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A DUAL VIEW OF THE CLIFFORD THEORY OF CHARACTERS OF FINITE GROUPS, II

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Introduction. This paper continues the analysis of Clifford theory for the case of a finite group G, K a normal subgroup of G and G/K abelian which was developed in [**7**]. In [**7**] the permutation actions of G/K on the characters of K and of $(G/K)^\wedge$ on the characters of G were studied in relation to their effects on induction and restriction of group characters. With χ an irreducible character of G, and σ an irreducible component of $\chi|_K$, the chain of subgroups $K \subseteq J(\chi) \subseteq I(\sigma) \subseteq G$ was investigated, where $I = I(\sigma)$ is the usual inertial subgroup for σ and $J = J(\chi)$ is a subgroup called the dual inertial group for χ. Corresponding to the orbit of χ under $(G/K)^\wedge$ and of σ under G/K we investigated a tableau of characters on J. In this paper a similar tableau is developed for I. A further subgroup M, called an intermediary subgroup, is introduced with $J \leqq M \leqq I$ which has the property that σ extends to a character ρ of M and $\rho^G = \chi$. There are in fact $e_K(\chi)$ such choices for ρ forming one orbit under the actions of I/M and of $(M/J)^\wedge$. (Here, the two types of actions are observed on the same set of characters.) The permutations involved are in fact identical, which leads to an isomorphism of I/M and $(M/J)^\wedge$. Thus also $I/M \cong M/J$. M is not unique and an example is given with two intermediary subgroups M_1, M_2 with $M_1/J \ncong M_2/J$.

Since writing [**7**], the author has become aware that some of the results on "dual Clifford theory" had been previously established in [**4**, Section 4] and [**5**, Section 3]; see also the more recent article [**9**, (Section 1)]. It should further be remarked that Dade, using a somewhat different approach, has also investigated the special properties of Clifford theory for G/K abelian in [**1**, Chapter 3].

1. Background. The notation in this paper will be the same as in [**7**]. All groups are finite and all characters come from representations over the complex numbers. As in [**7**], K is a normal subgroup of G, and the paper in general is concerned with the case that G/K is abelian.

As is well-known (see for example [**3**, Chapter V, Section 17]), if χ is an irreducible character of G and σ an irreducible component of $\chi|_K$ then the usual Clifford decomposition gives

$$(1) \qquad \chi|_K = e_K(\chi) \sum_{i=1}^{m} \sigma^{g_i}$$

where $g_1, \ldots, g_m$ are coset representatives for G modulo $I(\sigma)$, the inertial

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group of σ, and $e_K(\chi)$ is Clifford's index of ramification of χ with respect to K. If G/K is abelian, it follows from [**7** (see Theorem 3.2, (vi) for example)] that given σ, $e_K(\chi) = e_K(\chi')$ for any irreducible character χ' of G whose restriction to K contains σ. Hence $e = e_K(\chi)$ might be regarded dually as the "ramification index of σ with respect to G", independent of choice of χ.

G/K acts by conjugation on the characters of K while $(G/K)\hat{}$, the dual group of one dimensional characters of G/K, acts by multiplication on the characters of G. In Section 2 of [**7**], the effects of these actions on induction and restriction of group characters were compared and can be summarized in the following scheme:

THEOREM 1.1. *Let G/K be abelian. Let χ be an irreducible character of G, σ an irreducible character of K.*

I. (a) *$(G/K)\hat{}$ operates faithfully on $\chi \Leftrightarrow \chi|_K$ is irreducible.*

(b) *$(G/K)\hat{}$ fixes $\chi \Leftrightarrow e\chi$ is induced from an irreducible character of K. Here e is the ramification index of χ with respect to K.*

II. (a) *G/K operates faithfully on $\sigma \Leftrightarrow \sigma^G$ is irreducible.*

(b) *G/K fixes $\sigma \Leftrightarrow e\sigma$ is the restriction of an irreducible character of G. Here e is the ramification index of σ with respect to G.*

Proof. I(a) is Corollary 2.6, I(b) is Theorem 2.1, and II(a) is Theorem 2.7 (these references being to [**7**]). II(b) is seen as follows: Let χ be an irreducible character of G such that σ is a component of $\chi|_K$. Then in Equation (1), G/K fixes $\sigma \Leftrightarrow m = 1 \Leftrightarrow \chi|_K = \sigma$.

Remark. If G/K is cyclic, then it is known that $e = 1$ ([**7**, Lemma 1.1]; see [**2**, Theorem 9.12] for proof). Thus Theorem 1.1 in this case becomes precisely the "summary for G/K cyclic" given in [**7**, bottom of p. 260].

The following lemma will be useful later in the article.

LEMMA 1.2. *Let G/K be abelian, L any subgroup between G and K. Let $\lambda \in (L/K)\hat{}$ and $g \in G$. Then $\lambda^g = \lambda$.*

Proof. If $x \in L$ and $g \in G$, then $gxg^{-1} = kx$, $k \in K$ and $\lambda^g(x) = \lambda(gxg^{-1}) = \lambda(kx) = \lambda(k)\lambda(x) = \lambda(x)$.

In [**7**], the dual inertial group $J(\chi)$ with respect to K was defined as follows: Let $H(\chi) = \{\lambda \in (G/K)\hat{}\,|\lambda\chi = \chi\}$. Then

$$J(\chi) = \bigcap \{\text{Ker}\,\lambda : \lambda \in H(\chi)\}.$$

Various theorems concerning $J = J(\chi)$ were established in [**7**]. For example, $(I : J) = e^2$ where $I = I(\sigma)$ and $e = e_K(\chi) = e_J(\chi)$. There is a unique irreducible character ψ of J which is a component of $\chi|_J$ and such that $\psi|_K = \sigma$ and $\psi^G = e\chi$. This notation will be used throughout this paper.

2. The lower and upper tableaux. Let G/K be abelian, χ, ψ, σ, $I = I(\sigma)$, $J = J(\chi)$ be as described in Section 1. In [**7**, Theorem 3.2 (i)] it was observed that $I(\sigma) = I(\psi)$, the inertial subgroup of ψ in G. The following theorem gives a similar result for J.

THEOREM 2.1. *Let τ be any irreducible component of $\chi|_I$. Let $J(\tau)$ be the dual inertial group of τ* (defined with respect to the groups I and K). *Then $J(\tau) = J(\chi)$.*

Proof. By [**2**, Theorem 9.11] applied to χ on G and ψ on J there exists an irreducible character θ on $I = I(\psi)$ such that $\theta^G = \chi$ and ψ is a component of $\theta|_J$. Further $e_J(\chi) = e_J(\theta)$. Denote this as e, as usual. Then $\theta|_J = e\psi$ (Formula (1) of Section 1). And, $(\psi^I, \theta) = (\psi, \theta|_J) = e$, so $\deg \psi^I = [I : J] \deg \psi = e^2 \deg \psi = e \deg \theta$, so that $\psi^I = e\theta$.

This means that $(I/J)\hat{}$ fixes θ (by Theorem 1.1, I(b)). Letting $H(\theta) =$ subgroup of $(I/K)\hat{}$ which fixes θ, we have $(I/J)\hat{} \subseteq H(\theta)$. Then

$$J(\theta) = \cap \{\text{Ker } \lambda : \lambda \in H(\theta)\} \subseteq \cap \{\text{Ker } \lambda : \lambda \in (I/J)\hat{}\} = J(\chi).$$

But $[I : J(\chi)] = e^2 = [I : J(\theta)]$ by [**7**, Theorem 3.2 (iii)] so $J(\theta) = J(\chi)$.

Any irreducible component τ of $\chi|_I$ is of the form θ^g, $g \in G$ and $J(\theta) = J(\theta^g)$ is seen, using Lemma 1.2, as follows:

If $\lambda \in (I/K)\hat{}$, $\lambda\theta = \theta \Leftrightarrow (\lambda\theta)^g = \theta^g \Leftrightarrow \lambda^g\theta^g = \theta^g \Leftrightarrow \lambda\theta^g = \theta^g$. So $H(\theta) = H(\theta^g)$ and $J(\theta) = J(\theta^g)$.

COROLLARY 2.2. *With the same notation as above we have:*

(a) *there exists an irreducible character θ such that $\psi^I = e\theta$, $\theta|_J = e\psi$ and $\theta^G = \chi$;*

(b) $J(\lambda\theta) = J(\theta) = J(\theta^g) = J(\chi)$ *for any $\lambda \in (I/K)\hat{}$ and $g \in G$.*

Proof. For the first part of (b) note that if $\lambda' \in (I/K)\hat{}$, then $\lambda'\theta = \theta \Leftrightarrow \lambda'\lambda\theta = \lambda\theta$ so $H(\lambda\theta) = H(\theta)$ and $J(\lambda\theta) = J(\theta)$. The rest of the corollary was seen in the course of the proof of Theorem 2.1.

In [**7**] an m by r tableau (henceforth to be called the *lower tableau*) of distinct characters of J was described. Let $g_1, \ldots, g_m$ be a set of coset representatives of G modulo I. Then $\{\sigma_i = \sigma^{g_i} : i = 1, \ldots m\}$ is the orbit of σ under G/K (we may assume that $\sigma_1 = \sigma$). Let $\tau_1, \ldots, \tau_r$ be elements of $(G/K)\hat{}$ such that $\{\chi_i = \tau_i\chi : i = 1, \ldots, r\}$ is the set of (distinct) elements of the orbit of χ under $(G/K)\hat{}$ (we may assume $\chi_1 = \chi$). Let $\tau_i|_J = \lambda_i$, $i = 1, \ldots, r$. Then to each pair (χ_j, σ_i) is associated the unique character ψ_{ij} in the ith row and jth column of the lower tableau (this is a small change in notation from [**7**]) such that $\psi_{ij}|_K = \sigma_i$ and $(\psi_{ij})^G = e\chi_j$ ($i = 1, \ldots, m; j = 1, \ldots, r$). In fact $\psi_{ij} = (\lambda_j\psi)^{g_i} = \lambda_j\psi^{g_i}$ (note Lemma 1.2). There is considerable symmetry concerning this tableau; for example $J = J(\chi) = J(\chi_j)$ for $j = 1, \ldots, r$; $I = I(\sigma) = I(\sigma_i)$ $i = 1, \ldots, m$; $I = I(\psi) = I(\psi_{ij})$ for all i, j; $e = e_K(\chi) = e_K(\chi_j) = e_J(\chi_j)$ $j = 1, \ldots, r$ etc. And clearly, Theorem 2.1 and Corollary 2.2 apply with respect to any χ_i, σ_j, and the corresponding ψ_{ij}. Hence there corresponds to ψ_{ij} an irreducible character θ_{ij} on such that $(\psi_{ij})^I = e\theta_{ij}$,

$\theta_{ij}|_J = e\psi_{ij}$ and $(\theta_{ij})^G = \chi_i$. The rm distinct characters of I, θ_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, r$ form the *upper tableau*. Its properties, similar to those of the lower tableau, are described in the next theorem.

THEOREM 2.3. (a) *Let* $\lambda_j' = \tau_j|_I$, $j = 1, \ldots, r$. *Then* $\theta_{ij} = (\lambda_j'\theta)^{g_i} = \lambda_j'\theta^{g_i}$ *where* $\theta = \theta_{11}$ *corresponds to* χ *and* σ.

(b) *The elements of the jth column of the upper tableau are the distinct irreducible components of* $\chi_j|_I$ *and they form a faithful orbit under the action of* G/I.

(c) *The rows of the upper tableau are the orbits under the action of* $(I/K)\hat{}$.

Proof. (a) Since $\psi^I = e\theta$ and $\theta|_J = e\psi$, $(\lambda'_j\theta^{g_i})|_J = \lambda_j(e\psi^{g_i}) = e\psi_{ij}$, so by Frobenius Reciprocity $(\psi^I{}_{ij}, \lambda_j'\theta^{g_i}) = e$. Since $\psi^I{}_{ij} = e\theta_{ij}$ clearly $\theta_{ij} = \lambda_j'\theta^{g_i}$. This equals $(\lambda_j'\theta)^{g_i}$ by Lemma 1.2.

(b) Since $\theta^G = \chi$, $\chi|_I = \sum_{i=1}^m \theta^{g_i}$ and hence $\chi_j|_I = \tau_j\chi|_I = \lambda_j'\sum\theta^{g_i} = \sum_i\theta_{ij}$.

(c) The $r = [J : K]$ elements in any row clearly belong to an orbit under $(I/K)\hat{}$. Applying [**7**, Theorem 3.2] to θ^{g_i} and I (in place of χ and G) we see that $[J(\theta^{g_i}) : K]$ equals the size of the orbit under $(I/K)\hat{}$. This equals $[J : K]$ by Corollary 2.2 so the ith row forms the complete orbit.

3. The intermediary subgroups.

Definition. Let G be any finite group, K a normal subgroup, χ an irreducible character of G, and σ an irreducible component of $\chi|_K$. Let M be a subgroup such that $K \subseteq M \subseteq G$ with a character ρ such that $\rho|_K = \sigma$ and $\rho^G = \chi$. We call M an *intermediary subgroup* for χ and σ, with intermediary character ρ.

THEOREM 3.1. *Let* G/K *be abelian,* χ *an irreducible character of* G *and* σ *an irreducible component of* $\chi|_K$. *Then there exists an intermediary subgroup* M *for* χ *and* σ.

Proof. By induction on $[G : K]$. As noted earlier, we have $K \subseteq J \subseteq I \subseteq G$ and an irreducible character ψ on J such that ψ is an extension of σ and $\psi^G = e\chi$. By Corollary 2.2 (or directly from [**2**, Theorem 9.11]) there exists an irreducible character θ on I such that $\psi^I = e\theta$, $\theta|_J = e\psi$ and $\theta^G = \psi$. If either $I \neq G$ or $J \neq K$, then $[I : J] < [G : K]$ and by induction there exists a subgroup M, $J \subseteq M \subseteq I$ and character ρ on M such that ρ extends ψ and $\rho^I = \theta$. Then clearly ρ also extends σ and $\rho^G = (\rho^I)^G = \theta^G = \chi$.

Suppose now that $G = I$ and $K = J$. If $(I/J)\hat{}$ were cyclic, then I/J would also be cyclic and hence $I = J$ by [**7**, Theorem 3.4] and the case is trivial. Otherwise, let H be a non-trivial proper cyclic subgroup of $(I/J)\hat{}$ and let N be the subgroup of I such that $(N/J)^\perp = H$. Then $(I/N)\hat{} \cong H$ (see [**7**, Lemma 1.2]). Thus I/N is cyclic and $e_N(\chi) = 1$ (see "Remark" in section 1). Since $(I/N)\hat{} \subseteq (I/J)\hat{}$ and the elements of $(I/J)\hat{}$ all fix χ, we have that $\gamma^G = e_N(\chi)\chi = \chi$ where γ is an irreducible component of $\chi|_N$, by Theorem 1.1, I(b). Now $\chi|_J = e\sigma$ where $e = e_J(\chi)$ so σ is also a component of $\gamma|_J$. By induction, there exists a subgroup M, $J \subseteq M \subseteq N$, and an irreducible character ρ on M such that σ extends to ρ and $\rho^N = \gamma$. Hence $\rho^I = \gamma^I = \chi$.

Remark. Independently of the author, it has recently been shown by I. M. Isaacs and David Price that Theorem 3.1 (the existence of intermediary subgroups) in fact holds for a considerably wider class of groups G/K, including the case that G/K is supersolvable (see [**6**]).

THEOREM 3.2. *Let G/K be abelian with notation as above. Let M be any intermediary subgroup (with character ρ) for χ and σ. Then $J(\chi) \subseteq M \subseteq I(\sigma)$. Further if ψ is the unique irreducible character of J with $\psi^G = e\chi$ and $\psi|_J = \sigma$, and $\psi^I = e\theta$ (i.e., ψ and θ are the elements of the lower and upper tableaux corresponding to χ and σ as in Section 2) then M is an intermediary subgroup for θ and ψ with character ρ.*

Proof. Since ρ extends σ to M, if $g \in M$, $\rho^g = \rho$ so $\sigma^g = \sigma$, and $M \subseteq I(\sigma) = I$. Since $\rho^G = \chi$, if $\lambda \in (G/M)^\wedge$, $\lambda\chi = \chi$ hence

$$M = \cap \{\text{Ker}\,\lambda : \lambda \in (G/M)^\wedge\} \supseteq \cap \{\text{Ker}\,\lambda : \lambda \in (G/K)^\wedge \text{and } \lambda \text{ fixes } \chi\} = J(\chi).$$

Since ρ extends σ to M, $\rho|_J$ is an extension of σ to J, hence equals one of the characters in the top line of the lower tableau: i.e. $\rho|_J = \psi_{1j}$, say. Since $\rho^G = \chi$, $\chi|_M$ contains ρ, hence also ψ_{1j}. By [**7**, Theorem 3.1] $j = 1$ and $\psi_{1j} = \psi_{11} = \psi$, so $\rho|_J = \psi$. Since $\rho^G = \chi$ is irreducible, ρ^I is an irreducible character of I whose restriction to I contains ψ, hence $\rho^I = \theta$.

LEMMA 3.3. *Under the same hypotheses as in Theorem* 3.2, $[I : M] = [M : J] = e$.

Proof. $[I : J] = e^2$ by [**7**, Theorem 3.2 (iii)]. Since $\rho|_J = \psi$, $\rho^I = \theta$, $\psi^I = e\theta$ and $\theta|_J = e\psi$, we have $e \deg \psi = \deg \theta = \deg \rho^I = [I : M] \deg \psi$. Thus $[I : M] = e$ and $[M : J] = [I : J]/[I : M] = e$.

THEOREM 3.4. *Let G/K be abelian, χ an irreducible character on K, σ an irreducible component of $\chi|_K$. Let M be an intermediary subgroup for χ and σ. Then there exist precisely e intermediary characters $\rho_1, \ldots, \rho_e$ for χ and σ on M. They form an orbit under both the actions of I/M by conjugation and $(M/J)^\wedge$ by multiplication. Under these actions I/M and $(M/J)^\wedge$ are represented faithfully by the same regular permutation group; thus $I/M \cong (M/J)^\wedge$.*

Proof. Let ρ be an intermediary character for χ and σ on M. By Theorem 3.2 we may regard M as intermediary subgroup for ψ and θ where ψ is an irreducible character of $J = J(\chi) = J(\theta)$ (by Theorem 2.1) and θ is an irreducible character of $I = I(\sigma) = I(\psi)$. Also $\rho|_J = \psi$, $\rho^I = \theta$. Since ρ^I is irreducible, I/M operates faithfully on ρ yielding the orbit $\{\rho_1, \ldots, \rho_e\}$ with $e = [I : M]$ elements and these are the only characters which induce to θ. Since ρ is an extension of ψ, $(M/J)^\wedge$ operates faithfully on ρ, forming an orbit of $e = [M : J]$ elements which are precisely the extension of ψ to M. Let $\gamma \in (M/J)^\wedge$. Then

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$\gamma\rho$ is an element of the latter orbit. Extend γ to $\gamma' \in (I/J)^\wedge$. Then $(\gamma\rho)^I = \gamma'\rho^I = \gamma'\theta = \theta$ (by definition of $J = J(\theta)$). Hence $\gamma\rho = \rho^h$ some $h \in I$. We label the elements of $(M/J)^\wedge$ here $\gamma_1, \ldots, \gamma_e$ (so that $\gamma_i\rho = \rho_i$). Then choose coset representatives $h_1, \ldots, h_e$ of I modulo M such that $\gamma_i\rho = \rho^{h_i}$, $i = 1, 2, \ldots, e$. That this bijective correspondence of $(M/J)^\wedge$ with I/M is an isomorphism is seen as follows: Suppose that $h_ih_j = h_k$ modulo M. Since $(\gamma_i)^{h_i} = (\gamma_i)$ by Lemma 1.2, we have

$$\gamma_k\rho = \rho^{h_k} = \rho^{(h_ih_j)} = (\rho^{h_i})^{h_j} = (\gamma_i\rho)^{h_j} = (\gamma_i)^{h_j}\rho^{h_j} = \gamma_i\rho^{h_j} = \gamma_i\gamma_j\rho.$$

Hence $\gamma_k = \gamma_i\gamma_j$, since $(M/J)^\wedge$ acts faithfully on ρ. (Note: the initial choice of terminology in [**7**], i.e. that $\sigma^g(x) = \sigma(gxg^{-1})$ causes I/M apparently to act here as a permutation group on the right, but since I/M is abelian it may also be interpreted as affording permutations on the left).

COROLLARY 3.5. $I/M \cong M/J$.

THEOREM 3.6. *Let G/K be abelian and M an intermediary subgroup for χ and σ. Then for each χ_j in the orbit of χ under $(G/K)^\wedge$ and each σ_i in the orbit of σ under G/K, M is an intermediary subgroup with e intermediary characters. M is an intermediary subgroup for each corresponding pair θ_{ij} and ψ_{ij} from the upper and lower tableaux.*

Proof. Let ρ be a character on M which is intermediary for χ and σ. Then by Theorem 3.2 it is also intermediary for θ and ψ. Following the notation of Theorem 2.3, we let $\lambda_j' = \tau_j|_I$, $j = 1, \ldots, r$. Further let $\lambda_j'' = \lambda_j'|_M$, $j = 1, \ldots, r$. $\rho|_J = \psi$ so $\lambda_j''\rho^{g_i}|_J = \lambda_j\psi^{g_i} = \psi_{ij}$. It may be verified that $(\rho^{g_i})^I = (\rho^I)^{g_i}$. Then since $\rho^I = \theta$, $(\lambda_j''\rho^{g_i})^I = \lambda_j'(\rho^{g_i})^I = \lambda_j'(\rho^I)^{g_i} = \lambda_j'\theta^{g_i} = \theta_{ij}$. Thus $\lambda_j''\rho^{g_i}$ is an intermediary character for θ_{ij} and ψ_{ij}. It follows that it is also an intermediary character for χ_j and σ_i.

Remark. If S_{ij} denotes the set of e intermediary characters on M for θ_{ij} and ψ_{ij} then the union of the S_{ij} is a set S of *erm* characters on M corresponding to the entire tableaux. Each character σ_i has exactly *er* extensions to M $(\cup S_{ij} : i = 1, \ldots, r)$ while there are precisely *em* characters on M $(\cup S_{ij} : i = 1, \ldots, m)$ which give χ_j when induced to G.

4. An Example. It is known that $I/J \cong H_1 \times H_2$ (direct product) with $H_1 \cong H_2$ (see [**5**, p. 126]) and since $I/M \cong M/J$, where M is an intermediary subgroup, this suggests that I/J might be expressed as a direct product $(M/J) \times H_2$ with $H_2 \cong M/J$. That this is not true in general is shown by the following example.

Let G be a group of order 64, generated by elements a, b, c with $ac = ca$. $bc = cb$, $a^4 = c^2$, $b^4 = 1$, $b^{-1}ab = ac$ (group number 180 in [**8**]). Let $J = \langle c\rangle$, the subgroup generated by c. Let ψ be the linear character of $\langle c\rangle$ given by $\psi(c) = i$ and extend ψ to ρ on $M_1 = \langle a^2, b^2, c\rangle$ by setting $\rho(b^2) = 1$, $\rho(a^2) = -i$.

Then it can be checked that $\rho^G = \theta$ is irreducible, $G = I(\psi) = I$, M_1 is an intermediary subgroup for θ and ψ and I/J is not the direct product of M_1/J with any other subgroup. However if $M_2 = \langle a, c \rangle$, then it is now easily seen that if ρ' is any extension of ψ to M_2, we will have $(\rho')^G = \theta$ so that M_2 is also an intermediary subgroup. In this case, $I/J = (M_2/J) \times H/J$ with $H = \langle b, c \rangle$. Further, $M_1/J \not\cong M_2/J$ (the first being the Klein Four-group, while the second one is cyclic). [Added in proof: $c^4 = 1$; $J = J(\theta)$]

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