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AN EXTENSION OF A THEOREM OF JANKO ON FINITE GROUPS WITH NILPOTENT MAXIMAL SUBGROUPS

JOHN W. RANDOLPH

Throughout this paper G will denote a finite group containing a nilpotent maximal subgroup S and P will denote the Sylow 2-subgroup of S. The largest subgroup of S normal in G will be designated by core(S) and the largest solvable normal subgroup of G by rad(G). All other notation is standard.

Thompson [**6**] has shown that if $P = 1$ then G is solvable. Janko [**3**] then observed that G is solvable if P is abelian, a condition subsequently weakened by him [**4**] to the assumption that the class of P is $\leqq 2$. Our purpose is to demonstrate the sufficiency of a still weaker assumption about P.

THEOREM. *Let G be a finite group containing a nilpotent maximal subgroup S, and P denote the Sylow 2-subgroup of S. Suppose that any subgroup H of P satisfying the following three conditions is a normal subgroup of P.*

(1) *$H = P \cap Q$ for some Sylow 2-subgroup Q of G.*

(2) *$N_P(P \cap Q)$ and $N_Q(P \cap Q)$ are Sylow 2-subgroups of $N_G(P \cap Q)$.*

(3) *$C_P(P \cap Q) \subseteq P \cap Q$.*

Then G is solvable.

Proof. Assume the theorem is false. Let G be a minimal counter-example to the theorem, and distinguish two cases.

Case 1. rad$(G) = 1$. Then G has a non-abelian minimal normal subgroup and core$(S) = 1$. By [**5**], P is a non-abelian Sylow 2-subgroup of G and $S = P$.

Suppose a and b are two elements of P conjugate in G, with $a \neq 1$. Following [**1**, p. 239], we let F_c' be the family of all pairs $(H, N_G(H))$ where H is a subgroup of P such that there exists a Sylow 2-subgroup Q of G with $H = P \cap Q$ a tame intersection and with $C_P(H) \subseteq H$. By Theorem 5.2 of [**1**], F_c' is a weak conjugation family [**1**, p. 237]. Hence there exist elements $(H_i, N_G(H_i))$, $1 \leqq i \leqq n$, of F_c' and elements $x_1, \ldots, x_n, y$ of G such that $b = a^{x_1 \cdots x_n y}$, $x_i \in N_G(H_i)$, $1 \leqq i \leqq n$, $y \in N_G(P)$, $a \in H_1$ and $a^{x_1 \cdots x_i} \in H_{i+1}$, $1 \leqq i \leqq n - 1$.

Since P is a non-trivial maximal subgroup of G and since rad$(G) = 1$, then $P = N_G(P)$ and $y \in P$. Moreover, by hypothesis, $P \subseteq N_G(H_i)$ for each i, and since each $H_i \neq 1$ it follows that $P = N_G(H_i)$. Thus $x_i \in P$, $1 \leqq i \leqq n$, and a and b are conjugate in P.

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Now by a theorem of Frobenius ([**2**], p. 115), G has a normal 2-complement, say K. By the Feit-Thompson theorem, K, and hence G, is solvable, which is a contradiction.

Case 2. $\text{rad}(G) \neq 1$. Let $\text{rad}(G) = N$, and for $X \subseteq G$, let X^* denote the image of X in G/N. Since G is non-solvable then $N \subseteq S$, and S^* is nilpotent. A contradiction will arise (the solvability of G^* and hence of G) if we can show that the Sylow 2-subgroup $P^* = PN/N$ of S^* satisfies the hypothesis of the theorem.

Let Q be any Sylow 2-subgroup of G and suppose $N_{P^*}(P^* \cap Q^*)$ and $N_{Q^*}(P^* \cap Q^*)$ are Sylow 2-subgroups of $N_{G^*}(P^* \cap Q^*)$, and

$$C_{P^*}(P^* \cap Q^*) \subseteq P^* \cap Q^*.$$

Let $x^* \in P^* \cap Q^*$ so that $x^* = pN = qN$ for some $p \in P$ and some $q \in Q$. Thus $q = pr$ for some $r \in N$. Since P and N are subgroups of S, then $q \in S$. But q is a 2-element, so $q \in P$, the unique Sylow 2-subgroup of S. Hence $x^* \in (P \cap Q)^*$ and $P^* \cap Q^* = (P \cap Q)^*$. Therefore $N_{P^*}((P \cap Q)^*)$ and $N_{Q^*}((P \cap Q)^*)$ are Sylow 2-subgroups of $N_{G^*}((P \cap Q)^*)$, and

$$C_{P^*}((P \cap Q)^*) \subseteq (P \cap Q)^*.$$

Since $C_P(P \cap Q)^* \subseteq C_{P^*}((P \cap Q)^*)$, it follows that $C_P(P \cap Q) \subseteq (P \cap Q)N$.

Note that $P \cap Q = T$, the Sylow 2-subgroup of $(P \cap Q)N$, since $T \subseteq P$, as P is the Sylow 2-subgroup of PN, and since $T \subseteq Q$, as Q is a Sylow 2-subgroup of QN, and N normalizes T. But $C_P(P \cap Q) \subseteq (P \cap Q)N$ so $C_P(P \cap Q) \subseteq P \cap Q$. Moreover,

$$N_{P^*}((P \cap Q)^*) = N_P(P \cap Q)^*,$$
$$N_{Q^*}((P \cap Q)^*) = N_Q(P \cap Q)^*$$

and

$$N_{G^*}((P \cap Q)^*) = N_G(P \cap Q)^*.$$

For example, let $gN \in N_{G^*}((P \cap Q)^*)$ and $p \in P \cap Q$. Then $gpg^{-1} = \hat{p}r$ for some $\hat{p} \in P \cap Q$ and some $r \in N$. Therefore $gpg^{-1} \in P \cap Q$, the Sylow 2-subgroup of $(P \cap Q)N$, and $g \in N_G(P \cap Q)$. Hence

$$N_{G^*}((P \cap Q)^*) \subseteq N_G(P \cap Q)^*,$$

and since clearly $N_G(P \cap Q)^* \subseteq N_{G^*}((P \cap Q)^*)$, we conclude that $N_G(P \cap Q)^* = N_{G^*}((P \cap Q)^*)$.

Thus $N_P(P \cap Q)^*$ and $N_Q(P \cap Q)^*$ are Sylow 2-subgroups of $N_G(P \cap Q)^*$. We now observe that $N_P(P \cap Q)$ and $N_Q(P \cap Q)$ are Sylow 2-subgroups of $N_G(P \cap Q)$. For $N_P(P \cap Q) \subseteq R$ for some Sylow 2-subgroup R of $N_G(P \cap Q)$. Since $N_P(P \cap Q)^*$ is a Sylow 2-subgroup of $N_G(P \cap Q)^*$, $N_P(P \cap Q)^* = R^*$. It will follow that $N_P(P \cap Q)$ is a Sylow 2-subgroup of $N_G(P \cap Q)$ if we can show that $N_P(P \cap Q) \cap N = R \cap N$. Of course $N_P(P \cap Q) \cap N \subseteq R \cap N$. If $x \in R \cap N$ then x normalizes $P \cap Q$ since

$R \subseteq N_G(P \cap Q)$. But also x is a 2-element in S since $N \subset S$, so $x \in P$. Thus $x \in N_P(P \cap Q)$ and $N_P(P \cap Q)$ is a Sylow 2-subgroup of $N_G(P \cap Q)$. In the same way, $N_Q(P \cap Q)$ is a Sylow 2-subgroup of $N_G(P \cap Q)$.

Finally, $P \cap Q$ is normal in P by hypothesis, so $P^* \cap Q^*$ is normal in P^* as required. This completes the proof of the theorem.

COROLLARY 1. *If any subgroup H of P such that $C_P(H) \subseteq H$ is a normal subgroup of P, then G is solvable.*

COROLLARY 2. [**4**]. *If* $\mathrm{cl}(P) \leqq 2$, *then G is solvable.*

Proof. Suppose H is a subgroup of P such that $C_P(H) \subseteq H$. Then H contains the centre $Z(P)$ of P and, since $\mathrm{cl}(P) \leqq 2$, $H/Z(P)$ is normal in $P/Z(P)$. Thus H is normal in P and Corollary 1 applies.

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West Virginia University,
Morgantown, West Virginia