A NOTE ON STRONGLY REGULAR FUNCTION ALGEBRAS

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Let A be a uniformly closed subalgebra of $C(X)$, the algebra of all complex-valued continuous functions on a compact Hausdorff space X. If A separates the points of X and contains the constant functions, A is called a function algebra. The algebra A is said to be strongly regular on X if it has the following property.

Property. For each f in A, each point x in X, and every $\epsilon > 0$, there is a neighbourhood U of x and a function g in A with $g(y) = f(x)$ for all y in U and $|f(y) - g(y)| < \epsilon$ for all y in X.

That is, each function in A is uniformly approximable on X by functions in A which are constant near any point of X. Stated in terms of ideals, strong regularity means that, for each x, the ideal of functions vanishing in a neighbourhood of x is uniformly dense in the maximal ideal at x.

In this note we prove the following theorem.

THEOREM. *If A is a strongly regular function algebra on the unit interval* $[0, 1]$, *then* $A = C([0, 1])$.

In the proof we will need the notion of *Jensen measure,* namely, a probability measure μ on X which satisfies Jensen's inequality:

$$\log|f(m)| \leqq \int_X \log|f|\, d\mu, \qquad f \in A,$$

where m denotes a point in the maximal ideal space M_A of A. In (**1**) Bishop showed that each point in M_A has a Jensen measure on X, and it is easy to see that any such measure is a representing measure, i.e.

$$\int f\, d\mu = f(m), \qquad f \in A.$$

LEMMA. *If A is a strongly regular function algebra on X, then $M_A = X$.*

Proof. Let $m \in M_A$ and let μ be a Jensen measure for m on X. Let S denote the closed support of μ. Suppose that $x \in S$, $x \neq m$. Let $f \in A$ with $f(m) = 1$ and $f(x) = 0$. Choose $g \in A$ satisfying $g(y) = 0$ if $y \in U$, a neighbourhood of x, and $|g(y) - f(y)| < \epsilon$ for all $y \in X$. Then

$$|g(m) - f(m)| = |\int g\, dm - \int f\, dm| \leqq \int |g - f|\, dm < \epsilon$$

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so that $|g(m)| > 1 - \epsilon > 0$ (assume that $\epsilon < 1$). Hence,

$$-\infty < \log|g(m)| \leqq \int \log|g| \, dm = -\infty,$$

which is impossible. The last equality follows from the fact that g is identically zero in a neighbourhood of a point in the closed support of m. It follows that $S = \{m\}$ and $m \in X$.

COROLLARY 1. *If A is strongly regular on X, then A is normal on X.*

COROLLARY 2. *If A is strongly regular on X, then A is a local algebra on X, i.e. if f is a function in $C(X)$ which agrees locally with functions in A, then f must already be in A.*

The proofs of both corollaries follow immediately from the lemma and well-known facts about regular function algebras.

For terminology and pertinent information about peak points, peak sets, and orthogonal measures used in the proof of the theorem, the reader is referred to the expository paper of Wermer (**4**) and the richly informative paper of Glicksberg (**2**). For related work of the author on function algebras on the interval see (**5**).

Proof of the Theorem. Since the Šilov boundary of A coincides with $[0, 1]$, the set of peak points of A is dense in $[0, 1]$. Let x be a peak point of A distinct from 0 and 1, and let f be a function in A peaking at x, $f(x) = 1$, $|f(y)| < 1$, $y \neq x$. Let $\{(a_n, b_n)\}$ be a sequence of open intervals about x and $\{f_n\}$ a sequence of functions in A such that $f_n \equiv 1$ on (a_n, b_n) and $f_n \rightarrow f$ uniformly on X. Define a sequence of functions $\{g_n\}$ by

$$g_n(t) = \begin{cases} 1, & t > a_n, \\ f_n(t), & t < b_n. \end{cases}$$

Then, by Corollary 2, $\{g_n\} \subset A$ and $g_n \rightarrow g$ uniformly, where

$$g(t) = \begin{cases} 1, & t \geqq x, \\ f(t), & t \leqq x. \end{cases}$$

Hence, $[x, 1]$ is a peak set of A. Similarly, $[0, x]$ is a peak set of A. Since finite unions and countable intersections of peak sets are peak sets, it follows that every closed subset of $[0, 1]$ is a peak set. Now, if μ is an extreme point of the unit ball in the set of measures orthogonal to A, and F is a proper closed subset of the support of μ, then both $\mu|_F$ and $\mu - \mu|_F$ are orthogonal to A. If $\alpha = ||\mu|_F|| \neq 0$, then

$$||\mu - \mu|_F|| = 1 - \alpha \quad \text{and} \quad \mu = \alpha \frac{\mu|_F}{\alpha} + (1 - \alpha) \frac{\mu - \mu|_F}{1 - \alpha}.$$

Since μ is extreme, $\alpha = 0$. Hence, for each such closed set F, $||\mu|_F|| = 0$ and $\mu \equiv 0$. From standard methods in functional analysis we have $A = C([0, 1])$.

It has been conjectured that any strongly regular function algebra must coincide with $C(X)$. The most likely candidate for a counterexample would seem to be the normal algebra constructed by McKissick in (**3**). However, Wermer in a seminar at Brown University in 1967 sketched a proof showing that the McKissick algebra is not strongly regular. It would be interesting to see if a clever modification of the McKissick construction could provide a counterexample.

References

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