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SIMPLICITY OF REDUCED AMALGAMATED PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. We give sufficient conditions for the simplicity of reduced amalgamated products of C^*-algebras. We show that in some situations a minimal projection in a unital C^*-algebra A is minimal in a free product $A *_{\mathbb{C}} B$. We show that in certain situations if a minimal projection in A were minimal in a particular reduced free product of A and B then the reduced free product would be a simple C^*-algebra which has finite and infinite projections.

1. **Introduction.** Let A and B be unital C^*-algebras which have a common unital C^*-subalgebra C with $1_A = 1_B = 1_C$. Let $A *_C B$ denote the amalgamated product of A and B over C [Br]. If ϕ and ψ are conditional expectations from A and B onto C then one can form the reduced amalgamated product of A and B over C relative to ϕ and ψ [Voi]. This product will be denoted $\pi_{\phi*\psi}(A *_C B)$.

If G and S are discrete groups with a common subgroup H and ϕ, ψ are the natural conditional expectations from $C^*(G)$ and $C^*(S)$ onto $C^*(H)$, then

$$\pi_{\phi*\psi}\Big(C^*(G) *_{C^*(H)} C^*(S)\Big) \cong C^*_{\text{red}}(G *_H S).$$

If H is the trivial group and G, S are nontrivial, then $C^*_{\text{red}}(G *_H S)$ is simple unless $G = S = \mathbb{Z}_2$ [PS]. One method of producing a simple C^*-algebra with a specified property is to find groups G and S such that $C^*(G *_{\{e\}} S)$ has the desired property and then try to show that $C^*_{\text{red}}(G *_{\{e\}} S)$ has the same property. Because of the correspondence between the representations of $C^*(G)$ and the unitary representations of G it is often easier to verify a given property for the full group C^*-algebra than for the reduced group C^*-algebra. For example, consider the case of the free group on two generators $\mathbb{F}_2 = \mathbb{Z} *_{\{e\}} \mathbb{Z}$. It is not hard to see that $C^*(\mathbb{F}_2)$ is projectionless ([Coh], see also [Ch2]) but considerably more difficult to see that $C^*_{\text{red}}(\mathbb{F}_2)$ is projectionless ([PV], see also [Cu1], [Con]). Similarly it is more difficult to compute the K-groups of $C^*_{\text{red}}(\mathbb{F}_n)$ ([PV], see also [Cu1]) than to compute the K-groups of $C^*(\mathbb{F}_n)$ [Cu5].

In the case of amalgamated products of C^*-algebras over the complex numbers $\mathbb{C}$ Avitzour showed that $\pi_{\phi*\psi}(A *_C B)$ is simple provided ϕ and ψ are faithful traces and A and B have unitaries satisfying particular properties related to the traces [Av, Proposition 3.1 and Corollary]. We show that along with these unitary assumptions it is enough to assume

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that ϕ and ψ are faithful states. We extend these results to give sufficient conditions for reduced amalgamated products (not necessarily over $\mathbb{C}$) to be simple.

We also discuss how the simplicity results may eventually be used to solve certain open problems concerning simple C^*-algebras. For example, in Section 2 it is shown that in many cases the minimal projections in A and B are minimal in $A *_C B$ and in certain reduced products $\pi_{\phi*\psi}(A *_C B)$. Examples will be given where the minimality in $\pi_{\phi*\psi}(A *_C B)$ of a minimal projection p in A would make $\pi_{\phi*\psi}(A *_C B)$ the first known example of a simple C^*-algebra which has both finite and infinite projections. It would also follow that $p\pi_{\phi*\psi}(A *_C B)p$ is a simple C^*-algebra which is finite but not stably finite.

2. **Minimal projections in free products.** Roughly speaking, the amalgamated product of two C^*-algebras A and B over C is the C^*-algebra generated by monomials with letters alternating between elements of A and B. The elements of C can be identified as elements of A or B and can be combined with the surrounding letters. For a precise definition see [Br]. We will use the following universal property of the amalgamated product as our definition. $A *_C B$ contains copies of A and B and for every pair of representations α and β of A and B into a C^*-algebra E which agree on C there is a unique homomorphism $\alpha * \beta$ of $A *_C B$ into E which restricts to α on A and β on B. The representation is given on the monomials mentioned above by applying α to the letters in A and β to the letters in B. If C is the complex numbers $\mathbb{C}$ then we call the amalgamated product the (unital) free product of A and B.

Given a full or reduced amalgamated product, $A *_C B$ or $\pi_{\phi*\psi}(A *_C B)$ respectively, one may ask if a projection which is minimal in A or B is minimal in $A *_C B$ or $\pi_{\phi*\psi}(A *_C B)$. The relation between the reduced case and certain open problems concerning simple C^*-algebras will be discussed in Section 4. For now we will consider the case of full products with $C = \mathbb{C}1$ where 1 is the common unit of A and B. It is not the case that any minimal projection in A is minimal in $A *_{\mathbb{C}} B$ for any B. To see this, let A be any projectionless unital C^*-algebra, let B be any unital C^*-algebra which is not projectionless. Then 1_A is minimal in A, 1_B is not minimal in B, and hence $p = 1_A = 1_B$ is not minimal in $A *_C B$. The following propositions and examples show that in many cases the minimal projections in A and B are minimal in $A *_C B$. We first consider free products of the form $M_n *_{\mathbb{C}} A$ where M_n denotes the n by n matrices with complex entries. $\mathbb{C}$ is embedded unitally into M_n and A. For any unital C^*-algebra E which unitally contains an isomorphic copy of M_n the map $x \otimes y \longmapsto xy$ is an isomorphism of $M_n \otimes M_n^c$ and E. M_n^c denotes the relative commutant of M_n. Since the matrix unit e_{11} corresponds to $e_{11} \otimes 1$ under the isomorphism $M_n *_{\mathbb{C}} A \cong M_n \otimes M_n^c$ it follows that the cutdown of $M_n *_{\mathbb{C}} A$ by e_{11} is isomorphic to M_n^c. Thus the minimality of the projection e_{11} in $M_n *_{\mathbb{C}} A$ is equivalent to the projectionlessness of M_n^c.

Part (1) of the following proposition is due to J. Cuntz and is proven in [McC1]. The proof is repeated here because the same ideas are used in the proof of (2).

PROPOSITION 2.1. *Let A be a unital C^*-algebra. Suppose that either (1) or (2) holds.*
(1) A is projectionless.

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(2) There is a unital $*$-homomorphism from A onto $\mathbb{C}$. If Q is a projection in A and for all $1 \leq k \leq n-1$, there is a projection $P_k \in A$ so that $I_n \otimes (Q^{(k)} \oplus 0^{(n-k)})$ is path connected to $I_n \otimes P_k^{(n)}$ by a path of projections in $M_{n^2}(A)$, then $Q = 0$. (Here I_n denotes the n by n identity matrix and $X^{(j)}$ denotes the direct sum of j copies of X.)

Then the relative commutant M_n^c of M_n in $M_n *_{\mathbb{C}} A$ is projectionless.

PROOF. Let $w_t \in \mathcal{U}(M_n \otimes M_n)$ be a continuous path of unitaries from $w_0 = I_n \otimes I_n$ to a unitary w_1 such that $w_1(x \otimes y)w_1^* = y \otimes x$ for all $x, y \in M_n$. Let

$$\phi_t: M_n *_{\mathbb{C}} A \longrightarrow M_n \otimes (M_n *_{\mathbb{C}} A)$$

be defined by

$$\phi_t(z) = w_t(I_n \otimes z)w_t^* \quad z \in M_n$$
$$\phi_t(a) = I_n \otimes a \quad a \in A.$$

Then $\phi_0(x) = I_n \otimes x$ for all $x \in M_n *_{\mathbb{C}} A$. Also, $\phi_1(x) \in M_n \otimes A$ for all $x \in M_n *_{\mathbb{C}} A$. If $x \in M_n^c$, then $\phi_1(x)$ commutes with $\phi_1(M_n) = M_n \otimes 1$ and so $\phi_1(x) \in I_n \otimes A$. Hence if $P \in \mathrm{Proj}(M_n^c)$, then $\phi_t(P)$ is a norm-continuous path of projections from $\phi_0(P) = I_n \otimes P$ to $\phi_1(P) = I_n \otimes Q$ for some $Q \in \mathrm{Proj}(A)$. If (1) holds, then $Q = 0$ or $Q = 1$ and so $P = 0$ or $P = 1$. Suppose (2) holds. Let $\omega: A \longrightarrow \mathbb{C}$ be a unital $*$-homomorphism. For $1 \leq k \leq n-1$ let

$$\alpha_k: M_n *_{\mathbb{C}} A \longrightarrow M_n \otimes A$$

be defined by

$$\alpha_k(z) = z \otimes 1_A \quad z \in M_n$$
$$\alpha_k(a) = a^{(k)} \oplus \omega(a)^{(n-k)} \quad a \in A.$$

Since $\omega(Q) \in \mathrm{Proj}(\mathbb{C}) = \{0, 1\}$, we can assume $\omega(Q) = 0$ by replacing P with $1 - P$ if necessary. Since $P \in M_n^c$, $\alpha_k(P)$ commutes with $\alpha_k(M_n) = M_n \otimes 1_A$ and so $\alpha_k(P) = I_n \otimes \bar{P}_k = \bar{P}_k^{(n)}$ for some $\bar{P}_k \in \mathrm{Proj}(A)$. Because $I_n \otimes P$ is path connected to $I_n \otimes Q$ in $M_n \otimes M_n *_{\mathbb{C}} A$ we have that $(\mathrm{Id}_{M_n} \otimes \alpha_k)(I_n \otimes P)$ is path connected to $(\mathrm{Id}_{M_n} \otimes \alpha_k)(I_n \otimes Q)$. In other words, $I_n \otimes \bar{P}_k^{(n)}$ is path connected to $I_n \otimes (Q^{(k)} \oplus 0^{(n-k)})$ in $M_{n^2}(A)$. By (2) $Q = 0$. Hence $P = 0$. ∎

EXAMPLE 2.2. Let U_n^{nc} be the universal C^*-algebra generated by elements u_{ij}, $1 \leq i, j \leq n$, subject to the relations making the matrix $[u_{ij}]$ a unitary matrix [Br]. U_n^{nc} is isomorphic to the relative commutant of M_n in the free product $M_n *_{\mathbb{C}} C(\mathbb{T})$ where $\mathbb{T}$ denotes the unit circle (see [McC1, Proposition 2.2] for a proof). Since $C(\mathbb{T})$ is projectionless, U_n^{nc} is projectionless by Proposition 2.1 (1). This was originally shown in [McC1].

EXAMPLE 2.3. Let G_n^{nc} be the universal C^*-algebra generated by a multiplicative identity and elements p_{ij}, $1 \leq i, j \leq n$, subject to the relations making the matrix $[p_{ij}]$ a projection [Br]. G_n^{nc} is isomorphic to the relative commutant of M_n in the free product $M_n *_{\mathbb{C}} \mathbb{C}^2$. We claim that $\mathbb{C}^2$ satisfies property (2) of Proposition 2.1. The existence of the homomorphism onto $\mathbb{C}$ is obvious. Suppose Q is a projection in $\mathbb{C}^2$ satisfying the hypothesis of the second statement in property (2). Letting $k = 1$ we have that in $K_0(\mathbb{C}^2)$, $n[Q]_0 = n^2[P]_0$ for some projection P in $\mathbb{C}^2$. Since the K_0-classes of projections in $\mathbb{C}^2$ correspond to $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ under the isomorphism $K_0(\mathbb{C}^2) \cong \mathbb{Z}^2$ we must have $[Q]_0 = (0,0)$ and thus $Q = (0,0)$. Hence (2) holds and so G_n^{nc} is projectionless.

In certain cases the K-theory of amalgamated products can be computed. If retractions from A and B onto C exist then one has an exact sequence relating the K-groups of $A *_C B$ to those of A, B, and C [Cu5]. In the case $B = M_n$ and $C = \mathbb{C}$ one has a similar exact sequence if A has a $*$-homomorphism onto $\mathbb{C}$ and in fact $M_n *_{\mathbb{C}} A$ is KK-equivalent to A [McC3]. The existence of a similar exact sequence for other free products including $M_{n_1} *_{\mathbb{C}} M_{n_2}$ has recently been proved in [McC4]. In the above cases it follows that if τ is a trace on $M_n *_{\mathbb{C}} A$ then

$$\tau_*\big(K_0(A *_{\mathbb{C}} B)\big) \subset (\tau|_A)_*\big(K_0(A)\big) + (\tau|_B)_*\big(K_0(B)\big) \tag{2.1}$$

(see [McC3]).Thus it would be desirable to know when traces on A and B can be extended to $A *_C B$. Avitzour showed that any pair of states ϕ, ψ on A, B can be extended to a state $\phi * \psi$ on $A *_{\mathbb{C}} B$ in such a way that if $\phi * \psi$ is a trace if and only if ϕ and ψ are traces [Av, Proposition 1.4]. The problem is that $\phi * \psi$ is almost never faithful on $A *_{\mathbb{C}} B$. Since the range of a faithful trace on the K_0-group of a C^*-algebra gives information on the minimality of projections it would be advantageous to know whether or not faithful states and traces on A and B could be extended to faithful states and traces on $A *_{\mathbb{C}} B$. We will consider this question in the case where A and B are finite dimensional and $C = \mathbb{C}$.

A C^*-algebra is called *residually finite dimensional* if it has a separating family of finite dimensional representations. We will need the following result of Exel and Loring.

PROPOSITION 2.4 [EL, THEOREM 3.2]. *Let A and B be unital C^*-algebas. Then $A *_{\mathbb{C}} B$ is residually finite dimensional if and only if A and B are residually finite dimensional.*

We remark that if A is a separable residually finite dimensional C^*-algebra then the family of separating finite dimensional representations can be taken to be a countable family.

PROPOSITION 2.5. *Let A, B be finite dimensional C^*-algebras. Let ϕ, ψ be faithful states on A, B respectively. There exists a faithful state τ on $A *_{\mathbb{C}} B$ extending both ϕ and ψ. If ϕ and ψ are traces, then τ can be taken to be a trace.*

PROOF. By the remarks preceeding the statement of the proposition there is a countable separating family $\{\pi_n\}$ of finite dimensional representations of $A *_{\mathbb{C}} B$. Fix n. We can assume that there is an integer $N(n)$ so that π_n is unital and has the form

$$\pi_n : A *_{\mathbb{C}} B \longrightarrow M_{N(n)}.$$

Let σ_n be the unique trace on $M_{N(n)}$. Let $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_s$) be the diagonal matrix units in a fixed system of matrix units for A (resp. B). Choose $0 < \epsilon < 1$ so that

$$\epsilon\sigma_n\big(\pi_n(x_i)\big) < \phi(x_i) \quad 1 \le i \le r$$
$$\epsilon\sigma_n\big(\pi_n(y_j)\big) < \psi(y_j) \quad 1 \le j \le s.$$

Let $\alpha: A \longrightarrow \mathbb{C}$ be the faithful state defined by

$$\alpha(a) = \frac{\phi(a) - \epsilon\sigma_n\big(\pi_n(a)\big)}{1-\epsilon}.$$

Let $\beta: B \longrightarrow \mathbb{C}$ be the faithful state defined by

$$\beta(b) = \frac{\psi(b) - \epsilon\sigma_n\big(\pi_n(b)\big)}{1-\epsilon}.$$

Let $\phi: A *_{\mathbb{C}} B \longrightarrow A \otimes B$ be defined by

$$\phi(a) = a \otimes 1_B \quad a \in A$$
$$\phi(b) = 1_A \otimes b \quad b \in B.$$

Let

$$\theta_n: A *_{\mathbb{C}} B \longrightarrow M_{N(n)} \oplus (A \otimes B)$$

be given by

$$\theta_n(x) = \pi_n(x) \oplus \phi(x) \quad x \in A *_{\mathbb{C}} B.$$

Then $\{\theta_n\}$ is separating for $A *_{\mathbb{C}} B$ since $\{\pi_n\}$ is separating for $A *_{\mathbb{C}} B$. Let τ_n be the state on $M_{N(n)} \oplus (A \otimes B)$ defined by

$$\tau_n(x \oplus y) = \epsilon\sigma_n(x) + (1-\epsilon)(\alpha \otimes \beta)(y).$$

τ_n is faithful since α and β are faithful (and hence $\alpha \otimes \beta$ is faithful), σ_n is faithful, and $0 < \epsilon < 1$. Let τ be the state on $A *_{\mathbb{C}} B$ defined by

$$\tau(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \tau_n\big(\theta_n(x)\big).$$

Since $\{\theta_n\}$ is separating and each τ_n is faithful it follows that τ is faithful. A simple computation shows that τ extends both ϕ and ψ. It is easy to see that the above construction gives a trace if ϕ and ψ are traces. ∎

EXAMPLE 2.6. Consider again the example G_n^{nc} of Example 2.3. Let ϕ be the unique trace on M_n and let ψ be the faithful trace on $\mathbb{C}^2$ with weights $\{\frac{1}{n}, 1-\frac{1}{n}\}$. In [McC1] it was shown that $K_0(M_n *_{\mathbb{C}} \mathbb{C}^2) \cong \mathbb{Z}^2$ with generators $[e_{11}]_0$ and $[e_1]_0$ where $e_1 = (1,0) \in \mathbb{C}^2$ and e_{11} is a minimal projection in M_n. Thus if τ is a faithful tracial extension of ϕ and ψ to $M_n *_{\mathbb{C}} \mathbb{C}^2$ we have by (2.1) that the range of

$$\tau_*: K_0(M_n *_{\mathbb{C}} \mathbb{C}^2) \longrightarrow \mathbb{R}$$

is $\frac{1}{n}\mathbb{Z}$. Since $\tau(e_{11}) = \frac{1}{n}$, this gives another proof of the minimality of e_{11}. This proof, however, also gives the additional information that e_1 is minimal since $\tau(e_1) = \frac{1}{n}$.

3. **Simplicity of reduced amalgamated products.** We will now describe the reduced amalgamated product of two C^*-algebras over a common subalgebra as defined by Voiculescu [Voi]. Let A and B be unital C^*-algebras containing a common subalgebra C with $1_A = 1_B = 1_C$. Let ϕ and ψ be conditional expectations of A and B onto C.

DEFINITION 3.1 [VOI]. The *reduced amalgamated product of A and B over C relative to ϕ and ψ* is a triple (E, Φ, π) satisfying the following conditions. E is a C^*-algebra with C as a subalgebra. Φ is a conditional expectation of E onto C. π is a $*$-homomorphism from $A *_C B$ into E. Properties (a) through (d) also hold.

(a) $1_E \in C$, $\pi(c) = c$ for all $c \in C$, π is onto.

(b) If $\phi(a_k) = \psi(b_k) = 0$ then

$$\Phi\big(\pi(a_1)\pi(b_1)\cdots\pi(a_n)\pi(b_n)\big) = 0$$

and similarly for words beginning in B and/or ending in A.

(c)

$$\begin{aligned}\Phi\big(\pi(a)\big) &= \phi(a) \quad \text{for } a \in A\\ \Phi\big(\pi(b)\big) &= \psi(b) \quad \text{for } b \in B\end{aligned}$$

(d) If $x \in E$ satisfies $\Phi(a^*x^*xa) = 0$ for all $a \in E$ then $x = 0$.

The reduced amalgamated product is unique and a construction showing its existence can be found in [Voi]. We will denote Φ by $\widetilde{\phi * \psi}$ and π by $\pi_{\phi*\psi}$. Thus $E = \pi_{\phi*\psi}(A *_C B)$. Let $\phi * \psi = (\widetilde{\phi * \psi}) \circ \pi_{\phi*\psi}$ so that $\phi * \psi(a) = \phi(a)$ and $\phi * \psi(b) = \psi(b)$ for $a \in A$ and $b \in B$.

If we let π_Φ denote the GNS representation associated with $\Phi = \widetilde{\phi * \psi}$ then it follows from (d) that π_Φ is faithful on $\pi_{\phi*\psi}(A *_C B)$. Thus since $\Phi \circ \pi_{\phi*\psi} = \phi * \psi$ we have

$$\begin{aligned}\|\pi_{\phi*\psi}(x)\|^2 &= \left\|\pi_\Phi\big(\pi_{\phi*\psi}(x)\big)\right\|^2\\ &= \sup\left\{\frac{\Phi\big(\pi_{\phi*\psi}(y^*)\pi_{\phi*\psi}(x^*x)\pi_{\phi*\psi}(y)\big)}{\Phi\big(\pi_{\phi*\psi}(y^*y)\big)} : \Phi\big(\pi_{\phi*\psi}(y^*y)\big) \neq 0\right\}\\ &= \sup\left\{\frac{\phi * \psi(y^*x^*xy)}{\phi * \psi(y^*y)} : \phi * \psi(y^*y) \neq 0\right\}.\end{aligned} \tag{3.1}$$

From this it easily follows that $\|\pi_{\phi*\psi}(a)\| \geq \|\pi_\phi(a)\|$ for $a \in A$ and $\|\pi_{\phi*\psi}(b)\| \geq \|\pi_\psi(b)\|$ for $b \in B$. Hence A (resp. B) is faithfully contained in $\pi_{\phi*\psi}(A *_C B)$ if π_ϕ (resp. π_ψ) is faithful. In fact, if ϕ and ψ are faithful, then the algebraic free product $A *_C^{\text{alg}} B$ is faithfully contained in $\pi_{\phi*\psi}(A *_C B)$ if $C = \mathbb{C}$ [Av, Proposition 2.3]. However, it is not enough to assume that π_ϕ and π_ψ are faithful [Av, Example 3.3].

In [Av] Avitzour proved the following lemma which lead to results concerning the simplicity and uniqueness of traces in certain reduced free products of C^*-algebras.

LEMMA 3.2 [AV, PROPOSITION 3.1]. *Let A and B be C^*-algebras and ϕ, ψ states on them. Let a be a unitary in the kernel of ϕ and b a unitary in the kernel of ψ so that ϕ, ψ are invariant with respect to conjugation by a, b respectively. Let c be a unitary in B such that $\psi(c) = \psi(b^*c) = 0$. Then for all $x \in \pi_{\phi*\psi}(A *_{\mathbb{C}} B)$*

$$\widetilde{\phi * \psi}(x) \cdot 1 \in \overline{\mathrm{co}}\big\{u^*xu : u \in \mathcal{U}\big(\pi_{\phi*\psi}(A *_{\mathbb{C}} B)\big)\big\},$$

where $\overline{\mathrm{co}}(S)$ denotes the closed convex hull of S and $\mathcal{U}(A)$ denotes the unitary group of the C^*-algebra A.

It follows from this lemma that if $\widetilde{\phi * \psi}$ is faithful on $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ then $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ is simple. This is because for every nonzero positive element x in a closed two-sided ideal J the invertible element $\widetilde{\phi * \psi}(x) \cdot 1$ is in J. The problem is that it is not known whether or not $\widetilde{\phi * \psi}$ is faithful even if ϕ and ψ are faithful. It is known that if ϕ and ψ are faithful traces then $\widetilde{\phi * \psi}$ is a faithful trace [Av]. The following proposition shows that $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ is simple (under the assumption of the existence of the unitaries as in Lemma 3.2) whether or not $\widetilde{\phi * \psi}$ is a faithful trace.

PROPOSITION 3.3. *Let A, B, ϕ, ψ, a, b, c be as in Lemma 3.2. Then $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ is simple.*

PROOF. It is enough to show that $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ is algebraically simple. Since $\pi_{\phi*\psi}$ is onto, every two-sided ideal in $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ is of the form $\pi_{\phi*\psi}(J)$ for some two-sided ideal J in $A *_{\mathbb{C}} B$. So it is enough to show that if J is a two-sided ideal in $A *_{\mathbb{C}} B$ then $\pi_{\phi*\psi}(J)$ is either $\pi_{\phi*\psi}(A *_{\mathbb{C}} B)$ or 0. Let J be a two-sided ideal in $A *_{\mathbb{C}} B$ and assume that $\pi_{\phi*\psi}(J) \neq \pi_{\phi*\psi}(A *_{\mathbb{C}} B)$. Let $x \in J$. By Lemma 3.2 it follows that $\phi * \psi(x) \cdot 1 \in \overline{\pi_{\phi*\psi}(J)}$. If $\phi * \psi(x) \neq 0$ then $\phi * \psi(x) \cdot 1$ is an invertible element in the closure of the two-sided ideal $\pi_{\phi*\psi}(J)$. Since the set of invertible elements is open, $\pi_{\phi*\psi}(J)$ would contain an invertible element which would be a contradiction. Thus $\phi * \psi(x) = 0$ holds for all $x \in J$. So if $x \in J$ and $y \in A *_{\mathbb{C}} B$, then $y^*x^*xy \in J$ and thus $\phi * \psi(y^*x^*xy) = 0$. So by property (d) of the definition of reduced amalgamated products it follows that $\pi_{\phi*\psi}(x) = 0$. Hence $\pi_{\phi*\psi}(J) = 0$. ∎

We will now consider the case of reduced amalgamated products over subalgebras other than the complex numbers. Our plan of attack in determining sufficient conditions for the simplicity of reduced amalgamated products will proceed in two steps. First we will determine several properties of $\phi * \psi(J)$ for a two-sided ideal J of $A *_C B$ so that we will know in certain cases when $\phi * \psi(J)$ (and hence $\pi_{\phi*\psi}(J)$) must be zero. Then we will try to prove an analog of Lemma 3.2 so that we can conculde that $\phi * \psi(J) \subset \overline{\pi_{\phi*\psi}(J)}$ in some of these cases. Notice however that this containment is not enough to conclude that $\phi * \psi(J) \neq 0$ implies $\pi_{\phi*\psi}(J) = \pi_{\phi*\psi}(A *_C B)$ if $C \neq \mathbb{C}$. This is because for $x \geq 0$, $x \neq 0$, if $\phi * \psi(x) \neq 0$ we only have that $\phi * \psi(x)$ is a nonzero element of C in the ideal generated by $\pi_{\phi*\psi}(x)$ which may not be invertible. We will need the following two technical lemmas which are related to [Av, Proposition 1.4]. Let $A_0 = \ker \phi$, $B_0 = \ker \psi$, and $(A *_C B)_0 = \ker(\phi * \psi)$. A word consisting of one or more letters alternating in A_0 and B_0 is called a *reduced word*.

LEMMA 3.4. *Let $a \in A$, $a_1, a_2 \in A_0$.*
(1) If $aA_0 \subset A_0$, then $a(A *_C B)_0 \subset (A *_C B)_0$.
(2) If $A_0 a \subset A_0$, then $(A *_C B)_0 a \subset (A *_C B)_0$.
(3) If $a_1 A_0 a_2 \subset A_0$, then $a_1(A *_C B)_0 a_2 \subset (A *_C B)_0$.
The analogous results hold for B.

PROOF. We will prove (3) first. It is enough to show that $a_1 x a_2 \in (A *_C B)_0$ for a reduced word x since each element of $(A *_C B)_0$ can be approximated by a sum of reduced words [Av].

CASE 1. If x begins and ends in B_0, then $a_1 x a_2$ is a reduced word and hence $\phi * \psi(a_1 x a_2) = 0$ by property (b) of Definition 3.1.

CASE 2. If x begins with A_0 and ends with B_0, let $x = a_0 y$ where $a_0 \in A_0$ and y is a reduced word beginning and ending in B_0. Write $a_1 a_0$ as $a_{00} + \phi * \psi(a_1 a_0)$ where $\phi * \psi(a_{00}) = 0$. So $a_{00} y a_2$ and $y a_2$ are reduced words. Thus

$$\begin{aligned} \phi * \psi(a_1 x a_2) &= \phi * \psi(a_1 a_0 y a_2) \\ &= \phi * \psi(a_{00} y a_2) + \phi * \psi\big(\phi * \psi(a_1 a_0) y a_2\big) \\ &= 0 + \phi * \psi(a_1 a_0)\phi * \psi(y a_2) = 0. \end{aligned}$$

CASE 3. If x begins in B_0 and ends in A_0 let $x = y a_0$ where y is a reduced word beginning and ending in B_0 and $a_0 \in A_0$ and proceed as in Case 2.

CASE 4. If x begins and ends with A_0 and is not of length one, let $x = a_3 y a_4$ where $a_3, a_4 \in A_0$ and y is a reduced word beginning and ending in B_0. Write $a_1 a_3$ as $a_{11} + \phi * \psi(a_1 a_3)$ where $a_{11} \in A_0$ and $a_4 a_2$ as $a_{22} + \phi * \psi(a_4 a_2)$ where $a_{22} \in A_0$. Then $a_{11} y a_{22}$, $y a_{22}$, $a_{11} y$, y are all reduced words and are in the kernel of $\phi * \psi$. Hence

$$\begin{aligned} \phi * \psi(a_1 x a_2) &= \phi * \psi(a_1 a_3 y a_4 a_2) \\ &= \phi * \psi\Big(\big(a_{11} + \phi * \psi(a_1 a_3)\big) y \big(a_{22} + \phi * \psi(a_4 a_2)\big)\Big) \\ &= \phi * \psi(a_{11} y a_{22}) + \phi * \psi(a_1 a_3)\phi * \psi(y a_{22}) \\ &\quad + \phi * \psi(a_{11} y)\phi * \psi(a_4 a_2) + \phi * \psi(a_1 a_3)\phi * \psi(y)\phi * \psi(a_4 a_2) = 0. \end{aligned}$$

CASE 5. If $x \in A_0$ then $a_1 x a_2 \in A_0 \subset (A *_C B)_0$.

This concludes the proof of (3). If we add the additional hypothesis to (1) that $a \in A_0$ then (1) could be proven with the same techniques as in (3) except that there would be fewer cases to consider. So assume (1) holds under the additional hypothesis that $a \in A_0$. Let $a \in A$ and suppose $aA_0 \subset A_0$. Then $a = a_0 + \phi * \psi(a)$ where $a_0 \in A_0$. We have $\phi * \psi(a) A_0 \subset A_0$ since $\phi * \psi(a) \in C$ and $CA_0 \subset A_0$. Thus $a_0 A_0 \subset A_0$ and so $a_0(A *_C B)_0 \subset (A *_C B)_0$. We also have $\phi * \psi(a)(A *_C B)_0 \subset (A *_C B)_0$ and hence it follows that $a(A *_C B)_0 \subset (A *_C B)_0$. The proof of (2) is similar to the proof of (1). ∎

PROPOSITION 3.5. *Let $a \in A$, $a_1, a_2 \in A_0$.*
(1) If $aA_0 \subset A_0$, $aC \subset C$, then $\phi * \psi(ax) = a\phi * \psi(x)$ for all $x \in A *_C B$.

(2) If $A_0 a \subset A_0$, $Ca \subset C$, then $\phi * \psi(xa) = \phi * \psi(x)a$ for all $x \in A *_C B$.

(3) If $a_1 A_0 a_2 \subset A_0$, $a_1 C a_2 \subset C$, then $\phi * \psi(a_1 x a_2) = a_1 \phi * \psi(x) a_2$ for all $x \in A *_C B$. The analogous results hold for B.

PROOF. Write $x \in A *_C B$ as $x_0 + \phi * \psi(x)$ where $\phi * \psi(x_0) = 0$. In case (1) we have $\phi * \psi(ax_0) = 0$ by Proposition 3.3. Since $\phi * \psi$ fixes C and $a\phi * \psi(x) \in aC \subset C$ it follows that $\phi * \psi\big(a\phi * \psi(x)\big) = a\phi * \psi(x)$. Thus (1) holds. Case (2) is similar. In case (3), $\phi * \psi(a_1 x_0 a_2) = 0$ by Proposition 3.3. Also $\phi * \psi\big(a_1 \phi * \psi(x) a_2\big) = a_1 \phi * \psi(x) a_2$ since $a_1 \phi * \psi(x) a_2 \in a_1 C a_2 \subset C$ and $\phi * \psi$ fixes C. Thus (3) holds. ∎

DEFINITION 3.6. Let S_i be the set of elements in A satisfying the hypothesis of (i) of Proposition 3.5 for $i = 1, 2$. Let S_3 be the set of pairs (a_1, a_2) of elements of A satisfying the hypothesis of (3) of Proposition 3.5. Let T_1, T_2, T_3 be defined analogously for B.

COROLLARY 3.7. *If J is a two-sided ideal in $A *_C B$, then $\phi * \psi(J)$ is a two-sided ideal in C which is invariant under left multiplication by S_1 and T_1, right multiplication by S_2 and T_2, and left-right multiplication by pairs in S_3 and T_3.*

PROOF. The fact that $\phi * \psi(J)$ is a two-sided ideal in C follows from the fact that $\phi * \psi$ is a conditional expectation onto C. The invariance follows immediately from Proposition 3.5. ∎

Next we will establish an analog of Lemma 3.2 for amalgamated products. A few technicalities occur in the amalgamated product case which do not occur in the free product case. First of all the condition $\phi(a^* x a) = \phi(x)$ for all $x \in A$ for a fixed unitary $a \in A_0$ is too restrictive. This condition on a unitary $a \in A_0$ will be replaced by the weaker condition that $a^* A_0 a \subset A_0$. In the case where $C = \mathbb{C}$ these two conditions are equivalent. This follows easily by observing that $x = x_0 + \phi(x) \cdot 1$ with $x_0 \in A_0$. In the case where $C \neq \mathbb{C}$ we do not know that $\phi\big(a^* \phi(x) a\big) = \phi(x)$ since $\phi(x)$ may not commute with a. The following example shows that the two notions are not equivalent. Let ϕ be the conditional expectation from M_2 onto $\mathbb{C}^2$ given by

$$\phi([y_{ij}]) = (y_{11}, y_{22}).$$

The unitaries in the kernel of ϕ are exactly the 2×2 matrices with zeros on the diagonal and scalars of modulus one off the diagonal. Conjugation of a matrix $[y_{ij}]$ by one of these results in a matrix in which y_{11} and y_{22} are switched. Thus there are no unitaries a in the kernel of ϕ for which $\phi(a^* y a) = \phi(y)$ holds for all $y \in M_2$. However $\phi(y) = 0$ if and only if y has zeros on the diagonal and this property is preserved after the diagonal entries are switched. Thus $a^* \ker \phi a \subset \ker \phi$ holds for all unitaries $a \in \ker \phi$.

LEMMA 3.8 [AV, PROPOSITION 3.1]. *Suppose there are unitaries $a \in A_0$, $b \in B_0$, such that $aA_0 a^* \subset A_0$ and $bB_0 b^* \subset B_0$. Suppose there is a unitary $c \in B_0$ so that $b^* c \in B_0$. Let $V_{kj} = \pi_{\phi*\psi}\big((ba)^k cac(ab)^j\big)$. If x is a reduced word of length less than j then for any integers $1 \leq m_1 < m_2 < \cdots < m_N$,*

$$\left\| \frac{1}{N} \sum_{k=1}^{N} V_{m_k j} \pi_{\phi*\psi}(x) V_{m_k j}^* \right\| \leq \frac{2\|x\|}{N^{\frac{1}{2}}}. \tag{3.2}$$

Consequently,

$$0 \in \overline{\mathrm{co}}\{u\pi_{\phi*\psi}(x)u^* : u \in \mathcal{U}(S)\} \quad \textit{for all } x \in (A *_C B)_0,$$

where S is the group generated by $\pi_{\phi*\psi}(a)$, $\pi_{\phi*\psi}(b)$, and $\pi_{\phi*\psi}(c)$.

PROOF. Avitzour proved this theorem for $C = \mathbb{C}$. He showed that the inequality above holds if $m_k = k$ for each k. It follows from his proof that any sequence $(m_1, \ldots, m_N)$ will do. In Avitzour's version he assumed the conjugation invariance of ϕ with respect to a and the conjugation invariance of ψ with respect to b. An inspection of his proof shows that the weaker conditions $aA_0a^* \subset A_0$ and $bB_0b^* \subset B_0$ suffice for the case of an arbitrary C. ■

REMARK. Avitzour showed that $\phi * \psi(x) \cdot 1$ is in the closed convex hull mentioned in the statement of the lemma for all x, not just in the case $x \in (A *_C B)_0$. This follows in the case $C = \mathbb{C}$ by writing $x = x_0 + \phi * \psi(x) \cdot 1$ with $\phi * \psi(x_0) = 0$ and using the fact that

$$\frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\big(\phi * \psi(x) \cdot 1\big)V_{m_k j}^* = \phi * \psi(x) \cdot 1.$$

This may not be true if $C \neq \mathbb{C}$ since $\phi * \psi(x)$ may not commute with the unitaries $V_{m_k j}$. This is the reason why a sequence $(m_1, \ldots, m_n)$ was used instead of $(1, \ldots, N)$. As we shall see later in an example it may be possible for $\phi * \psi(x)$ to commute with a sequence of V_{ij}'s but not all of them.

COROLLARY 3.9. *Suppose that for every $c_0 \in C$, $j \in \mathbb{N}$, there is a sequence of integers $0 < m_1 < m_2 < \cdots$ so that $\pi_{\phi*\psi}(c_0)$ commutes with $V_{m_k j}$ for all k. Then*

$$\phi * \psi(x) \in \overline{\mathrm{co}}\Big\{u^*\pi_{\phi*\psi}(x)u : u \in \mathcal{U}\big(\pi_{\phi*\psi}(A *_C B)\big)\Big\}.$$

PROOF. Let $x \in A *_C B$. Write $x = x_0 + \phi * \psi(x)$ for some $x_0 \in (A *_C B)_0$. We will identify $\pi_{\phi*\psi}\big(\phi * \psi(x)\big)$ with $\phi * \psi(x)$. Let $\epsilon > 0$. Since x_0 can be approximated by a sum of reduced words it follows from (3.2) that we can choose $j, N \in \mathbb{N}$ so that

$$\Big\|\frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\pi_{\phi*\psi}(x_0)V_{m_k j}^*\Big\| \leq \epsilon$$

for any sequence $m_1 < \cdots < m_N$. Choose the sequence m_j as in the hypothesis with respect to $c = \phi * \psi(x)$ and the chosen j. Then

$$\begin{aligned}
&\Big\|\phi * \psi(x) - \frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\pi_{\phi*\psi}(x)V_{m_k j}^*\Big\| \\
&\quad = \Big\|\phi * \psi(x) - \frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\pi_{\phi*\psi}(x_0)V_{m_k j}^* - \frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\phi * \psi(x)V_{m_k j}^*\Big\| \\
&\quad = \Big\|\phi * \psi(x) - \frac{1}{N}\sum_{k=1}^{N} V_{m_k j}\pi_{\phi*\psi}(x_0)V_{m_k j}^* - \phi * \psi(x)\Big\| \leq \epsilon.
\end{aligned}$$

■

PROPOSITION 3.10. *Let J be a two-sided ideal in $A *_C B$. Suppose A and B have unitaries as in Lemma 3.8 and the hypothesis on C in Corollary 3.9 is satisfied. Then $\phi * \psi(J) \subset \overline{\pi_{\phi*\psi}(J)}$. Moreover, if C contains no proper two-sided ideals invariant under left (resp. right) multiplication by elements of S_1 and T_1 (resp. S_2 and T_2) and left-right multiplication by pairs of elements of S_3 and T_3, then $\pi_{\phi*\psi}(A *_C B)$ is simple.*

PROOF. The containment $\phi * \psi(J) \subset \overline{\pi_{\phi*\psi}(J)}$ follows from Corollary 3.9. By Corollary 3.7 $\phi*\psi(J)$ must be either 0 or C. By (3.1) we have $\phi*\psi(J) = 0$ implies $\pi_{\phi*\psi}(J) = 0$. Since $1 \in C$ and $\phi * \psi(J) \subset \overline{\pi_{\phi*\psi}(J)}$, $\phi * \psi(J) = C$ implies $\pi_{\phi*\psi}(J) = \pi_{\phi*\psi}(A *_C B)$. Since every two-sided ideal in $\pi_{\phi*\psi}(A *_C B)$ is of the form $\pi_{\phi*\psi}(J)$ for some two-sided ideal J in $A *_C B$ it follows that $\pi_{\phi*\psi}(A *_C B)$ is simple. ∎

COROLLARY 3.11. *Suppose A and B have unitaries a, b, and c as in Lemma 3.8. Suppose C is simple and commutes with a, b, and c. Then $\pi_{\phi*\psi}(A *_C B)$ is simple.*

EXAMPLE 3.12. Consider the amalgamated product $M_{m+n} *_{\mathbb{C}^2} M_2$. A pair $(\alpha, \beta) \in \mathbb{C}^2$ is identified with the diagonal matrix in M_{m+n} having m copies of α and n copies of β and with the diagonal matrix in M_2 having diagonal entries α and β. It was shown in [McC2] that the relative commutant of M_{m+n} in this product is isomorphic to the C^*-algebra $U^{\mathrm{nc}}_{(m,n)}$. This C^*-algebra is the rectangular version of $U^{\mathrm{nc}}_n = U^{\mathrm{nc}}_{(n,n)}$ and a generalization of the Cuntz algebra $O_n = U^{\mathrm{nc}}_{(1,n)}$. Namely, $U^{\mathrm{nc}}_{(m,n)}$ is the universal C^*-algebra generated by elements u_{ij}, $1 \le i \le m$, $1 \le j \le n$, subject to the relations making the $m \times n$ matrix $[u_{ij}]$ a unitary matrix. Let ϕ be the conditional expectation from M_{m+n} onto $\mathbb{C}^2$ given by

$$\phi([x_{ij}]) = \Big(\frac{1}{m}\sum_{k=1}^{m} x_{kk}, \frac{1}{n}\sum_{l=m+1}^{m+n} x_{ll}\Big).$$

Let ψ be the conditional expectation from M_2 onto $\mathbb{C}^2$ given by

$$\psi([y_{ij}]) = (y_{11}, y_{22}).$$

Let $E = \pi_{\phi*\psi}(M_{m+n} *_{\mathbb{C}^2} M_2)$, with $m > 1$ and $n > 1$. We will show that E is simple. We do not consider the cases $m = 1$ and $n = 1$ since $M_{m+1} *_{\mathbb{C}^2} M_2 \cong M_{m+1} \otimes O_m$ and $M_{1+n} *_{\mathbb{C}^2} M_2 \cong M_{1+n} \otimes O_n$ [Br] which implies that E is simple in these cases. Let $\lambda_n = e^{\frac{2\pi i}{n}}$. Let u_n and v_n denote the following $n \times n$ unitary matrices.

$$u_n = \begin{bmatrix} 1 & & & \\ & \lambda_n & & \\ & & \ddots & \\ & & & \lambda_n^{n-1} \end{bmatrix} \quad v_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}.$$

Let $a = v_2$, $b = u_m \oplus u_n$, $c = v_m \oplus v_n$. The matrices a, b, and c are unitaries with a in the kernel of ψ and b, c, b^*c are in the kernel of ϕ. We have $\psi(a^*ya) = 0$ if $\psi(y) = 0$ for $y \in M_2$ and $\phi(b^*xb) = \phi(x)$ for $x \in M_{m+n}$. We now will show that the condition on C in Corollary 3.9 holds. Let $c_0 = (\alpha, \beta) \in \mathbb{C}^2$. Then b and c commute with c_0 under the identification of $\mathbb{C}^2$ with a sublagebra of M_{m+n}. Since $a^*c_0a = (\beta, \alpha)$ we do not have that c_0 commutes with a. However c_0 commutes with even powers of a. Since $V_{kj} = (ba)^k cac(ab)^j$ it follows that c_0 commutes with V_{kj} if $k+j$ is odd (since there are $k+j+1$ occurrences of a in V_{kj}). So for a fixed j we can choose $(m_1,\ldots,m_N)$ to be $(2,4,\ldots,2N)$ if j is odd and $(1,3,\ldots,2N+1)$ if j is even. Note that $a^*Ca \subset C$ and so $(a^*,a) \in T_3$. The only two proper ideals of $\mathbb{C}^2$ are $\mathbb{C}\oplus 0$ and $0\oplus\mathbb{C}$ and neither of these is invariant under conjugation by a. Thus there are no proper two sided ideals of $\mathbb{C}^2$ which are invariant under left-right multiplication by T_3. So by Proposition 3.10, $\pi_{\phi*\psi}(M_{m+n} *_{\mathbb{C}^2} M_2)$ is simple.

4. **Relations to open problems concerning simple C^*-algebras.** We now discuss the possibilities of certain simple reduced amalgamated products being the solutions to open problems in the theory of simple C^*-algebras. We begin by recalling some relevant properties of C^*-algebras. Two projections p and q in a C^*-algebra A are said to be (Murray-von Neumann) *equivalent* if there is an element $x \in A$ such that $x^*x = p$ and $xx^* = q$. A projection p is said to be *infinite* if it is equivalent to a proper subprojection of itself. Otherwise p is said to be *finite*. If all of the nonzero projections in a simple C^*-algebra A are infinite then A is said to be *purely infinite* [Cu2]. If all projections in A are finite (equivalently 1_A is finite if A is unital) then A is said to be *finite*. If $M_n(A)$ is finite for all n then A is said to be *stably finite*. If the invertible elements of A are dense in A then A is said to have *topological stable rank one* ($\mathrm{tsr}(A) = 1$) [Rf]. It is known that $\mathrm{tsr}(A) = 1$ implies that A is stably finite [Rf, Proposition 3.1, Theorem 3.3]. Obviously if A is stably finite then A is finite. The reverse implication is false in general. There is no known example of a simple C^*-algebra which is known to be neither of topological stable rank one nor purely infinite. Consequently there is no known example of a simple C^*-algebra which has both finite and infinite nonzero projections. It also follows that there is no known example of a simple finite C^*-algebra which is not stably finite. In the case of $\mathrm{tsr}(A) = 1$ for a simple C^*-algebra it is known that the map

$$\mathcal{U}(A)/\mathcal{U}_0(A) \longrightarrow K_1(A) \tag{4.1}$$

is an isomorphism [Rf, Corollary 4.10, Theorem 10.10]. In the above map $\mathcal{U}(A)$ denotes the unitary group of A and $\mathcal{U}_0(A)$ denotes the connected component of the identity in $\mathcal{U}(A)$. This is also known for purely infinite C^*-algebras [Cu2, Theorem 1.9]. It follows from this and the previous remarks that there is no known example of a simple C^*-algebra for which the above map is known not to be an isomorphism.

EXAMPLE 4.1. Consider the free product $M_n *_{\mathbb{C}} C(\mathbb{T})$ and the relative commutant $U_n^{\mathrm{nc}} \cong M_n^c$ discussed in Example 2.2. Let ϕ denote the unique trace on M_n and ψ denote the canonical faithful trace on $C(\mathbb{T})$ given by

$$\psi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,dt.$$

Equivalently, ψ is the trace on $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ given by $\psi(n) = \delta_{n0}$. Then by Proposition 3.3, $\pi_{\phi*\psi}\big(M_n *_{\mathbb{C}} C(\mathbb{T})\big)$ is simple. Let $U_{n,\mathrm{red}^{\mathrm{nc}}}$ denote the relative commutant of

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$\pi_{\phi*\psi}(M_n)$ in this reduced free product. Then since $M_n \otimes U_{n,\mathrm{red}^{\mathrm{nc}}} \cong \pi_{\phi*\psi}\big(M_n *_{\mathbb{C}} C(\mathbb{T})\big)$ it follows that $U_{n,\mathrm{red}^{\mathrm{nc}}}$ is simple. The map in (4.1) is not onto for $A = U_n^{\mathrm{nc}}$. To see this we will use the fact that $K_1\big(C(S^3)\big) \cong \mathbb{Z}$ with a generator v in $\mathcal{U}\Big(M_2\big(C(S^3)\big)\Big)$ [Cl]. The topological fact that every continuous function from S^3 into $\mathbb{T}$ is homotopic to a constant map [Sp, 7.2.12] can be restated by saying that the unitary group of $C(S^3)$ is path connected. Let

$$V = \begin{bmatrix} v & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathcal{U}\Big(M_n\big(C(S^3)\big)\Big).$$

Then the map which takes u_{ij} onto the (i,j)-th element of V induces a homomorphism $\sigma\colon U_n^{\mathrm{nc}} \longrightarrow C(S^3)$. Since $K_1(U_n^{\mathrm{nc}}) \cong \mathbb{Z}$ [Ph, Lemmas 4.1,4.3] with $[[u_{ij}]]_1$ as a generator, the map

$$\sigma_*\colon K_1(U_n^{\mathrm{nc}}) \longrightarrow K_1\big(C(S^3)\big)$$

is an isomorphism. If $w \in \mathcal{U}(U_n^{\mathrm{nc}})$, then $\sigma(w) \in \mathcal{U}(S^3)$ is connected to the identity by a path of unitaries. Thus $\sigma_*[w]_1 = 0$ and so $[w]_1 = 0$. Hence the map in (4.1) is the zero map. It was shown in [McC3] that the natural map

$$\pi_{\phi*\psi}\colon M_n *_{\mathbb{C}} C(\mathbb{T}) \longrightarrow \pi_{\phi*\psi}\big(M_n *_{\mathbb{C}} C(\mathbb{T})\big)$$

induces an isomorphism of K-theory from the full to the reduced free product. It follows from a more recent result of the author that $\pi_{\phi*\psi}$ is a KK-equivalence [McC4]. Hence $\pi_{\phi*\psi}$ induces a KK-equivalence of U_n^{nc} and $U_{n,\mathrm{red}^{\mathrm{nc}}}$. It would seem reasonable that the map in (4.1) is zero if A is the simple C^*-algebra $U_{n,\mathrm{red}^{\mathrm{nc}}}$.

EXAMPLE 4.2. Let $C^*(S)$ denote the universal C^*-algebra generated by a nonunitary isometry S [Co]. Since $S^*S = 1$ and $SS^* \neq 1$ it follows that 1 is equivalent to a proper subprojection and hence 1 is an infinite projection in $C^*(S)$. Consider the free product $A = M_n *_{\mathbb{C}} \big(C^*(S) \oplus C^*(S)\big)$ for $n > 1$. Let ψ denote the unique trace on M_n. Since $C^*(S)$ is separable it has a faithful state ϕ_0. Let ϕ denote the faithful state on $C^*(S) \oplus C^*(S)$ given by $\phi(a \oplus b) = \frac{1}{2}\big(\phi_0(a) + \phi_0(b)\big)$. Unitaries $b, c \in M_n$ as in Lemma 2.2 can be found and $a \in \mathcal{U}\big(C^*(S) \oplus C^*(S)\big)$ can be taken to be $1 \oplus -1$. Thus the reduced free product $\pi_{\phi*\psi}(A)$ is simple by Proposition 3.3. Since $C^*(S) \oplus C^*(S)$ is faithfully contained in $\pi_{\phi*\psi}(A)$ by the remarks following Definition 3.1 it follows that 1 is an infinite projection in $\pi_{\phi*\psi}(A)$. If $\pi_{\phi*\psi}(e_{11})$ were minimal in $\pi_{\phi*\psi}(A)$, then $\pi_{\phi*\psi}(A)$ would be a simple C^*-algebra having finite and infinite projections. If this were the case then since $\pi_{\phi*\psi}(M_n)^c$ is isomorphic to the cutdown of $\pi_{\phi*\psi}(A)$ by $\pi_{\phi*\psi}(e_{11})$ it would follow that $\pi_{\phi*\psi}(M_n)^c$ is projectionless and hence finite. However $M_n \otimes \pi_{\phi*\psi}(M_n)^c \cong \pi_{\phi*\psi}(A)$ is infinite as was pointed out above. Thus $\pi_{\phi*\psi}(M_n)^c$ is not stably finite. So $\pi_{\phi*\psi}(M_n)^c$ would be an example of a simple finite C^*-algebra which is not stably finite.

EXAMPLE 4.3. Consider the product $M_{m+n} *_{\mathbb{C}^2} M_2$ of Example 3.12 with $m < n$. Let $U_{(m,n),\mathrm{red}}^{\mathrm{nc}}$ denote the relative commutant of $\pi_{\phi*\psi}(M_{m+n})$ in $\pi_{\phi*\psi}(M_{m+n} *_{\mathbb{C}^2} M_2)$. Then

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$U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ is simple because $M_{m+n} \otimes U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ is isomorphic to the simple C^*-algebra $\pi_{\phi*\psi}(M_{m+n} *_{\mathbb{C}^2} M_2)$. The $m \times m$ matrix $x = [u_{ij}]$, $1 \leq i, j \leq m$, satisfies $x^*x = I_m$ and $xx^* \neq I_m$ since $m < n$. Thus $M_m(U^{\mathrm{nc}}_{(m,n),\mathrm{red}})$ is not finite and hence $U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ is not stably finite. So if $\pi_{\phi*\psi}(e_{11}) \in \pi_{\phi*\psi}(M_{m+n})$ were minimal, then $\pi_{\phi*\psi}(M_{m+n})^c \cong U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ would be projectionless. Thus $U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ would be simple, finite, and not stably finite. Also $M_m \otimes U^{\mathrm{nc}}_{(m,n),\mathrm{red}}$ would be simple with a finite projection $e_{11} \otimes 1$ and an infinite projection $I_m \otimes 1$.

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