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SUBEXPONENTIAL TAILS OF DISCOUNTED AGGREGATE CLAIMS IN A TIME-DEPENDENT RENEWAL RISK MODEL

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Abstract

Consider a continuous-time renewal risk model with a constant force of interest. We assume that claim sizes and interarrival times correspondingly form a sequence of independent and identically distributed random pairs and that each pair obeys a dependence structure described via the conditional tail probability of a claim size given the interarrival time before the claim. We focus on determining the impact of this dependence structure on the asymptotic tail probability of discounted aggregate claims. Assuming that the claim size distribution is subexponential, we derive an exact locally uniform asymptotic formula, which quantitatively captures the impact of the dependence structure. When the claim size distribution is extended regularly varying tailed, we show that this asymptotic formula is globally uniform.

Keywords: Asymptotics; dependence; discounted aggregate claim; extended regular variation; subexponentiality; uniformity

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1. Introduction

The renewal risk model has played a fundamental role in classical and modern risk theory since it was introduced by Sparre Andersen in the middle of the last century as a natural generalization of the compound Poisson risk model. In the standard framework of the renewal risk model, both claim sizes X_k, $k = 1, 2, \dots$, and interarrival times θ_k, $k = 1, 2, \dots$, form a sequence of independent and identically distributed (i.i.d.) random variables, the two sequences are also mutually independent. Denote by X and θ the generic random variables of the claim sizes and interarrival times, respectively. Being equipped with other modeling factors (such as initial surplus and premium incomes) and incorporated with some economic factors (such as interests, dividends, taxes, and returns on investments), this model provides a good mechanism for describing nonlife insurance business.

It should be noted that these independence assumptions are not made for practical relevance but mainly for mathematical tractability. Among these assumptions, the one in which complete independence between the claim size X and the interarrival time θ is assumed is especially

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unrealistic in almost all kinds of insurance. Consider the situation where the deductible retained by insureds is raised, then the interarrival time will increase because small claims will be ruled out, while the likelihood of a large claim will increase if X is new worse than used or decrease if X is new better than used. However, it is usually challenging to obtain results without such independence assumptions because most available methods will fail to work.

During the last few years things have started to change. In particular, in ruin theory, various nonstandard extensions to the renewal risk model have been proposed to appropriately relax these independence assumptions. Initiated in Albrecher and Teugels (2006), there have been a series of papers devoted to ruin-related problems of an extended renewal risk model in which (X_k, θ_k), $k = 1, 2, \dots$, are assumed to be i.i.d. copies of a generic pair (X, θ) with dependent components X and θ. An advantage of this risk model is that independence between the increments of the surplus process over claim arrival times is preserved. Albrecher and Teugels (2006) considered the case in which (X, θ) follows an arbitrary dependence structure, through a copula, and they derived explicit exponential estimates for finite-time and infinite-time ruin probabilities in the case of light-tailed claim sizes. Boudreault *et al.* (2006) proposed a dependence structure in which X conditional on θ has a density function equal to a mixture of two arbitrary density functions, and they studied the Gerber–Shiu expected discounted penalty function and measured the impact of the dependence structure on the ruin probability via the comparison of Lundberg coefficients. Cossette *et al.* (2008) assumed a generalized Farlie–Gumbel–Morgenstern copula to describe the dependence structure of (X, θ), and they derived the Laplace transform of the Gerber–Shiu function. Badescu *et al.* (2009) assumed that (X, θ) follows a bivariate phase-type distribution, and they employed the existing connection between risk processes and fluid flows to the analysis of various ruin-related quantities. Recently, Asimit and Badescu (2010) introduced a general dependence structure for (X, θ), via the conditional tail probability of X given θ, and they studied the tail behavior of discounted aggregate claims in the compound Poisson risk model in the presence of a constant force of interest and heavy-tailed claim sizes.

Other nonstandard extensions to the renewal risk model, not within the abovementioned framework, can also be found in Asmussen *et al.* (1999), Albrecher and Boxma (2004), (2005), and Biard *et al.* (2008), among others.

In this paper we use the same dependence structure as proposed in Asimit and Badescu (2010) for the generic random pair (X, θ). That is, we assume that the claim size X and the interarrival time θ fulfill the relation

$$\Pr(X > x \mid \theta = t) \sim \Pr(X > x)h(t), \qquad t \geq 0, \tag{1.1}$$

for some measurable function $h(\cdot)\colon [0, \infty) \longmapsto (0, \infty)$, where '$\sim$' means that the quotient of both sides tends to 1 as $x \to \infty$. When t is not a possible value of θ, that is, $\Pr(\theta \in \Delta) = 0$ for some open interval Δ containing t, the conditional probability in (1.1) is simply understood as unconditional and, therefore, $h(t) = 1$. In fact, in our main results below, whenever $h(t)$ appears, it is multiplied by $\Pr(\theta \in \mathrm{d}t)$. Hence, the function $h(t)$ at such a point t can be assigned any positive value without affecting our final results. As discussed in Asimit and Badescu (2010) (see also Section 3 below), relation (1.1) defines a general dependence structure which is easily verifiable for some commonly used bivariate copulas, and allows both positive and negative dependencies. It is also very convenient when dealing with the tail behavior of the sum or product of two dependent random variables. For instance, consider the discounted value $X\mathrm{e}^{-r\theta}$ with $r \geq 0$. If relation (1.1) holds uniformly for $t \in [0, \infty)$, which is often the case in concrete examples, then integrating both sides with respect to $\Pr(\theta \in \mathrm{d}t)$ leads to $\mathrm{E}\, h(\theta) = 1$.

Hence, by conditioning on θ we obtain

$$\Pr(X\mathrm{e}^{-r\theta} > x) \sim \int_{0-}^{\infty} \Pr(X > x\mathrm{e}^{rt})h(t)\Pr(\theta \in \mathrm{d}t) = \Pr(X\mathrm{e}^{-r\theta^*} > x), \tag{1.2}$$

where θ^* is a random variable, independent of X, with a proper distribution given by

$$\Pr(\theta^* \in \mathrm{d}t) = h(t)\Pr(\theta \in \mathrm{d}t).$$

The analysis in relation (1.2) shows that the dependence structure defined by (1.1) can be easily dissolved and its impact on the tail behavior of quantities under consideration can be easily captured.

Consider the renewal risk model in which (X_k, θ_k), $k = 1, 2, \dots$, are i.i.d. copies of a generic pair (X, θ) fulfilling the dependence structure defined by (1.1). In this paper, assuming a constant force of interest $r \geq 0$ and heavy-tailed claim sizes, we study the tail behavior of discounted aggregate claims and derive exact asymptotic formulae. We establish local uniformity for the obtained asymptotic formulae for the subexponential case and global uniformity for the extended regularly varying case. More importantly, in comparison with the corresponding existing results of Tang (2007) and Hao and Tang (2008) in the case of independent X and θ, our formulae successfully capture the impact of the dependence structure of (X, θ).

The asymptotic behavior of the finite-time and infinite-time ruin probabilities of the renewal risk model of standard structure with a constant force of interest $r > 0$ has been extensively investigated in the literature. The reader is referred to Asmussen (1998), Klüppelberg and Stadtmüller (1998), Kalashnikov and Konstantinides (2000), Konstantinides *et al.* (2002), Tang (2005), and Wang (2008). It is worthwhile noting that, if both the force of interest $r > 0$ and the premium rate $c \geq 0$ are constant and the claim size distribution is subexponential, then the tail probability of the discounted aggregate claims up to a finite or infinite time is asymptotically equivalent to the probability of ruin by the same time. This is because the amount of discounted aggregate premiums is always bounded by a finite constant c/r and, thus, it does not affect the asymptotic behavior of a subexponential tail. Owing to this reason, our results in this paper can be straightforwardly translated into finite-time and infinite-time ruin probabilities. In comparison with the corresponding results of the works cited above, our formulae also explicitly show the impact of the dependence structure of (X, θ) on the ruin.

The rest of this paper consists of four sections. In Section 2 we show two main results after briefly introducing necessary preliminaries about the renewal risk model and subexponential distributions. In Section 3 we verify the local and global uniformity of relation (1.1) through copulas. In Sections 4 and 5 we prove Theorems 2.1 and 2.2, respectively.

2. Main results

Throughout this paper, all limit relationships hold as $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$ satisfying

$$l_1 = \liminf_{x\to\infty} \frac{a(x)}{b(x)} \leq \limsup_{x\to\infty} \frac{a(x)}{b(x)} = l_2,$$

we write $a(x) \gtrsim b(x)$ if $l_1 \geq 1$, $a(x) \lesssim b(x)$ if $l_2 \leq 1$, $a(x) \sim b(x)$ if $l_1 = l_2 = 1$, and $a(x) \asymp b(x)$ if $0 < l_1 \leq l_2 < \infty$. We frequently equip limit relationships with certain

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uniformity, which is crucial for our purpose. For instance, for two positive bivariate functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we say that $a(x,t) \lesssim b(x,t)$ holds uniformly for $t \in \Delta \neq \varnothing$ if

$$\limsup_{x\to\infty} \sup_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \le 1.$$

Consider the renewal risk model in which (X_k, θ_k), $k = 1, 2, \dots$, are i.i.d. copies of a generic pair (X, θ) with marginal distributions F and G on $[0,\infty)$. To avoid triviality, both F and G are assumed to be nondegenerate at 0. Denote by $\tau_k = \sum_{i=1}^{k} \theta_i$, $k = 1, 2, \dots$, the claim arrival times, with $\tau_0 = 0$. Then the number of claims by time t is

$$N_t = \#\{\tau_k \le t \colon k = 1, 2, \dots\}, \qquad t \ge 0,$$

which forms an ordinary renewal counting process with a finite mean function:

$$\lambda_t = \mathrm{E}\, N_t = \sum_{k=1}^{\infty} \Pr(\tau_k \le t), \qquad t \ge 0.$$

In this way, the amount of aggregate claims is a random sum of the form $X(t) = \sum_{k=1}^{N_t} X_k$ for $t \ge 0$, where, here and throughout, a summation over an empty index set produces a value 0. Assuming a constant force of interest $r \ge 0$, the amount of discounted aggregate claims by time t is expressed as

$$D_r(t) = \int_{0-}^{t} \mathrm{e}^{-rs}\, \mathrm{d}X(s) = \sum_{k=1}^{\infty} X_k \mathrm{e}^{-r\tau_k} \mathbf{1}_{\{\tau_k \le t\}}, \qquad t \ge 0, \tag{2.1}$$

where the symbol $\mathbf{1}_E$ denotes the indicator function of an event E.

When studying the tail probability of $D_r(t)$, it is natural to restrict the region of the variable t to

$$\Lambda = \{t \colon 0 < \lambda_t \le \infty\}.$$

With $\underline{t} = \inf\{t \colon \Pr(\theta \le t) > 0\}$, it is clear that $\Lambda = [\underline{t}, \infty]$ if $\Pr(\theta = \underline{t}) > 0$ while $\Lambda = (\underline{t}, \infty]$ if $\Pr(\theta = \underline{t}) = 0$. For notational convenience, we write $\Lambda_T = [0, T] \cap \Lambda$ for every fixed $T \in \Lambda$.

We consider only the case of heavy-tailed claim size distributions. One of the most important classes of heavy-tailed distributions is the class $\mathcal{S}$ of subexponential distributions. By definition, a distribution F on $[0,\infty)$ is said to be subexponential if $\bar{F}(x) = 1 - F(x) > 0$ for all $x \ge 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\bar{F}(x)} = n$$

holds for all (or, equivalently, for some) $n = 2, 3, \dots$, where F^{n*} denotes the n-fold convolution of F. The class $\mathcal{S}$ contains a lot of important distributions such as Pareto, lognormal, and heavy-tailed Weibull distributions. See Embrechts *et al.* (1997) for a review of subexponential distributions in the context of insurance and finance.

A useful subclass of $\mathcal{S}$ is the class of distributions with extended regularly varying (ERV) tails, characterized by the relations $\bar{F}(x) > 0$ for all $x \ge 0$ and

$$y^{-\beta} \le \liminf_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \le \limsup_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \le y^{-\alpha}, \qquad y \ge 1, \tag{2.2}$$

for some $0 < \alpha \le \beta < \infty$. We signify the regularity property in (2.2) as $F \in \mathrm{ERV}(-\alpha, -\beta)$, so that ERV is the union of all ERV$(-\alpha, -\beta)$ over the range $0 < \alpha \le \beta < \infty$. In particular, when $\alpha = \beta$, the class ERV$(-\alpha, -\beta)$ coincides with the famous class $\mathcal{R}_{-\alpha}$ of distributions with regularly varying tails. Thus, if $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ then

$$\lim_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}, \qquad y > 0. \tag{2.3}$$

Write $\mathcal{R}$ as the union of all $\mathcal{R}_{-\alpha}$ over the range $0 < \alpha < \infty$.

For a distribution $F \in$ ERV$(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$, by Proposition 2.2.3 of Bingham *et al.* (1989) we know that, for every $\varepsilon > 0$ and $b > 1$, there is some $x_0 > 0$ such that the inequalities

$$\frac{1}{b}(y^{-\beta-\varepsilon} \wedge y^{-\alpha+\varepsilon}) \le \frac{\bar{F}(xy)}{\bar{F}(x)} \le b(y^{-\beta-\varepsilon} \vee y^{-\alpha+\varepsilon}) \tag{2.4}$$

hold whenever $x > x_0$ and $xy > x_0$. In particular, when $\alpha = \beta$, the inequalities in (2.4) reduce to the well-known Potter's bounds for the class $\mathcal{R}$. That is, if $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$ then, for every $\varepsilon > 0$ and $b > 1$, there is some $x_0 > 0$ such that the inequalities

$$\frac{1}{b}(y^{-\alpha-\varepsilon} \wedge y^{-\alpha+\varepsilon}) \le \frac{\bar{F}(xy)}{\bar{F}(x)} \le b(y^{-\alpha-\varepsilon} \vee y^{-\alpha+\varepsilon}) \tag{2.5}$$

hold whenever $x > x_0$ and $xy > x_0$. In addition, by Theorem 1.5.2 of Bingham *et al.* (1989), the convergence in relation (2.3) is uniform for $y \in [\varepsilon, \infty)$ for every fixed $\varepsilon > 0$; that is,

$$\lim_{x\to\infty} \sup_{y\in[\varepsilon,\infty)} \left| \frac{\bar{F}(xy)}{\bar{F}(x)} - y^{-\alpha} \right| = 0. \tag{2.6}$$

As mentioned in Section 1, a standing assumption on the dependence structure of (X, θ) in this paper is the following.

(A1) There is some measurable function $h(\cdot)\colon [0,\infty) \longmapsto (0,\infty)$ such that relation (1.1) holds locally uniformly for $t \in \Lambda$ (that is, it holds uniformly for $t \in \Lambda_T$ for every $T \in \Lambda$).

To achieve global uniformity of the obtained asymptotic formula, we need to strengthen assumption (A1) to the following.

(A2) There is some measurable function $h(\cdot)\colon [0,\infty) \longmapsto (0,\infty)$ such that relation (1.1) holds uniformly for $t \in \Lambda$.

In addition to assumption (A1) or (A2), we also need to assume the following.

(B) Either $\underline{t} > 0$, or $\underline{t} = 0$ and there is some $t^* \in \Lambda$ such that $\inf_{0\le t\le t^*} h(t) > 0$.

Note that t^* appearing in assumption (B) can be chosen to be 0 if $\Pr(\theta = 0) > 0$, and in this case the restriction $\inf_{0\le t\le t^*} h(t) > 0$ is redundant since $h(0) > 0$ by assumption (A1) or (A2).

We remark that these assumptions on the dependence structure of (X, θ) are close to minimum for establishing a locally or globally uniform asymptotic formula.

The first main result of this paper is as follows.

Theorem 2.1. *Consider the discounted aggregate claims described in relation (2.1) with* $r \geq 0$. *If* $F \in \mathscr{S}$ *and assumptions (A1) and (B) hold, then the relation*

$$\Pr(D_r(t) > x) \sim \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s \tag{2.7}$$

holds locally uniformly for $t \in \Lambda$, *where*

$$\tilde{\lambda}_t = \int_{0-}^{t} (1 + \lambda_{t-u})h(u)G(\mathrm{d}u). \tag{2.8}$$

If, in addition to the other conditions of Theorem 2.1, $F \in \mathscr{R}_{-\alpha}$ for some $0 < \alpha < \infty$, then applying the uniformity of relation (2.3) as explained in (2.6) to relation (2.7), we find that the relation

$$\Pr(D_r(t) > x) \sim \bar{F}(x) \int_{0-}^{t} \mathrm{e}^{-\alpha rs}\, \mathrm{d}\tilde{\lambda}_s \tag{2.9}$$

holds locally uniformly for $t \in \Lambda$.

Consider the uniformity of relation (2.7) on Λ_T for some $T \in \Lambda$. Under assumption (A1), integrating both sides of (1.1) with respect to $\Pr(\theta \in \mathrm{d}t)$ over the range $[0, T]$ leads to $0 < \mathrm{E}\, h(\theta)\, \mathbf{1}_{\{\theta \leq T\}} \leq 1$. Similarly as in (1.2), we introduce an independent random variable θ^* with a proper distribution given by

$$\Pr(\theta^* \in \mathrm{d}t) = \frac{h(t)}{\mathrm{E}\, h(\theta)\, \mathbf{1}_{\{\theta \leq T\}}} G(\mathrm{d}t), \qquad t \in [0, T].$$

Construct a delayed renewal counting process $\{N_t^*,\ t \geq 0\}$ with interarrival times $\theta^*, \theta_k,\ k = 2, 3 \ldots,$ and a mean function λ_t^*. It is easy to see that

$$\tilde{\lambda}_t = \lambda_t^*\, \mathrm{E}\, h(\theta)\, \mathbf{1}_{\{\theta \leq T\}}, \qquad t \in \Lambda_T;$$

that is, $\tilde{\lambda}_t$ is proportional to the mean function of a delayed renewal counting process whose first interarrival time is affected by the dependence structure of (X, θ).

Next we establish global uniformity for relation (2.7). For this purpose, we need to restrict the claim size distribution to the class ERV. Note that if $r = 0$ then $D_r(t)$ diverges to ∞ almost surely as $t \to \infty$ and, hence, it is not possible to establish the global uniformity for relation (2.7). For this reason, we assume $r > 0$ in the following second main result.

Theorem 2.2. *Consider the discounted aggregate claims described in relation (2.1) with* $r > 0$. *If* $F \in \mathrm{ERV}$ *and assumptions (A2) and (B) hold, then relation (2.7) holds uniformly for* $t \in \Lambda$.

Similarly as above, if, in addition to the other conditions of Theorem 2.2, $F \in \mathscr{R}_{-\alpha}$ for some $0 < \alpha < \infty$, then, by (2.5) and (2.6), it is easy to verify that relation (2.9) holds uniformly for $t \in \Lambda$. In particular, taking $t = \infty$ in relation (2.9) yields the more transparent asymptotic formula

$$\Pr(D_r(\infty) > x) \sim \bar{F}(x) \frac{\mathrm{E}\, h(\theta)\mathrm{e}^{-\alpha r\theta}}{1 - \mathrm{E}\, \mathrm{e}^{-\alpha r\theta}}.$$

Moreover, under assumption (A2), which implies that $\mathrm{E}\, h(\theta) = 1$, $\tilde{\lambda}_t$ is equal to the mean function of a delayed renewal counting process whose first interarrival time θ^* follows

$$\Pr(\theta^* \in \mathrm{d}t) = h(t)G(\mathrm{d}t).$$

When X and θ are independent, $h(t) \equiv 1$ and $\tilde{\lambda}_t \equiv \lambda_t$ for all $t \in \Lambda$. Hence, Theorem 2.1 of Hao and Tang (2008) corresponds to our Theorem 2.1 for the case of independent X and θ. The expression of $\tilde{\lambda}_t$ given in (2.8) for the general case of dependent X and θ shows that our results successfully capture the impact of the dependence structure of (X, θ) on the tail behavior of the discounted aggregate claims.

Asimit and Badescu (2010) studied the same problem. Our work extends theirs in the following three directions: (i) they considered the compound Poisson risk model while we consider the renewal risk model; (ii) they derived results for $F \in \mathscr{S}$ when $r = 0$ and for $F \in \mathscr{R}$ when $r > 0$, both of which are covered and unified by our Theorem 2.1; (iii) their formulae hold for a fixed time t while ours are equipped with local or global uniformity in time t, which greatly enhances the theoretical and applied interests of the results.

Restricted to the compound Poisson risk model with $F \in \mathscr{R}_{-\alpha}$ for some $0 < \alpha < \infty$, assuming that the Poisson intensity $\lambda > 0$, (2.9) immediately gives

$$\Pr(D_r(t) > x) \sim K_{r,t}\bar{F}(x), \qquad t > 0, \tag{2.10}$$

with

$$K_{r,t} = \int_0^t \left(\left(1 + \frac{\lambda}{\alpha r}\right)\mathrm{e}^{-\alpha r u} - \frac{\lambda}{\alpha r}\mathrm{e}^{-\alpha r t}\right) h(u)\lambda \mathrm{e}^{-\lambda u}\,\mathrm{d}u.$$

Theorem 3.2 of Asimit and Badescu (2010) also gives relation (2.10) but with a coefficient

$$K^*_{r,t} = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda t}\lambda^n \int \cdots \int_{\Omega_{n,t}} \sum_{i=1}^{n} h(s_i)\exp\left(-\alpha r \sum_{j=1}^{i} s_j\right) \prod_{i=1}^{n} \mathrm{d}s_i,$$

where $\Omega_{n,t} = \{(s_1, \dots, s_n) \in [0, t]^n \colon \sum_{i=1}^n s_i \le t\}$. It is not hard to verify that the two coefficients are actually the same though they look quite different. Indeed, recall that, for the Poisson process $\{N_t,\ t \ge 0\}$, the interarrival times $\theta_1, \dots, \theta_n$ conditional on $(N_t = n)$ have a joint distribution on $\Omega_{n,t}$ given by $\Pr(\theta_1 \in \mathrm{d}s_1, \dots, \theta_n \in \mathrm{d}s_n \mid N_t = n) = (n!/t^n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n$. With the help of this property, we have

$$\begin{aligned}
K^*_{r,t} &= \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t} \int \cdots \int_{\Omega_{n,t}} \sum_{i=1}^{n} h(s_i)\exp\left(-\alpha r \sum_{j=1}^{i} s_j\right) \frac{n!}{t^n} \prod_{i=1}^{n} \mathrm{d}s_i \\
&= \sum_{n=1}^{\infty} \Pr(N_t = n)\,\mathrm{E}\left(\sum_{i=1}^{n} h(\theta_i)\mathrm{e}^{-\alpha r \tau_i} \,\middle|\, N_t = n\right) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathrm{E}\, h(\theta_i)\mathrm{e}^{-\alpha r \tau_i}\,\mathbf{1}_{\{N_t = n\}} \\
&= \sum_{i=1}^{\infty} \mathrm{E}\, h(\theta_i)\mathrm{e}^{-\alpha r \tau_i}\,\mathbf{1}_{\{\tau_i \le t\}} \\
&= \int_0^t h(u)\mathrm{e}^{-\alpha r u} G(\mathrm{d}u) + \sum_{i=2}^{\infty} \int_0^t \int_0^{t-u} h(u)\mathrm{e}^{-\alpha r(u+v)} \Pr(\tau_{i-1} \in \mathrm{d}v) G(\mathrm{d}u) \\
&= \int_0^t h(u)\mathrm{e}^{-\alpha r u}(\lambda \mathrm{e}^{-\lambda u}\,\mathrm{d}u) + \int_0^t \int_0^{t-u} h(u)\mathrm{e}^{-\alpha r(u+v)}(\lambda\,\mathrm{d}v)(\lambda \mathrm{e}^{-\lambda u}\,\mathrm{d}u).
\end{aligned}$$

Thus, $K^*_{r,t} = K_{r,t}$.

3. Verification of the assumptions on dependence

This section concerns verification of our assumptions on the dependence structure of (X, θ). We carry on this discussion through copulas. The reader is referred to Joe (1997) or Nelsen (2006) for a comprehensive treatment of copulas.

For simplicity, assume that H, the joint distribution of (X, θ), has continuous marginal distributions F and G. Then, by Sklar's theorem, there is a unique copula $C(u, v)\colon [0, 1]^2 \mapsto [0, 1]$, which is the joint distribution of the uniform variates $F(X)$ and $G(\theta)$, such that

$$H(x, t) = C(F(x), G(t)).$$

The corresponding survival copula is defined to be

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \qquad (u, v) \in [0, 1]^2,$$

which is such that

$$\bar{H}(x, t) = \Pr(X > x, \theta > t) = \hat{C}(\bar{F}(x), \bar{G}(t)).$$

Assume that the copula $C(u, v)$ is absolutely continuous; hence, so is the survival copula $\hat{C}(u, v)$. Denote by $\hat{c}(u, v)$ the density of the survival copula. Then the function $h(\cdot)$ defined in (1.1), if it exists, is equal to

$$h(t) = \lim_{u \to 0+} \frac{\partial \hat{C}(u, v)/\partial v}{u}\bigg|_{v=\bar{G}(t)} = \hat{c}(0+, \bar{G}(t)), \qquad t > 0. \tag{3.1}$$

In the rest of this section, the variables t and v are always connected through the identity $v = \bar{G}(t)$, as indicated in (3.1). In terms of the survival copula $\hat{C}(u, v)$, the local uniformity of relation (1.1), as required by assumption (A1), can be restated as

$$\lim_{u \to 0+} \sup_{v \in [\delta, 1]} \left| \frac{\partial \hat{C}(u, v)/\partial v}{uh(t)} - 1 \right| = \lim_{u \to 0+} \sup_{v \in [\delta, 1]} \left| \frac{(1/u)\int_0^u \hat{c}(s, v)\,\mathrm{d}s}{\hat{c}(0+, v)} - 1 \right| = 0 \tag{3.2}$$

for $\delta \in (0, 1)$, and the global uniformity of relation (1.1), as required by assumption (A2), can be restated as

$$\lim_{u \to 0+} \sup_{v \in (0, 1]} \left| \frac{\partial \hat{C}(u, v)/\partial v}{uh(t)} - 1 \right| = \lim_{u \to 0+} \sup_{v \in (0, 1]} \left| \frac{(1/u)\int_0^u \hat{c}(s, v)\,\mathrm{d}s}{\hat{c}(0+, v)} - 1 \right| = 0. \tag{3.3}$$

So far, the function $h(\cdot)$ and the local/global uniformity of relation (1.1) have been expressed through the survival copula and its density.

Recall that an Archimedean copula is of the form

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \qquad (u, v) \in [0, 1]^2,$$

where $\varphi(\cdot)\colon [0, 1] \longmapsto [0, \infty]$, called the generator of $C(u, v)$, is a strictly decreasing and convex function with $0 < \varphi(0) \le \infty$ and $\varphi(1) = 0$, while $\varphi^{[-1]}(\cdot)$ is the pseudo-inverse of $\varphi(\cdot)$, equal to $\varphi^{-1}(t)$ when $0 \le t \le \varphi(0)$ and equal to 0 otherwise. This copula has a simple structure, as its definition shows, and it possesses a lot of nice properties. If the generator $\varphi(\cdot)$ is twice differentiable then the copula density has a transparent form; see relation (4.3.6) of Nelsen (2006). In this case, recalling (3.1), it follows that

$$h(t) = \hat{c}(0+, v) = -\frac{\varphi'(1-)\varphi''(1 - v)}{(\varphi'(1 - v))^2}.$$

Moreover, it should not be hard to construct some general conditions on the generator $\varphi(\cdot)$ to guarantee relations (3.2) and (3.3).

Next, we reexamine the three examples given in Asimit and Badescu (2010).

Example 3.1. The Ali–Mikhail–Haq copula is of the form

$$C(u, v) = \frac{uv}{1-\gamma(1-u)(1-v)}, \qquad \gamma \in [-1, 1).$$

Direct calculation shows that

$$\frac{\partial \hat{C}(u, v)}{\partial v} = \frac{u + \gamma u(1-2v) - \gamma u^2(1-\gamma v^2)}{(1-\gamma uv)^2}.$$

Then, by relation (3.1), $h(t) = 1 + \gamma(1-2v)$. It follows that

$$\left|\frac{\partial \hat{C}(u, v)/\partial v}{uh(t)} - 1\right| = \frac{|v(2-\gamma uv)(1+\gamma(1-2v)) - 1 + \gamma v^2|}{(1-\gamma uv)^2(1+\gamma(1-2v))}|\gamma|u.$$

Hence, relation (3.2) holds when $\gamma \in [-1, 1)$ and relation (3.3) holds when $\gamma \in (-1, 1)$.

Example 3.2. The Farlie–Gumbel–Morgenstern copula is of the form

$$C(u, v) = uv + \gamma uv(1-u)(1-v), \qquad \gamma \in [-1, 1].$$

Following the same lines as in Example 3.1, we have, respectively,

$$\frac{\partial \hat{C}(u, v)}{\partial v} = u + \gamma u(1-u)(1-2v), \qquad h(t) = 1 + \gamma(1-2v),$$

and

$$\left|\frac{\partial \hat{C}(u, v)/\partial v}{uh(t)} - 1\right| = \frac{|1-2v|}{1+\gamma(1-2v)}|\gamma|u.$$

Hence, relation (3.2) holds when $\gamma \in [-1, 1)$ and relation (3.3) holds when $\gamma \in (-1, 1)$.

Example 3.3. The Frank copula is of the form

$$C(u, v) = -\frac{1}{\gamma}\ln\left(1 + \frac{(\mathrm{e}^{-\gamma u}-1)(\mathrm{e}^{-\gamma v}-1)}{\mathrm{e}^{-\gamma}-1}\right), \qquad \gamma \neq 0.$$

A direct but rather tedious calculation gives

$$\frac{\partial \hat{C}(u, v)}{\partial v} = \frac{\mathrm{e}^{\gamma}(1-\mathrm{e}^{\gamma u})}{\mathrm{e}^{\gamma}(1-\mathrm{e}^{\gamma}) + (\mathrm{e}^{\gamma u}-\mathrm{e}^{\gamma})(\mathrm{e}^{\gamma v}-\mathrm{e}^{\gamma})}.$$

It follows that

$$h(t) = \frac{\gamma \mathrm{e}^{\gamma(1-v)}}{\mathrm{e}^{\gamma}-1}$$

and

$$\left|\frac{\partial \hat{C}(u, v)/\partial v}{uh(t)} - 1\right| = \frac{|\gamma u(\mathrm{e}^{\gamma(1-v)}(\mathrm{e}^{\gamma u}-1) + (\mathrm{e}^{\gamma}-\mathrm{e}^{\gamma u})) - (\mathrm{e}^{\gamma u}-1)(\mathrm{e}^{\gamma}-1)|}{\gamma u(\mathrm{e}^{\gamma(1-v)}(\mathrm{e}^{\gamma u}-1) + (\mathrm{e}^{\gamma}-\mathrm{e}^{\gamma u}))}.$$

Hence, relations (3.2) and (3.3) hold for all $\gamma \neq 0$.

4. Proof of Theorem 2.1

4.1. Lemmas

It is well known that every subexponential distribution F is long tailed, denoted as $F \in \mathcal{L}$, in the sense that the relation

$$\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = 1$$

holds for all (or, equivalently, for some) $y \neq 0$; see Lemma 2 of Chistyakov (1964) or Lemma 1.3.5(a) of Embrechts *et al.* (1997).

We first show an elementary result regarding long-tailed distributions.

Lemma 4.1. *We have $F \in \mathcal{L}$ if and only if there is a function $l(\cdot)\colon (0,\infty) \mapsto (0,\infty)$ satisfying*

(i) $l(x) < x/2$ *for all* $x > 0$,

(ii) $l(x) \to \infty$,

(iii) $l(\cdot)$ *is slowly varying at* ∞,

such that, for every $K > 0$, $\bar{F}(x - Kl(x)) \sim \bar{F}(x)$.

Proof. The 'if' assertion is trivial, so we prove only the 'only if' assertion. Let $F \in \mathcal{L}$. It is easy to see that there is a positive function $l_1(\cdot)$ satisfying $l_1(x) \to \infty$, $l_1(x) < x^2/4$ for all $x > 0$, and $\bar{F}(x - l_1(x)) \sim \bar{F}(x)$. Furthermore, by Lemma 3.2 of Tang (2008), there is a slowly varying function $l(\cdot)\colon (0,\infty) \mapsto (0,\infty)$ satisfying $l(x) \to \infty$ and $l(x) \le l_1(x)^{1/2}$ for all $x > 0$. This function $l(\cdot)$ fulfills all the requirements in Lemma 4.1.

Lemma 4.2 below forms the main ingredient of the proof of Theorem 2.1. It deals with the tail probability of the sum of n random variables equipped with a certain dependence structure. A similar problem was considered in Proposition 2.1 of Foss and Richards (2010). However, a close look reveals that their Proposition 2.1 and our Lemma 4.2 below are essentially different.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with common distribution $F \in \mathcal{S}$. Recall Proposition 5.1 of Tang and Tsitsiashvili (2003), which shows that, for arbitrarily fixed $0 < a \le b < \infty$, the relation

$$\Pr\biggl(\sum_{k=1}^{n} w_k X_k > x\biggr) \sim \sum_{k=1}^{n} \Pr(w_k X_k > x) \tag{4.1}$$

holds uniformly for $(w_1, \ldots, w_n) \in [a,b]^n$. Hence, for the case of independent X and θ, Lemma 4.2 below immediately follows by conditioning on $(\theta_1, \ldots, \theta_n)$. However, for the general case of dependent X and θ, this lemma is a nontrivial consequence of Proposition 5.1 of Tang and Tsitsiashvili (2003).

Hereafter, for notational convenience, for every $t \in \Lambda$ and $n = 1, 2, \ldots$, we write $t_n = \sum_{i=1}^{n} s_i$ and $\Omega_{n,t} = \{(s_1, \ldots, s_n) \in [0,t]^n : t_n \le t\}$.

Lemma 4.2. *Recall the renewal risk model introduced in Section 2 with $r \ge 0$. If $F \in \mathcal{S}$ and assumption (A1) holds, then, for every $n = 1, 2, \ldots$, it holds locally uniformly for $t \in \Lambda$ that*

$$\Pr\biggl(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = n\biggr) = (1 + o(1)) \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = n). \tag{4.2}$$

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Proof. Note that the event $(N_t = n)$ in relation (4.2) could have a probability of 0. In the proof below, we still use the notation $a(x, t) \sim b(x, t)$ even though the two functions $a(x, t)$ and $b(x, t)$ could be simultaneously equal to 0. No confusion should occur since the precise meaning of the notation is $a(x, t) = (1 + o(1))b(x, t)$.

We need to prove that relation (4.2) holds uniformly for $t \in \Lambda_T$ for arbitrarily fixed $T \in \Lambda$. We proceed by induction. Trivially, the assertion holds for $n = 1$. Now we assume by induction that the assertion holds for some positive integer $n = m - 1$, and we prove it for $n = m$; that is, we aim at the relation

$$\Pr\left(\sum_{k=1}^{m} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m\right) \sim \sum_{k=1}^{m} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m) \tag{4.3}$$

with the required uniformity for $t \in \Lambda_T$.

Recall the function $l(\cdot)$ given in Lemma 4.1. According to the value of the sum $\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k}$ belonging to $(0, l(x)]$, $(x - l(x), \infty)$, and $(l(x), x - l(x)]$, we split the probability on the left-hand side of (4.3) into three parts as

$$\Pr\left(\sum_{k=1}^{m} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m\right) = I_1(x, m, t) + I_2(x, m, t) + I_3(x, m, t).$$

For $I_1(x, m, t)$, it holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned}
I_1(x, m, t) &\le \Pr(X_m \mathrm{e}^{-r\tau_m} > x - l(x),\ N_t = m) \\
&= \int \cdots \int_{\Omega_{m,t}} \Pr(X_m\, \mathrm{e}^{-rt_m} > x - l(x) \mid \theta_m = s_m)\bar{G}(t - t_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&\sim \int \cdots \int_{\Omega_{m,t}} \Pr(X_m \mathrm{e}^{-rt_m} > x) h(s_m)\bar{G}(t - t_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&\sim \int \cdots \int_{\Omega_{m,t}} \Pr(X_m\, \mathrm{e}^{-rt_m} > x \mid \theta_m = s_m)\bar{G}(t - t_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&= \Pr(X_m \mathrm{e}^{-r\tau_m} > x,\ N_t = m),
\end{aligned} \tag{4.4}$$

where at the third and fourth steps we used assumption (A1) and Lemma 4.1. As can be seen from the second step to the last step, the derivation of (4.4) mainly involves eliminating the slowly varying function $l(x)$. For $I_2(x, m, t)$, by the induction assumption and the same idea used to derive (4.4), we have, uniformly for $t \in \Lambda_T$,

$$\begin{aligned}
I_2(x, m, t) &\le \Pr\left(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > x - l(x),\ N_t = m\right) \\
&= \int_{0-}^{t} \Pr\left(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > x - l(x),\ N_{t-s_m} = m - 1\right) G(\mathrm{d}s_m) \\
&\sim \sum_{k=1}^{m-1} \int_{0-}^{t} \Pr(X_k \mathrm{e}^{-r\tau_k} > x - l(x),\ N_{t-s_m} = m - 1) G(\mathrm{d}s_m)
\end{aligned}$$

$$\sim \sum_{k=1}^{m-1} \int_{0-}^{t} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_{t-s_m} = m-1) G(\mathrm{d}s_m)$$
$$= \sum_{k=1}^{m-1} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m). \tag{4.5}$$

Now we focus on $I_3(x, m, t)$. It holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned}
I_3(x, m, t) &\le \Pr\bigg(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} + X_m \mathrm{e}^{-r\theta_m} > x,\ \sum_{k=1}^{m-1} X_k\, \mathrm{e}^{-r\tau_k} > l(x), \\
&\qquad X_m \mathrm{e}^{-r\theta_m} > l(x),\ N_t = m\bigg) \\
&= \int_{0-}^{t} \int_{l(x)}^{\infty} \Pr\bigg(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > (x-y) \vee l(x),\ N_{t-s_m} = m-1\bigg) \\
&\qquad \times \Pr(X_m \mathrm{e}^{-rs_m} \in \mathrm{d}y \mid \theta_m = s_m) G(\mathrm{d}s_m) \\
&\sim \sum_{k=1}^{m-1} \int_{0-}^{t} \int_{l(x)}^{\infty} \Pr(X_k \mathrm{e}^{-r\tau_k} > (x-y) \vee l(x),\ N_{t-s_m} = m-1) \\
&\qquad \times \Pr(X_m \mathrm{e}^{-rs_m} \in \mathrm{d}y \mid \theta_m = s_m) G(\mathrm{d}s_m) \\
&\sim \sum_{k=1}^{m-1} \int \cdots \int_{\Omega_{m,t}} \Pr(X_k \mathrm{e}^{-rt_k} + X_m \mathrm{e}^{-rs_m} > x,\ X_k \mathrm{e}^{-rt_k} > l(x),\ X_m \mathrm{e}^{-rs_m} > l(x)) \\
&\qquad \times \bar{G}(t - t_m) h(s_k) h(s_m) \prod_{i=1}^{m} G(\mathrm{d}s_i),
\end{aligned}$$

where at the third step we used the induction assumption and at the last step we applied assumption (A1) twice. For every $k = 1, \dots, m-1$, we note that, uniformly for $(s_1, \dots, s_m) \in \Omega_{m,t}$,

$$\begin{aligned}
&\Pr(X_k \mathrm{e}^{-rt_k} + X_m \mathrm{e}^{-rs_m} > x,\ X_k \mathrm{e}^{-rt_k} > l(x),\ X_m \mathrm{e}^{-rs_m} > l(x)) \\
&\quad \le \Pr(X_k \mathrm{e}^{-rt_k} + X_m \mathrm{e}^{-rs_m} > x) - \Pr(X_k \mathrm{e}^{-rt_k} > x,\ X_m \mathrm{e}^{-rs_m} \le l(x)) \\
&\qquad - \Pr(X_m \mathrm{e}^{-rs_m} > x,\ X_k \mathrm{e}^{-rt_k} \le l(x)) \\
&\quad = o(1)(\Pr(X_k \mathrm{e}^{-rt_k} > x) + \Pr(X_m \mathrm{e}^{-rs_m} > x)),
\end{aligned}$$

where at the last step we applied Proposition 5.1 of Tang and Tsitsiashvili (2003) as summarized in (4.1) above. Substituting these estimates into $I_3(x, m, t)$, we have, uniformly for $t \in \Lambda_T$,

$$\begin{aligned}
I_3(x, m, t) &= o(1) \sum_{k=1}^{m-1} \int \cdots \int_{\Omega_{m,t}} \Pr(X_k \mathrm{e}^{-rt_k} > x) \bar{G}(t - t_m) h(s_k) h(s_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&\quad + o(1) \sum_{k=1}^{m-1} \int \cdots \int_{\Omega_{m,t}} \Pr(X_m\, \mathrm{e}^{-rs_m} > x) \bar{G}(t - t_m) h(s_k) h(s_m) \prod_{i=1}^{m} G(\mathrm{d}s_i).
\end{aligned}$$

Since the distribution F has an ultimate tail, for an arbitrary function $a(x) = o(1)$, we can always find some positive function $l^*(x)$, which diverges to ∞ but slowly enough, such that

$$a(x) = o(1)\bar{F}(l^*(x)).$$

Using this idea and assumption (A1), we have, uniformly for $t \in \Lambda_T$,

$$\begin{aligned}
I_3(x, m, t) &= o(1) \sum_{k=1}^{m-1} \int \cdots \int_{\Omega_{m,t}} \Pr(X_k \mathrm{e}^{-rt_k} > x,\ X_m > l^*(x))\bar{G}(t - t_m) \\
&\qquad\qquad \times h(s_k)h(s_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&\quad + o(1)\sum_{k=1}^{m-1} \int \cdots \int_{\Omega_{m,t}} \Pr(X_m \mathrm{e}^{-rs_m} > x,\ X_k > l^*(x))\bar{G}(t - t_m) \\
&\qquad\qquad \times h(s_k)h(s_m) \prod_{i=1}^{m} G(\mathrm{d}s_i) \\
&= o(1) \sum_{k=1}^{m-1} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ X_m > l^*(x),\ N_t = m) \\
&\quad + o(1) \sum_{k=1}^{m-1} \Pr(X_m \mathrm{e}^{-r\theta_m} > x,\ X_k > l^*(x),\ N_t = m) \\
&= o(1) \sum_{k=1}^{m} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m).
\end{aligned} \tag{4.6}$$

A combination of (4.4), (4.5), and (4.6) gives the upper-bound version of (4.3).

The corresponding lower-bound version of (4.3) can be easily established. In fact,

$$\begin{aligned}
&\Pr\biggl(\sum_{k=1}^{m} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m\biggr) \\
&\quad \geq \Pr\biggl(\biggl(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > x\biggr) \cup (X_m \mathrm{e}^{-r\tau_m} > x),\ N_t = m\biggr) \\
&\quad = \Pr\biggl(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m\biggr) + \Pr(X_m \mathrm{e}^{-r\tau_m} > x,\ N_t = m) \\
&\qquad - \Pr\biggl(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > x,\ X_m \mathrm{e}^{-r\tau_m} > x,\ N_t = m\biggr) \\
&\quad = J_1(x, m, t) + \Pr(X_m \mathrm{e}^{-r\tau_m} > x,\ N_t = m) - J_2(x, m, t).
\end{aligned} \tag{4.7}$$

As in dealing with $I_2(x, m, t)$, by the induction assumption we have, uniformly for $t \in \Lambda_T$,

$$J_1(x, m, t) \sim \sum_{k=1}^{m-1} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m). \tag{4.8}$$

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Furthermore, as in dealing with $I_3(x, m, t)$, it holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned}J_2(x, m, t) &\le \Pr\biggl(\sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} + X_m \mathrm{e}^{-r\theta_m} > x,\ \sum_{k=1}^{m-1} X_k \mathrm{e}^{-r\tau_k} > l(x),\\ &\qquad\qquad X_m \mathrm{e}^{-r\theta_m} > l(x),\ N_t = m\biggr)\\ &= o(1) \sum_{k=1}^{m} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = m).\end{aligned} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) gives the lower-bound version of (4.3). This completes the proof of Lemma 4.2.

Following the same lines of the proof of Lemma 4.2 with some obvious modifications, we can obtain the following result.

Lemma 4.3. *Under the same conditions of Lemma 4.2, for every $n = 1, 2, \dots$, it holds locally uniformly for $t \in \Lambda$ that*

$$\Pr\biggl(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\theta_k} > x,\ \tau_n \le t\biggr) = (1 + o(1))n \Pr(X_1 \mathrm{e}^{-r\theta_1} > x,\ \tau_n \le t). \tag{4.10}$$

For a distribution $F \in \mathcal{S}$, the well-known Kesten's inequality states that, for every $\varepsilon > 0$, there is some constant $K = K_\varepsilon > 0$ such that the inequality

$$\overline{F^{n*}}(x) \le K(1 + \varepsilon)^n \bar{F}(x)$$

holds for all $n = 1, 2, \dots$ and $x \ge 0$. For its proof, see Athreya and Ney (1972, p. 149) or Lemma 1.3.5(c) of Embrechts *et al.* (1997). In the following lemma, we establish an inequality of Kesten's type for the probability on the left-hand side of (4.10).

Lemma 4.4. *Recall the renewal risk model introduced in Section 2 with $r \ge 0$. If $F \in \mathcal{S}$, $\underline{t} = 0$, and assumptions (A1) and (B) hold, then, for every $\varepsilon > 0$ and $T \in \Lambda$, there is some constant $K = K_{r,\varepsilon,T} > 0$ such that the inequality*

$$\Pr\biggl(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\theta_k} > x,\ \tau_n \le t\biggr) \le K(1 + \varepsilon)^n \Pr(X_1 \mathrm{e}^{-r\theta_1} > x,\ \tau_n \le t)$$

holds for all $n = 1, 2, \dots$, $x \ge 0$, and $t \in \Lambda_T$.

Proof. For every $\varepsilon > 0$, by Lemma 4.3, there is some constant $x_0 > 0$ such that, for all $x > x_0$ and $t \in \Lambda_T$,

$$\begin{aligned}&\Pr(X_1 \mathrm{e}^{-r\theta_1} + X_2 \mathrm{e}^{-r\theta_2} > x,\ X_2 \mathrm{e}^{-r\theta_2} \le x,\ \tau_2 \le t)\\ &\quad = \Pr(X_1 \mathrm{e}^{-r\theta_1} + X_2 \mathrm{e}^{-r\theta_2} > x,\ \tau_2 \le t) - \Pr(X_2 \mathrm{e}^{-r\theta_2} > x,\ \tau_2 \le t)\\ &\quad \le (1 + \varepsilon) \Pr(X_1 \mathrm{e}^{-r\theta_1} > x,\ \tau_2 \le t).\end{aligned} \tag{4.11}$$

By assumption (A1), the constant x_0 above can be chosen so large that, for all $t \in \Lambda_T$,

$$\Pr(X_1 \mathrm{e}^{-r\theta_1} > x_0 \mid \theta_1 = t) \ge \tfrac{1}{2} \bar{F}(x_0 \mathrm{e}^{rT}) h(t). \tag{4.12}$$

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Write

$$a_n = \sup_{x\geq 0,\, t\in\Lambda_T} \frac{\Pr(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\theta_k} > x,\ \tau_n \leq t)}{\Pr(X_1\mathrm{e}^{-r\theta_1} > x,\ \tau_n \leq t)}.$$

We start by evaluating a_{n+1}. It is clear that

$$\Pr\bigg(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ \tau_{n+1} \leq t\bigg) = \Pr\bigg(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ X_{n+1}\mathrm{e}^{-r\theta_{n+1}} \leq x,\ \tau_{n+1} \leq t\bigg) + \Pr(X_{n+1}\mathrm{e}^{-r\theta_{n+1}} > x,\ \tau_{n+1} \leq t).$$

Conditioning on (X_{n+1}, θ_{n+1}) and noting the definition of a_n, we have

$$\begin{aligned}\Pr\bigg(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ X_{n+1}\mathrm{e}^{-r\theta_{n+1}} \leq x,\ \tau_{n+1} \leq t\bigg)&\\ \leq a_n \Pr(X_1\mathrm{e}^{-r\theta_1} + X_{n+1}\mathrm{e}^{-r\theta_{n+1}} > x,\ X_{n+1}\mathrm{e}^{-r\theta_{n+1}} \leq x,\ \tau_{n+1} \leq t).&\end{aligned}$$

Using (4.11), we know that, for all $x > x_0$ and $t \in \Lambda_T$,

$$\begin{aligned}&\Pr(X_1\mathrm{e}^{-r\theta_1} + X_{n+1}\mathrm{e}^{-r\theta_{n+1}} > x,\ X_{n+1}\mathrm{e}^{-r\theta_{n+1}} \leq x,\ \tau_{n+1} \leq t)\\ &\quad = \int\cdots\int_{\Omega_{n-1,t}} \Pr(X_1\mathrm{e}^{-r\theta_1} + X_2\mathrm{e}^{-r\theta_2} > x,\ X_2\mathrm{e}^{-r\theta_2} \leq x,\ \tau_2 \leq t - t_{n-1}) \prod_{i=1}^{n-1} G(\mathrm{d}s_i)\\ &\quad \leq (1+\varepsilon)\Pr(X_1\mathrm{e}^{-r\theta_1} > x,\ \tau_{n+1} \leq t).\end{aligned}$$

Hence,

$$\sup_{x>x_0,\, t\in\Lambda_T} \frac{\Pr(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ \tau_{n+1} \leq t)}{\Pr(X_1\mathrm{e}^{-r\theta_1} > x,\ \tau_{n+1} \leq t)} \leq (1+\varepsilon)a_n + 1. \tag{4.13}$$

When $x \leq x_0$, by inequality (4.12), it holds for all $t \in \Lambda_T$ that

$$\begin{aligned}\frac{\Pr(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ \tau_{n+1} \leq t)}{\Pr(X_1\mathrm{e}^{-r\theta_1} > x,\ \tau_{n+1} \leq t)} &\leq \frac{\Pr(\tau_{n+1} \leq t)}{\Pr(X_1\mathrm{e}^{-r\theta_1} > x_0,\ \tau_{n+1} \leq t)}\\ &\leq \big(\tfrac{1}{2}\bar{F}(x_0\mathrm{e}^{rT})\big)^{-1} \frac{\int_{0-}^{t} G^{n*}(t-s)G(\mathrm{d}s)}{\int_{0-}^{t} G^{n*}(t-s)h(s)G(\mathrm{d}s)}.\end{aligned}$$

By assumption (B), there is some constant $0 \leq t^* \in \Lambda$ such that $h(s)$ is away from 0 for $s \in [0, t^*]$. We have

$$\begin{aligned}\frac{\int_{0-}^{t} G^{n*}(t-s)G(\mathrm{d}s)}{\int_{0-}^{t} G^{n*}(t-s)h(s)G(\mathrm{d}s)} &\leq \frac{\int_{0-}^{t\wedge t^*} G^{n*}(t-s)G(\mathrm{d}s) + \int_{t\wedge t^*}^{t} G^{n*}(t-s)G(\mathrm{d}s)\,\mathbf{1}_{\{t>t^*\}}}{\int_{0-}^{t\wedge t^*} G^{n*}(t-s)h(s)G(\mathrm{d}s)}\\ &\leq \Big(\inf_{s\in[0,t^*]} h(s)\Big)^{-1} + \frac{\bar{G}(t^*)}{\int_{0-}^{t^*} h(s)G(\mathrm{d}s)}.\end{aligned}$$

Therefore, there is some $0 < L < \infty$ such that

$$\sup_{x\leq x_0,\, t\in\Lambda_T} \frac{\Pr(\sum_{k=1}^{n+1} X_k\mathrm{e}^{-r\theta_k} > x,\ \tau_{n+1} \leq t)}{\Pr(X_1\mathrm{e}^{-r\theta_1} > x,\ \tau_{n+1} \leq t)} \leq L. \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$a_{n+1} \leq (1+\varepsilon)a_n + 1 + L.$$

This recursive inequality with initial value $a_1 = 1$ completes the proof of Lemma 4.4.

4.2. Proof of Theorem 2.1

We follow the proof of Theorem 2.1 of Hao and Tang (2008), but we need to overcome some technical difficulties due to the dependence structure of (X, θ).

Let us prove that relation (2.7) holds uniformly for $t \in \Lambda_T$ for arbitrarily fixed $T \in \Lambda$. Choose some large positive integer N and write

$$\begin{aligned}\Pr(D_r(t) > x) &= \bigg(\sum_{n=N+1}^{\infty} + \sum_{n=1}^{N}\bigg) \Pr\bigg(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = n\bigg) \\ &= I_1(x,t) + I_2(x,t). \end{aligned} \tag{4.15}$$

First, we look at $I_1(x,t)$. Note that if $\underline{t} > 0$ then $I_1(x,t)$ vanishes for some large N. Thus, we can assume that $\underline{t} = 0$. Applying Lemma 4.4, for every $\varepsilon > 0$, there is some constant $K > 0$ such that, for all $t \in \Lambda_T$,

$$\begin{aligned} I_1(x,t) &\leq \sum_{n=N+1}^{\infty} \Pr\bigg(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\theta_k} > x,\ \tau_n \leq t\bigg) \\ &\leq K \sum_{n=N+1}^{\infty} (1+\varepsilon)^n \Pr(X_1 \mathrm{e}^{-r\theta_1} > x,\ \tau_n \leq t). \end{aligned}$$

By assumption (A1), it follows that, uniformly for $t \in \Lambda_T$,

$$\begin{aligned} I_1(x,t) &\lesssim K \sum_{n=N+1}^{\infty} (1+\varepsilon)^n \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs_1}) h(s_1) \Pr(N_{t-s_1} \geq n-1) G(\mathrm{d}s_1) \\ &\leq K \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs}) h(s) G(\mathrm{d}s) \sum_{n=N+1}^{\infty} (1+\varepsilon)^n \Pr(N_T \geq n-1). \end{aligned}$$

It is well known that the moment generating function of N_T is analytic in a neighborhood of 0; see, e.g. Stein (1946). Thus, we may choose some $\varepsilon > 0$ sufficiently small such that the series in the last step above converges. Therefore, for every $0 < \delta < 1$, we can find some large positive integer N such that, uniformly for $t \in \Lambda_T$,

$$I_1(x,t) \lesssim \delta \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s. \tag{4.16}$$

Next, we turn to $I_2(x,t)$. By Lemma 4.2, it holds uniformly for $t \in \Lambda_T$ that

$$I_2(x,t) \sim \bigg(\sum_{n=1}^{\infty} - \sum_{n=N+1}^{\infty}\bigg) \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = n) = I_{21}(x,t) - I_{22}(x,t). \tag{4.17}$$

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For $I_{21}(x, t)$, by interchanging the order of the sums and then conditioning on τ_{k-1} and θ_k, we obtain, uniformly for $t \in \Lambda_T$,

$$\begin{aligned}
I_{21}(x, t) &= \sum_{k=1}^{\infty} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ \tau_k \le t) \\
&\sim \sum_{k=1}^{\infty} \int_{0-}^{t} \int_{0-}^{t-u} \bar{F}(x\mathrm{e}^{r(u+v)}) \Pr(\tau_{k-1} \in \mathrm{d}v) h(u) G(\mathrm{d}u) \\
&= \int_{0-}^{t} \left(\bar{F}(x\mathrm{e}^{ru}) + \int_{0-}^{t-u} \bar{F}(x\mathrm{e}^{r(u+v)})\, \mathrm{d}\lambda_v \right) h(u) G(\mathrm{d}u) \\
&= \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs}) h(s) G(\mathrm{d}s) + \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\left(\int_{0-}^{s} \lambda_{s-u} h(u) G(\mathrm{d}u) \right) \\
&= \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s,
\end{aligned} \tag{4.18}$$

where for the fourth step we used integration by parts with possible jumps; see, e.g. Equation (1.20) of Klebaner (2005). For $I_{22}(x, t)$, it holds uniformly for $t \in \Lambda_T$ that

$$\begin{aligned}
I_{22}(x, t) &= \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x,\ N_t = n) \\
&\lesssim \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs_k}) \Pr(N_{t-s_k} = n-1) h(s_k) G(\mathrm{d}s_k) \\
&= \sum_{n=N+1}^{\infty} n \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs}) \Pr(N_{t-s} = n-1) h(s) G(\mathrm{d}s) \\
&\le \mathrm{E}(1 + N_T)\, \mathbf{1}_{\{N_T \ge N\}} \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs}) h(s) G(\mathrm{d}s).
\end{aligned}$$

Hence, we can find some large positive integer N such that, uniformly for $t \in \Lambda_T$,

$$I_{22}(x, t) \lesssim \delta \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s. \tag{4.19}$$

Substituting (4.18) and (4.19) into (4.17) yields, uniformly for $t \in \Lambda_T$,

$$(1 - \delta) \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s \lesssim I_2(x, t) \lesssim \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\, \mathrm{d}\tilde{\lambda}_s. \tag{4.20}$$

Substituting (4.16) and (4.20) into (4.15) and noting the arbitrariness of δ, we complete the proof of Theorem 2.1.

5. Proof of Theorem 2.2

5.1. Lemmas

The following lemma describes the closure of the class ERV under the product of two dependent random variables.

Lemma 5.1. *Recall the renewal risk model introduced in Section 2 with* $r \geq 0$*. If* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \leq \beta < \infty$ *and assumption (A2) holds, then for every* $k = 1, 2, \ldots$*, the distribution of* $X_k \mathrm{e}^{-r\tau_k}$ *still belongs to* $\mathrm{ERV}(-\alpha, -\beta)$ *and*

$$\Pr(X_k \mathrm{e}^{-r\tau_k} > x) \asymp \bar{F}(x). \tag{5.1}$$

Proof. For every $k = 1, 2, \ldots$, conditioning on τ_{k-1} and θ_k, and recalling assumption (A2), we have

$$\Pr(X_k \mathrm{e}^{-r\tau_k} > x) \sim \int_{0-}^{\infty} \int_{0-}^{\infty} \bar{F}(x \mathrm{e}^{r(u+v)}) h(u) G(\mathrm{d}u) \Pr(\tau_{k-1} \in \mathrm{d}v). \tag{5.2}$$

It follows from (5.2) and (2.2) that, for every $y \geq 1$,

$$\limsup_{x\to\infty} \frac{\Pr(X_k \mathrm{e}^{-r\tau_k} > xy)}{\Pr(X_k \mathrm{e}^{-r\tau_k} > x)} \leq \limsup_{x\to\infty} \sup_{s\geq 0} \frac{\bar{F}(xy\mathrm{e}^{rs})}{\bar{F}(x\mathrm{e}^{rs})} \leq y^{-\alpha}$$

and

$$\liminf_{x\to\infty} \frac{\Pr(X_k \mathrm{e}^{-r\tau_k} > xy)}{\Pr(X_k \mathrm{e}^{-r\tau_k} > x)} \geq \liminf_{x\to\infty} \inf_{s\geq 0} \frac{\bar{F}(xy\mathrm{e}^{rs})}{\bar{F}(x\mathrm{e}^{rs})} \geq y^{-\beta}.$$

Therefore, the distribution of $X_k \mathrm{e}^{-r\tau_k}$ belongs to $\mathrm{ERV}(-\alpha, -\beta)$. Moreover, applying (2.4) to (5.2), we obtain relation (5.1).

The following lemma is interesting in its own right and it will be the main ingredient of the proof of Theorem 2.2.

Lemma 5.2. *Recall the renewal risk model introduced in Section 2 with* $r > 0$*. If* $F \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \leq \beta < \infty$ *and assumption (A2) holds, then it holds uniformly for* $n = 1, 2, \ldots$ *that*

$$\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x\right) \sim \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x). \tag{5.3}$$

Proof. By Lemma 5.1, the distributions of $X_k \mathrm{e}^{-r\tau_k}$, $k = 1, 2, \ldots$, belong to $\mathrm{ERV}(-\alpha, -\beta)$ and, for $1 \leq i \neq j \leq n$,

$$\Pr(X_i \mathrm{e}^{-r\tau_i} > x,\ X_j \mathrm{e}^{-r\tau_j} > x) \leq (\bar{F}(x))^2 = o(\Pr(X_i \mathrm{e}^{-r\tau_i} > x) + \Pr(X_j \mathrm{e}^{-r\tau_j} > x)).$$

Hence, it follows from Theorem 3.1 of Chen and Yuen (2009) that relation (5.3) holds for every fixed $n = 1, 2, \ldots$.

Now we turn to the required uniformity of relation (5.3). Trivially, it holds for every $k = n+1, n+2, \ldots$ that $X_k \mathrm{e}^{-r\tau_k} \leq X_k \mathrm{e}^{-r\tau_{k-1}}$, where X_k and $\mathrm{e}^{-r\tau_{k-1}}$ on the right-hand side are independent. Hence, following the same lines of the proof of Theorem 3.1 of Tang and Tsitsiashvili (2004) with some obvious modifications (see also Section 4 of Chen and Ng (2007)), we obtain

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{1}{\bar{F}(x)} \Pr\left(\sum_{k=n+1}^{\infty} X_k \mathrm{e}^{-r\tau_k} > x\right) = \lim_{n\to\infty} \limsup_{x\to\infty} \sum_{k=n+1}^{\infty} \frac{\Pr(X_k \mathrm{e}^{-r\tau_k} > x)}{\bar{F}(x)} = 0.$$

This means that, for every $\delta > 0$, there is some large positive integer n_0 such that

$$\Pr\left(\sum_{k=n_0+1}^{\infty} X_k \mathrm{e}^{-r\tau_k} > x\right) + \sum_{k=n_0+1}^{\infty} \Pr(X_k \mathrm{e}^{-r\tau_k} > x) \lesssim \delta \bar{F}(x). \tag{5.4}$$

By relation (5.3) with $n = n_0$, the first assertion of Lemma 5.1, and relation (5.4), it holds, for arbitrarily fixed $0 < \varepsilon < 1$ and uniformly for $n > n_0$, that

$$\begin{aligned}\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x\right) &\le \Pr\left(\sum_{k=1}^{n_0} X_k \mathrm{e}^{-r\tau_k} > (1-\varepsilon)x\right) + \Pr\left(\sum_{k=n_0+1}^{\infty} X_k \mathrm{e}^{-r\tau_k} > \varepsilon x\right) \\ &\lesssim (1-\varepsilon)^{-\beta} \sum_{k=1}^{n_0} \Pr(X_k \mathrm{e}^{-r\tau_k} > x) + \delta \varepsilon^{-\beta} \bar{F}(x) \\ &\le (1-\varepsilon)^{-\beta} \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x) + \delta \varepsilon^{-\beta} \bar{F}(x).\end{aligned} \tag{5.5}$$

Symmetrically, it holds uniformly for $n > n_0$ that

$$\begin{aligned}\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x\right) &\ge \Pr\left(\sum_{k=1}^{n_0} X_k \mathrm{e}^{-r\tau_k} > x\right) \\ &\sim \left(\sum_{k=1}^{n} - \sum_{k=n_0+1}^{n}\right) \Pr(X_k \mathrm{e}^{-r\tau_k} > x) \\ &\gtrsim \sum_{k=1}^{n} \Pr(X_k \mathrm{e}^{-r\tau_k} > x) - \delta \bar{F}(x).\end{aligned} \tag{5.6}$$

By relation (5.1) and the arbitrariness of ε and δ, we conclude from (5.5) and (5.6) that relation (5.3) holds uniformly for $n > n_0$. The uniformity of relation (5.3) for $1 \le n \le n_0$ is obvious since it holds for every fixed $n = 1, 2, \ldots$. This completes the proof of Lemma 5.2.

5.2. Proof of Theorem 2.2

Following the proof of Lemma 4.2 of Hao and Tang (2008), we have

$$\lim_{t\to\infty} \limsup_{x\to\infty} \frac{\int_t^{\infty} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s}{\int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s} = 0.$$

Thus, for every $\delta > 0$, there is some $T_0 \in \Lambda$ such that

$$\int_{T_0}^{\infty} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s \lesssim \delta \int_{0-}^{T_0} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s. \tag{5.7}$$

On the one hand, by Theorem 2.1 and relation (5.7), it holds uniformly for $t \in (T_0, \infty]$ that

$$\begin{aligned}\Pr(D_r(t) > x) &\ge \Pr(D_r(T_0) > x) \\ &\sim \left(\int_{0-}^{t} - \int_{T_0}^{t}\right) \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s \\ &\gtrsim (1-\delta) \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s.\end{aligned} \tag{5.8}$$

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On the other hand, by Lemma 5.2 and assumption (A2),

$$\begin{aligned}\Pr(D_r(\infty) > x) &\sim \sum_{k=1}^{\infty} \Pr(X_k \mathrm{e}^{-r\tau_k} > x) \\ &= \Pr(X_1 \mathrm{e}^{-r\theta_1} > x) + \int_{0-}^{\infty} \Pr(X_k \mathrm{e}^{-r\theta_k} > x\mathrm{e}^{rv})\,\mathrm{d}\lambda_v \\ &\sim \int_{0-}^{\infty} \bar{F}(x\mathrm{e}^{rs})h(s)G(\mathrm{d}s) + \int_{0-}^{\infty}\int_{0-}^{\infty} \bar{F}(x\mathrm{e}^{r(u+v)})h(u)G(\mathrm{d}u)\,\mathrm{d}\lambda_v \\ &= \int_{0-}^{\infty} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s. \end{aligned} \tag{5.9}$$

Hence, by (5.9) and (5.7), it holds uniformly for $t \in (T_0, \infty]$ that

$$\begin{aligned}\Pr(D_r(t) > x) &\le \Pr(D_r(\infty) > x) \\ &\sim \bigg(\int_{0-}^{t} + \int_{t}^{\infty}\bigg) \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s \\ &\lesssim (1+\delta) \int_{0-}^{t} \bar{F}(x\mathrm{e}^{rs})\,\mathrm{d}\tilde{\lambda}_s. \end{aligned} \tag{5.10}$$

By the arbitrariness of δ in (5.8) and (5.10), we prove the uniformity of (2.7) for $t \in (T_0, \infty]$. Recall that Theorem 2.1 already shows the local uniformity of (2.7). This completes the proof of Theorem 2.2.

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