\ (x, (p-2)y, z)^{p^{\alpha_p}} = 1 \text{ (equivalent to } (x_1, x_2, \cdots, x_p)^{\alpha_p} = 1) \\ (x, py)^{p^{\alpha_{p+1}}} = 1 \end{cases}$$

*or the set $S$ together with the law $L$:*

$$L: \qquad (x, (p-1)y)^{p^{\alpha_{p-1}}} = 1$$

*for some suitable set of indices $\alpha_i$, $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_{p+1} > 0$, all indices being minimal.*

*In the case that L is present, $\alpha_{p+1} \leqq \alpha_p - 1$, and in the case that $\alpha_1 = \alpha_2 = \cdots = \alpha_p$, L is present.*

Again the question of the existence of such varieties must be considered, but in this case the construction has already been done: for $S$ without $L$ (and hence $\alpha_p < \alpha_1$) the construction of a two-generator generating group as in [1] proceeds even with the restriction eased to $c \leqq p+1$ rather than $c < p$, and for $S$ with $L$ (including $\alpha_1 = \alpha_p$) the construction for Theorem 3 extends easily to a class $p+1$ group by adjoining two more cyclic groups $\langle B_p \rangle$ and $\langle C_p \rangle$ of orders $p^{\alpha_{p+1}}$ ($\alpha_{p+1} < \alpha_p$) in the direct product to give $G'$, and extending the automorphisms in the obvious manner. Thus:

THEOREM 5. *For each possible variety mentioned in Theorem 4, there is a single group which generates it; the group is on 2 or 3 generators, depending on the absence or presence of L.*

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Again, the lattice of such varieties is given simply by considering the minimal indices in the basis laws; it is distributive, and in it can be included the lattice of all varieties of finite-exponent metabelian $p$-groups of class $\leqq p+1$. The varieties are simply named as $[\alpha_1, \alpha_2, \cdots \alpha_p, q, \alpha_{p+1}]$, with $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_p \geqq q \geqq \alpha_{p+1} \geqq 0$, with $\alpha_p \geqq q \geqq \alpha_p - 1$, and the restrictions that $q = \alpha_p - 1$ whenever $\alpha_1 = \alpha_p$. If $\alpha_{p+1} = 0$, we interpret the variety as of class $p$; if $\alpha_{c+1} = \alpha_{c+2} = \cdots = \alpha_p = 0$, we interpret the variety as of class $c$:

$$[\alpha_1, \cdots, q, \alpha_{p+1}] \vee [\alpha_1', \cdots, q', \alpha_{p+1}'] = [\max(\alpha_1, \alpha_1'), \cdots, \max(q, q'), \max(\alpha_{p+1}, \alpha_{p+1}')]$$

$$[\alpha_1, \cdots, q, \alpha_{p+1}] \wedge [\alpha_1', \cdots, q', \alpha_{p+1}'] = [\min(\alpha_1, \alpha_1'), \cdots, \min(q, q'), \min(\alpha_{p+1}, \alpha_{p+1}')].$$

It remains to establish Lemmas 3.1 and 3.2.

PROOF OF 3.1. Take first the law $(x, (p-1)y)^{p^\lambda} = 1$ in $F/\gamma_{p+1}(F)$. If this is not a law in $F/\gamma_{p+2}(F)$, there must be some relation

$$(g_1, (p-1)g_2)^{p^\lambda} = \prod_n C_n$$

amongst a subset $\{g_1, g_2, \cdots\}$ of the free generators of $F$, where the $C_n$ are commutator words in $g_1, g_2, \cdots$, of weight at least $p+2$. This relation leads to a law in $F$, hence a law in $F/\gamma_{p+2}(F)$, by reading $x$ for $g_1$, $y$ for $g_2$, $z$ for $g_3, \cdots$. Setting every variable, except $x$ and $y$, equal to 1, we then have a law

$$(x, (p-1)y)^{p^\lambda} = \prod_{k=1}^{p} (x, y, (k-1)y, (p-k)x)^{\beta_k}$$

in $F/\gamma_{p+2}(F)$. (We note that then $(x, (p-1)y, z)^{p^\lambda} = 1$, so the exponent of $\gamma_{p+1}(F/\gamma_{p+2}(F))$ is at most $p^\lambda$.) By writing the law in the form $\prod_{k=1}^{p} W_k = 1$, where $W_{p-1} = (x, (p-1)y)^{-p^\lambda}(x, (p-1)y, x)^{\beta_{p-1}}$, but $W_k = (x, y, (k-1)y, (p-k)x)$ for $k \neq p-1$, and by applying the Stirling manipulation, we derive $W_p^p = 1$ and $W_{p-1}^{(p-1)!}\, W_p^S = 1$, where $S$ is a Stirling number divisible by $p$. Thus $W_{p-1} = 1$ is a law in $F/\gamma_{p+2}(F)$. Applying the Stirling manipulation again, (this time, on the number of $x$-entries), we have the required result. The result for the law $(x, (p-2)y, z)^{p^\mu} = 1$ is obtained similarly, but requires extra collection steps for the three variables involved.

PROOF OF 3.2. That each factor in the given law is identically 1 is a simple consequence of the Stirling manipulation, and the involvement of Stirling numbers divisible by $p$. This gives $p^\mu | \beta_k$ by the minimality of $\mu$. That 'each of these laws is a consequence of $(x, py)^{p^\mu} = 1$' is simply Lemma 1.2. That '$(x, ky, (p-k)x)^n = 1$ implies $(x, py)^n = 1$' is a simple application of the Stirling manipulation after substitution of $xy$ for $y$.

## References

[1] W. Brisley, 'On varieties of metabelian $p$-groups and their laws', *Journ. Aust. Math. Soc.* VII (part 1) (1967), 64–80.
[2] N. D. Gupta and M. F. Newman, 'On metabelian Groups', *J. Aust. Math. Soc.* 6 (1966), 362–368.
[3] H. Meier-Wunderli, 'Metabelsche Gruppen', *Comment. Math. Helvet.* 25 (1951), 1–10.
[4] W. Brisley and I. D. Macdonald, 'Two classes of metabelian $p$-groups', *Math. Z.* 112 (1969), 5–12.

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