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Abstract

We prove a Knörrer-periodicity-type equivalence between derived factorization categories of gauged Landau–Ginzburg models, which is an analogy of a theorem proved by Shipman and Isik independently. As an application, we obtain a gauged Landau–Ginzburg version of Orlov's theorem describing a relationship between categories of graded matrix factorizations and derived categories of hypersurfaces in projective spaces, by combining the above Knörrer periodicity type equivalence and the theory of variations of geometric invariant theory quotients due to Ballard, Favero and Katzarkov.

1. Introduction

1.1 Background and motivation

When X is a scheme, G is an affine algebraic group acting on X, $\chi : G \to \mathbb{G}_m$ is a character, and $W : X \to \mathbb{A}^1$ is a χ-semi-invariant regular function, we call data $(X, \chi, W)^G$ a *gauged Landau–Ginzburg (LG) model*. Following Positselski [Pos11, EP15], we consider *the derived factorization category* of $(X, \chi, W)^G$, denoted by

$$\mathrm{Dcoh}_G(X, \chi, W).$$

Derived factorization categories are simultaneous generalizations of bounded derived categories of coherent sheaves on schemes, and of categories of (graded) matrix factorizations of (homogeneous) polynomials. Orlov proved the following semi-orthogonal decompositions between bounded derived categories of hypersurfaces in projective spaces and categories of graded matrix factorizations [Orl09].

Theorem 1.1 [Orl09, Theorem 40]. *Let $X \subset \mathbb{P}_k^{N-1}$ be the hypersurface defined by a section $f \in \Gamma(\mathbb{P}_k^{N-1}, \mathcal{O}(d))$. Denote by F the corresponding homogeneous polynomial.*

(1) *If $d < N$, there is a semi-orthogonal decomposition*

$$\mathrm{D^b}(\mathrm{coh}X) = \langle \mathcal{O}_X(d - N + 1), \ldots, \mathcal{O}_X, \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F)\rangle.$$

(2) *If $d = N$, there is an equivalence*

$$\mathrm{D^b}(\mathrm{coh}X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F).$$

(3) *If $d > N$, there is a semi-orthogonal decomposition*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F) = \langle k, \ldots, k(N - d + 1), \mathrm{D^b}(\mathrm{coh}X)\rangle.$$

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While Orlov's approach was algebraic, there are geometric approaches to the above theorem [Shi12, BFK12, BDFIK14], where a version of Knörrer periodicity [Shi12, Isi13] and homological variations of geometric invariant theory (GIT) quotients [Seg11, Hal15, BFK12] are the main tools. Combinations of Knörrer periodicity and the theory of variations of GIT quotients also imply homological projective dualities [BDFIK13, ADS15, ST14, Ren15].

In this article, we prove another version of Knörrer periodicity [Knö87], which is a derived (or global) version, and we combine it with the theory of variations of GIT quotients by [BFK12] to obtain a gauged LG version of Orlov's theorem.

1.2 Main results

Let k be an algebraically closed field of characteristic zero. Let X be a smooth quasi-projective variety over k, and let G be a reductive affine algebraic group acting on X. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of finite rank, and choose a G-invariant regular section $s \in \Gamma(X, \mathcal{E}^\vee)^G$. Denote by $Z \subset X$ the zero scheme of s. Let $\chi : G \to \mathbb{G}_m$ be a character of G, and set $\mathcal{E}(\chi) := \mathcal{E} \otimes \mathcal{O}(\chi)$, where $\mathcal{O}(\chi)$ is the G-equivariant invertible sheaf corresponding to χ. Then $\mathcal{E}(\chi)$ induces a vector bundle $\mathrm{V}(\mathcal{E}(\chi))$ over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi)$. Let $q : \mathrm{V}(\mathcal{E}(\chi)) \to X$ and $p : \mathrm{V}(\mathcal{E}(\chi))|_Z \to Z$ be natural projections, and let $i : \mathrm{V}(\mathcal{E}(\chi))|_Z \to \mathrm{V}(\mathcal{E}(\chi))$ be a natural inclusion. The regular section s induces a χ-semi-invariant regular function $Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1$. The first main result in this paper is the following.

Theorem 1.2. *Let $W : X \to \mathbb{A}^1$ be a χ-semi-invariant regular function, such that the restricted function $W|_Z : Z \to \mathbb{A}^1$ is flat. Then there is an equivalence*

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W|_Z) \xrightarrow{\sim} \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, q^*W + Q_s).$$

The above result is an analogy of Shipman's and Isik's result, where they consider the case when $G = \mathbb{G}_m$, the G-action on X is trivial, $\chi = \mathrm{id}_{\mathbb{G}_m}$, and $W = 0$ (see [Shi12, Isi13]). Furthermore, the above theorem can be considered as a generalization of Knörrer periodicity to a derived and G-equivariant version. The proof of the above theorem is quite different from Shipman's and Isik's proofs, and we consider relative singularity categories introduced in [EP15], which are equivalent to derived factorization categories, and use results from [Orl06].

To state the next result, let S be a smooth quasi-projective variety over k with a $\mathbb{G}_m$-action, and let $W : S \to \mathbb{A}^1$ be a $\chi_1 := \mathrm{id}_{\mathbb{G}_m}$-semi-invariant regular function which is flat. Let $d > 1$ and $N > 0$ be positive integers, and consider $\mathbb{G}_m$-actions on $\mathbb{A}^N_S := S \times \mathbb{A}^N_k$ and on $\mathbb{P}^{N-1}_S := S \times \mathbb{P}^{N-1}_k$ given by

$$\mathbb{G}_m \times \mathbb{A}^N_S \ni t \times (s, v_1, \ldots, v_N) \mapsto (t^d \cdot s, tv_1, \ldots, tv_N) \in \mathbb{A}^N_S,$$
$$\mathbb{G}_m \times \mathbb{P}^{N-1}_S \ni t \times (s, v_1 : \cdots : v_N) \mapsto (t \cdot s, v_1 : \cdots : v_N) \in \mathbb{P}^{N-1}_S.$$

Denote by the same notation $W : \mathbb{A}^N_S \to \mathbb{A}^1$ and $W : \mathbb{P}^{N-1}_S \to \mathbb{A}^1$ the pull-backs of $W : S \to \mathbb{A}^1$ by the natural projections, respectively. Combining the above-derived Knörrer periodicity with the theory of variations of GIT quotients, we obtain the following gauged LG version of the Orlov's theorem.

Theorem 1.3. *Let $X \subset \mathbb{P}^{N-1}_S$ be the hypersurface defined by a $\mathbb{G}_m$-invariant section $f \in \Gamma(\mathbb{P}^{N-1}_S, \mathcal{O}(d))^{\mathbb{G}_m}$, and assume that the morphism $W : \mathbb{P}^{N-1}_S \to \mathbb{A}^1$ is flat on X. Denote by $F : \mathbb{A}^N_S \to \mathbb{A}^1$ the regular function induced by f.*

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(1) *If $d < N$, there are fully faithful functors*

$$\Phi : \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) = \langle \Upsilon_{d-N+1}, \ldots, \Upsilon_0, \Phi(\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F))\rangle,$$

where Υ_i denotes the essential image of the composition $(-)\otimes\mathcal{O}(i)\circ\Upsilon$.

(2) *If $d = N$, there is an equivalence*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F).$$

(3) *If $d > N$, there are fully faithful functors*

$$\Psi : \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F),$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W+F) = \langle \Upsilon_0, \ldots, \Upsilon_{N-d+1}, \Psi(\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W))\rangle,$$

where Υ_i denotes the essential image of the composition $(-)\otimes\mathcal{O}(\chi_i)\circ\Upsilon$.

Since we have an equivalence

$$\mathrm{D^b}(\mathrm{coh}X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, 0),$$

where the $\mathbb{G}_m$-action on X is trivial, we can view Orlov's theorem as the case when $S = \operatorname{Spec} k$ and $W = 0$ in the above theorem.

1.3 Notation and conventions

- For an integer $n \in \mathbb{Z}$, we denote by $\chi_n : \mathbb{G}_m \to \mathbb{G}_m$ the character of the one-dimensional algebraic torus $\mathbb{G}_m$ defined by
$$\chi_n(t) := t^n.$$
- For a character $\chi : G \to \mathbb{G}_m$ of an algebraic group G, we denote by $\mathcal{O}(\chi)$ the G-equivariant invertible sheaf $(\mathcal{O}_X, \lambda : \pi^*\mathcal{O}_X \xrightarrow{\sim} \sigma^*\mathcal{O}_X)$ associated to χ, where $\pi : G\times X \to X$ and $\sigma : G \times X \to X$ are the projection and the morphism defining the G-action, respectively. For any $g \in G$, $\lambda_g := \lambda|_{\{g\}\times X} : \mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$ is given as the composition
$$\mathcal{O}_X \xrightarrow{\chi(g)} \mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$$
of the multiplication by $\chi(g) \in \mathbb{G}_m$ and the natural isomorphism $\mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$.

For a G-equivariant quasi-coherent sheaf F on a G-scheme, we set
$$F(\chi) := F \otimes \mathcal{O}(\chi).$$
- Throughout this article, unless stated otherwise, all schemes and categories are over an algebraically closed field k of characteristic zero.

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2. Derived factorization categories

In this section, we provide the definition of the derived factorization category, which is introduced by Positselski (cf. [Pos11, EP15]), and its basic properties.

2.1 Derived factorization categories

Definition 2.1. A *gauged LG model* is data $(X,\chi,W)^G$ with X a scheme, $\chi : G\to\mathbb{G}_m$ a character of G, G an affine algebraic group acting on X, and $W : X\to\mathbb{A}^1$ a χ-semi-invariant regular function, i.e. $W(g\cdot x)=\chi(g)W(x)$ for any $g\in G$ and any $x\in X$. If G is trivial, we denote the gauged LG model by (X,W), and call it an *LG model*.

For a gauged LG model, we consider its factorizations which are 'twisted' complexes.

Definition 2.2. Let $(X,\chi,W)^G$ be a gauged LG model. A *factorization* F of $(X,\chi,W)^G$ is a sequence

$$F=(F_1\xrightarrow{\varphi_1^F}F_0\xrightarrow{\varphi_0^F}F_1(\chi)),$$

where F_i is a G-equivariant quasi-coherent sheaf on X and φ_i^F is a G-invariant homomorphism for $i=0,1$ such that $\varphi_0^F\circ\varphi_1^F=W\cdot\mathrm{id}_{F_1}$ and $\varphi_1^F(\chi)\circ\varphi_0^F=W\cdot\mathrm{id}_{F_0}$. Equivariant quasi-coherent sheaves F_0 and F_1 in the above sequence are called *components* of the factorization F.

Definition 2.3. For a gauged LG model $(X,\chi,W)^G$, we define a differential graded (dg) category

$$\mathrm{Qcoh}_G(X,\chi,W)$$

whose objects are factorizations of $(X,\chi,W)^G$, and whose complexes of morphisms are defined as follows: for two objects $E,F\in\mathrm{Qcoh}_G(X,\chi,W)$, we define the complex $\mathrm{Hom}(E,F)^\bullet$ of morphisms from E to F as the following graded vector space

$$\mathrm{Hom}(E,F)^\bullet:=\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}(E,F)^n$$

with a differential $d^i:\mathrm{Hom}(E,F)^i\to\mathrm{Hom}(E,F)^{i+1}$ given by

$$d^i(f):=\varphi^F\circ f-(-1)^if\circ\varphi^E$$

where

$$\mathrm{Hom}(E,F)^{2m}:=\mathrm{Hom}(E_1,F_1(\chi^m))\oplus\mathrm{Hom}(E_0,F_0(\chi^m)),$$
$$\mathrm{Hom}(E,F)^{2m+1}:=\mathrm{Hom}(E_1,F_0(\chi^m))\oplus\mathrm{Hom}(E_0,F_1(\chi^{m+1})).$$

We define dg full subcategories $\mathrm{coh}_G(X,\chi,W)$, $\mathrm{Inj}_G(X,\chi,W)$, $\mathrm{LFr}_G(X,\chi,W)$, and $\mathrm{lfr}_G(X,\chi,W)$ of $\mathrm{Qcoh}_G(X,\chi,W)$ whose objects are factorizations whose components are coherent, injective quasi-coherent, locally free, and locally free of finite ranks, respectively. If G is trivial, dropping G and χ from each notation, we denote the dg categories by $\mathrm{Qcoh}(X,W)$, $\mathrm{coh}(X,W)$, etc.

The dg category $\mathrm{Qcoh}_G(X,\chi,W)$ induces two categories

$$Z^0(\mathrm{Qcoh}_G(X,\chi,W))\quad\text{and}\quad H^0(\mathrm{Qcoh}_G(X,\chi,W)).$$

Objects of these categories are same as objects of $\mathrm{Qcoh}_G(X, \chi, W)$, and sets of morphisms are defined as

$$\mathrm{Hom}_{Z^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F) := Z^0(\mathrm{Hom}(E, F)^\bullet) = \mathrm{Ker}(d^0),$$
$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F) := H^0(\mathrm{Hom}(E, F)^\bullet) = \mathrm{Ker}(d^0)/\mathrm{Im}(d^{-1}).$$

Remark 2.4. We can write down the above sets of morphisms as follows: $\mathrm{Hom}_{Z^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F)$ is the set of pairs (f_1, f_0) such that $f_i \in \mathrm{Hom}_{\mathrm{Qcoh}_G X}(E_i, F_i)$ and that the following diagram is commutative.

$$\begin{array}{ccccc} E_1 & \xrightarrow{\varphi_1^E} & E_0 & \xrightarrow{\varphi_0^E} & E_1(\chi) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1(\chi)} \\ F_1 & \xrightarrow{\varphi_1^F} & F_0 & \xrightarrow{\varphi_0^F} & F_1(\chi) \end{array}$$

The set of morphisms in the category $H^0(\mathrm{Qcoh}_G(X, \chi, W))$ can be described as the set of homotopy equivalence (denoted by $\sim$) classes in $\mathrm{Hom}_{Z^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F)$,

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F) = \mathrm{Hom}_{Z^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F)/\sim,$$

where two morphisms $f = (f_1, f_0)$ and $g = (g_1, g_0)$ in $\mathrm{Hom}_{Z^0(\mathrm{Qcoh}_G(X,\chi,W))}(E, F)$ are *homotopy equivalent* if there exist two morphisms

$$h_0 : E_0 \to F_1 \quad \text{and} \quad h_1 : E_1(\chi) \to F_0$$

such that $f_0 = \varphi_1^F h_0 + h_1 \varphi_0^E$ and $f_1(\chi) = \varphi_0^F h_1 + h_0(\chi)\varphi_1^E(\chi)$.

We easily see the following result (cf. [Hir17, Proposition 3.5]).

Proposition 2.5. *The categories* $Z^0(\mathrm{Qcoh}_G(X, \chi, W))$ *and* $Z^0(\mathrm{coh}_G(X, \chi, W))$ *are abelian, and the categories* $Z^0(\mathrm{LFr}_G(X, \chi, W))$ *and* $Z^0(\mathrm{lfr}_G(X, \chi, W))$ *are exact.*

We next define the totalizations of bounded complexes of factorizations.

Definition 2.6. Let $F^\bullet = (\cdots \to F^i \xrightarrow{\delta^i} F^{i+1} \to \cdots)$ be a bounded complex of $Z^0(\mathrm{Qcoh}_G(X, \chi, W))$. For $l = 0, 1$, set

$$T_l := \bigoplus_{i+j=-l} F^i_{\overline{j}}(\chi^{\lceil j/2 \rceil}),$$

and let

$$t_l : T_l \to T_{\overline{l+1}}$$

be a G-invariant homomorphism given by

$$t_l|_{F^i_{\overline{j}}(\chi^{\lceil j/2 \rceil})} := \delta^i_{\overline{j}}(\chi^{\lceil j/2 \rceil}) + (-1)^i \varphi^{F^i}_{\overline{j}}(\chi^{\lceil j/2 \rceil}),$$

where $\overline{n}$ is n modulo 2, and $\lceil m \rceil$ is the minimum integer which is greater than or equal to a real number m. We define the *totalization* $\mathrm{Tot}(F^\bullet) \in Z^0(\mathrm{Qcoh}_G(X, \chi, W))$ of $F^\bullet$ as

$$\mathrm{Tot}(F^\bullet) := (T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} T_1(\chi)).$$

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In what follows, we will recall that the category $H^0(\mathrm{Qcoh}_G(X,\chi,W))$ has a structure of a triangulated category.

Definition 2.7. We define an automorphism T on $H^0(\mathrm{Qcoh}_G(X,\chi,W))$, which is called the *shift functor*, as follows. For an object $F \in H^0(\mathrm{Qcoh}_G(X,\chi,W))$, we define an object $T(F)$ as

$$T(F) := (F_0 \xrightarrow{-\varphi_0^F} F_1(\chi) \xrightarrow{-\varphi_1^F(\chi)} F_0(\chi))$$

and for a morphism $f = (f_1, f_0) \in \mathrm{Hom}(E,F)$, we set $T(f) := (f_0, f_1(\chi)) \in \mathrm{Hom}(T(E), T(F))$. For any integer $n \in \mathbb{Z}$, denote by $(-)[n]$ the functor $T^n(-)$.

Definition 2.8. Let $f : E \to F$ be a morphism in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. We define its *mapping cone* $\mathrm{Cone}(f)$ to be the totalization of the complex

$$(\cdots \to 0 \to E \xrightarrow{f} F \to 0 \to \cdots)$$

with F in degree zero.

A *distinguished triangle* is a sequence in $H^0(\mathrm{Qcoh}_G(X,\chi,W))$ which is isomorphic to a sequence of the form

$$E \xrightarrow{f} F \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} E[1],$$

where i and p are natural injection and projection, respectively.

The following is well known.

Proposition 2.9. *We denote by $H^0(\mathrm{Qcoh}_G(X,\chi,W))$ a triangulated category with respect to its shift functor and its distinguished triangles defined above. Full subcategories $H^0(\mathrm{coh}_G(X,\chi,W))$, $H^0(\mathrm{Inj}_G(X,\chi,W))$, $H^0(\mathrm{LFr}_G(X,\chi,W))$, and $H^0(\mathrm{lfr}_G(X,\chi,W))$ are full triangulated subcategories.*

Following Positselski [Pos11, EP15], we define derived factorization categories.

Definition 2.10. Denote by $\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W))$ the smallest thick subcategory of $H^0(\mathrm{coh}_G(X,\chi,W))$ containing all totalizations of short exact sequences in $Z^0(\mathrm{coh}_G(X,\chi,W))$. We define the *derived factorization category* of $(X,\chi,W)^G$ as the Verdier quotient

$$\mathrm{Dcoh}_G(X,\chi,W) := H^0(\mathrm{coh}_G(X,\chi,W))/\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W)).$$

Similarly, consider thick full subcategories $\mathrm{Acycl}(\mathrm{Qcoh}_G(X,\chi,W))$, $\mathrm{Acycl}(\mathrm{LFr}_G(X,\chi,W))$ and $\mathrm{Acycl}(\mathrm{lfr}_G(X,\chi,W))$ of $H^0(\mathrm{Qcoh}_G(X,\chi,W))$, $H^0(\mathrm{LFr}_G(X,\chi,W))$, and $H^0(\mathrm{lfr}_G(X,\chi,W))$ respectively, and denote the corresponding Verdier quotients by

$$\mathrm{DQcoh}_G(X,\chi,W) := H^0(\mathrm{Qcoh}_G(X,\chi,W))/\mathrm{Acycl}(\mathrm{Qcoh}_G(X,\chi,W)),$$
$$\mathrm{DLFr}_G(X,\chi,W) := H^0(\mathrm{LFr}_G(X,\chi,W))/\mathrm{Acycl}(\mathrm{LFr}_G(X,\chi,W)),$$
$$\mathrm{Dlfr}_G(X,\chi,W) := H^0(\mathrm{lfr}_G(X,\chi,W))/\mathrm{Acycl}(\mathrm{lfr}_G(X,\chi,W)).$$

Objects in $\mathrm{Acycl}(\mathrm{Qcoh}_G(X,\chi,W))$ are called *acyclic*.

Denote by $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X,\chi,W))$ (respectively $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{LFr}_G(X,\chi,W))$) the smallest thick subcategory of the triangulated category $H^0(\mathrm{Qcoh}_G(X,\chi,W))$ (respectively $H^0(\mathrm{LFr}_G(X,\chi,W))$) which is closed under taking small direct sums and contain all totalizations of short exact

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sequences in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$ (respectively $Z^0(\mathrm{LFr}_G(X,\chi,W))$). Denote the Verdier quotients by

$$\mathrm{D^{co}Qcoh}_G(X,\chi,W) := H^0(\mathrm{Qcoh}_G(X,\chi,W))/\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W)),$$
$$\mathrm{D^{co}LFr}_G(X,\chi,W) := H^0(\mathrm{LFr}_G(X,\chi,W))/\mathrm{Acycl^{co}}(\mathrm{LFr}_G(X,\chi,W)).$$

Objects in $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W))$ are called *coacyclic*.

If G is trivial, we drop G and χ from the above notation, and denote each triangulated category by $\mathrm{Dcoh}(X,W)$, $\mathrm{DQcoh^{co}}(X,W)$, etc.

Remark 2.11. If X is a regular Noetherian scheme of finite Krull dimension, then $\mathrm{Acycl}(\mathrm{Qcoh}_G(X,\chi,W))$ is cocomplete, i.e. admits arbitrary direct sums (cf. [LS16, Corollary 2.23]). Hence, in that case, we have

$$\mathrm{D^{co}Qcoh}_G(X,\chi,W) = \mathrm{DQcoh}_G(X,\chi,W).$$

The following lemmas ensure the existence of derived functors between derived factorization categories.

LEMMA 2.12. *Assume that the scheme X is Noetherian. The natural functor*

$$H^0(\mathrm{Inj}_G(X,\chi,W)) \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)$$

is an equivalence.

Proof. Since the abelian category $\mathrm{Qcoh}_G X$ of G-equivariant quasi-coherent sheaves is a locally Noetherian Grothendieck category, it has enough injective objects, and coproducts of injective objects are injective. Hence, the result follows from [BDFIK16, Cororally 2.25]. □

LEMMA 2.13 [BFK14, Proposition 3.14]. *Assume that X is a smooth variety. Then the natural functor*

$$\mathrm{DLFr}_G(X,\chi,W) \to \mathrm{DQcoh}_G(X,\chi,W)$$

is an equivalence. This equivalence induces an equivalence

$$\mathrm{Dlfr}_G(X,\chi,W) \to \mathrm{Dcoh}_G(X,\chi,W).$$

2.2 Derived categories and derived factorization categories

In this section, we recall that derived factorization categories are generalizations of bounded derived categories of coherent sheaves on schemes.

Consider trivial $\mathbb{G}_m$-action on a scheme X and an exact functor between abelian categories

$$\Upsilon : \mathrm{Ch}(\mathrm{Qcoh}X) \to Z^0(\mathrm{Qcoh}_{\mathbb{G}_m}(X,\chi_1,0))$$

given by

$$\Upsilon(F^\bullet, d_F^\bullet) := \left(\bigoplus_{i\in\mathbb{Z}} F^{2i-1}(\chi_{-i}) \xrightarrow{\oplus d_F^{2i-1}(\chi_{-i})} \bigoplus_{i\in\mathbb{Z}} F^{2i}(\chi_{-i}) \xrightarrow{\oplus d_F^{2i}(\chi_{-i})} \bigoplus_{i\in\mathbb{Z}} F^{2i-1}(\chi_{-i+1})\right).$$

Then it is easy to see that the exact functor Υ is an equivalence, and it induces an equivalence between triangulated categories:

$$\Upsilon : \mathrm{K}(\mathrm{Qcoh}X) \to H^0(\mathrm{Qcoh}_{\mathbb{G}_m}(X,\chi_1,0)).$$

Since the triangulated equivalence preserves coacyclic objects, we obtain the following.

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Proposition 2.14. *There is an equivalence*

$$\Upsilon : \mathrm{D}^{\mathrm{co}}(\mathrm{Qcoh}X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_{\mathbb{G}_m}(X, \chi_1, 0),$$

which induces an equivalence between full subcategories

$$\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}X) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, 0).$$

Remark 2.15. See [Pos12, Appendix A] for the definition and basic properties of coderived categories. By [Pos12, Theorem 5.9.1(b)], if X is Noetherian, the thick subcategory $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}X)$ of $\mathrm{D}^{\mathrm{co}}(\mathrm{Qcoh}X)$ is the full subcategory of compact objects. However, when X is a singular variety, there is an object in $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}X)$ which is not compact in the usual derived category $\mathrm{D}(\mathrm{Qcoh}X)$. This is a remarkable difference between the usual derived category and the coderived category. On the other hand, if X is a smooth variety, these two kinds of derived categories are equivalent (see the argument in the proof of [Pos12, Theorem 5.5.1(c)]).

2.3 Derived functors between derived factorization categories

We quickly review derived functors between derived factorization categories. See, for example, [LS16, BFK14] or [Hir17] for more details.

2.3.1 *Direct images and inverse images.* Let X and Y be Noetherian schemes, and let G be an affine algebraic group acting on X and Y. Let $f : X \to Y$ be a G-equivariant morphism, and choose a χ-semi-invariant function $W : Y \to \mathbb{A}^1$.

The morphism f naturally induces a dg functor

$$f_* : \mathrm{Qcoh}_G(X, \chi, f^*W) \to \mathrm{Qcoh}_G(Y, \chi, W)$$

defined by

$$f_*F := \left(f_*(F_1) \xrightarrow{f_*(\varphi_1^F)} f_*(F_0) \xrightarrow{f_*(\varphi_0^F)} f_*(F_1)(\chi)\right).$$

By Lemma 2.12, we can derive the dg functor f_* to obtain an exact functor

$$\mathbf{R}f_* : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, f^*W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(Y, \chi, W).$$

If f is a proper morphism, it preserves factorizations whose components are coherent sheaves:

$$\mathbf{R}f_* : \mathrm{Dcoh}_G(X, \chi, f^*W) \to \mathrm{Dcoh}_G(Y, \chi, W).$$

The morphism f also induces another dg functor

$$f^* : \mathrm{Qcoh}_G(Y, \chi, W) \to \mathrm{Qcoh}_G(X, \chi, f^*W),$$

defined by

$$f^*E := \left(f^*(E_1) \xrightarrow{f^*(\varphi_1^E)} f^*(E_0) \xrightarrow{f^*(\varphi_0^E)} f^*(E_1)(\chi)\right).$$

If Y is a smooth variety, by Lemma 2.13, we have the derived functor of f^*

$$\mathbf{L}f^* : \mathrm{DQcoh}_G(Y, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, f^*W).$$

This functor maps coherent factorizations to coherent factorizations:

$$\mathbf{L}f^* : \mathrm{Dcoh}_G(Y, \chi, W) \to \mathrm{Dcoh}_G(X, \chi, f^*W).$$

It is standard that the direct image $\mathbf{R}f_*$ and the inverse image $\mathbf{L}f^*$ are adjoint.

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Remark 2.16. If f is an affine morphism, we do not need to take the derived functor, and the dg functor f_* naturally defines an exact functor

$$f_* : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, f^*W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(Y, \chi, W).$$

Similarly, if f is a flat morphism, we do not have to assume that Y is a smooth variety and take the derived functor, and the dg functor f^* naturally defines an exact functor

$$f^* : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(Y, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, f^*W).$$

2.3.2 *Tensor products.* Let $(X, \chi, W)^G$ be a gauged LG model, and let $V : X \to \mathbb{A}^1$ be another χ-semi-invariant regular function. Fix an object $F \in \mathrm{Qcoh}_G(X, \chi, V)$. We define a dg functor

$$(-) \otimes F : \mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{Qcoh}_G(X, \chi, W + V)$$

given by

$$E \otimes F := \left(\bigoplus_{i=0,1} (F_i \otimes E_{\overline{i+1}}) \xrightarrow{\varphi_1^{E \otimes F}} \bigoplus_{i=0,1} (F_i \otimes E_i)(\chi^i) \xrightarrow{\varphi_0^{E \otimes F}} \bigoplus_{i=0,1} (F_i \otimes E_{\overline{i+1}})(\chi) \right),$$

where $\overline{n}$ is n modulo 2, and

$$\varphi_1^{E \otimes F} = \begin{pmatrix} \varphi_1^E \otimes 1 & 1 \otimes \varphi_1^F \\ -1 \otimes \varphi_0^F & \varphi_0^E \otimes 1 \end{pmatrix}$$

and

$$\varphi_0^{E \otimes F} = \begin{pmatrix} \varphi_0^E \otimes 1 & -(1 \otimes \varphi_1^F)(\chi) \\ 1 \otimes \varphi_0^F & (\varphi_1^E \otimes 1)(\chi) \end{pmatrix}.$$

By Lemma 2.13, if X is a smooth variety, we have the derived functor

$$(-) \otimes^{\mathbf{L}} F : \mathrm{DQcoh}_G(X, \chi, W) \to \mathrm{DQcoh}_G(X, \chi, W + V).$$

If F is a coherent factorization, this functor preserves coherent factorizations:

$$(-) \otimes^{\mathbf{L}} F : \mathrm{Dcoh}_G(X, \chi, W) \to \mathrm{Dcoh}_G(X, \chi, W + V).$$

Consider a natural exact functor between abelian categories

$$\tau : \mathrm{Qcoh}_G(X) \to Z^0(\mathrm{Qcoh}_G(X, \chi, 0))$$

defined by

$$\tau(F) := (0 \to F \to 0).$$

Then we denote by Σ the following composition of functors

$$\Sigma : \mathrm{D^b}(\mathrm{Qcoh}_G X) \xrightarrow{\tau} \mathrm{D^b}(Z^0(\mathrm{Qcoh}_G(X, \chi, 0))) \xrightarrow{\mathrm{Tot}} \mathrm{DQcoh}_G(X, \chi, 0).$$

For a complex $F^\bullet \in \mathrm{D^b}(\mathrm{Qcoh}_G X)$, we define the tensor product

$$(-) \otimes^{\mathbf{L}} F^\bullet : \mathrm{DQcoh}_G(X, \chi, W) \to \mathrm{DQcoh}_G(X, \chi, W)$$

by the following

$$(-) \otimes^{\mathbf{L}} F^\bullet := (-) \otimes^{\mathbf{L}} \Sigma(F^\bullet).$$

Remark 2.17. If the components of F are flat sheaves, we do not have to assume that X is a smooth variety and take the derived functor, and the dg functor $(-) \otimes F$ induces an exact functor

$$(-) \otimes F : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W + V).$$

Furthermore, if F is a coherent factorization, the tensor product preserves coherent factorizations:

$$(-) \otimes F : \mathrm{Dcoh}_G(X, \chi, W) \to \mathrm{Dcoh}_G(X, \chi, W + V).$$

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2.3.3 *Integral functors.* We define integral functors between derived factorization categories. For simplicity, we consider the case when G is trivial. Let X_1 and X_2 be Gorenstein quasi-projective schemes, and let $W_i : X_i \to \mathbb{A}^1$ be a regular function. We denote the projection by $\pi_i : X_1 \times X_2 \to X_i$ for each $i = 1, 2$.

In order to define integral functors, we need the following lemma.

LEMMA 2.18 [EP15, Corollaries 2.3.e and 2.4.a]. *Let (X, W) be a LG model. Assume that the scheme X is a Gorenstein separated scheme of finite Krull dimension with an ample line bundle. Then the functor*

$$\mathrm{D^{co}LFr}(X, W) \to \mathrm{D^{co}Qcoh}(X, W)$$

induced by the embedding of dg functor $\mathrm{LFr}(X, W) \to \mathrm{Qcoh}(X, W)$ is an equivalence.

Now we define integral functors. Let $P \in \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_2^*W - \pi_1^*W)$ be an object. Since X_1 and X_2 are Gorenstein, so is $X_1 \times X_2$ (cf. [TY03]). By the above lemma, we have the derived tensor product

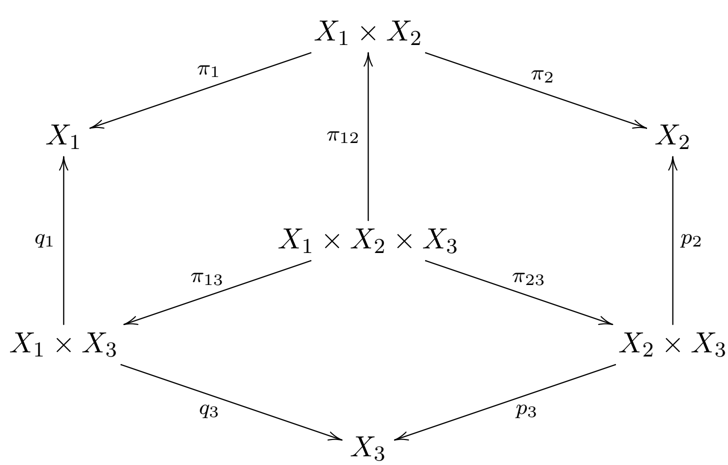
$$(-) \otimes^{\mathbf{L}} P : \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_1^*W) \to \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_2^*W).$$

DEFINITION 2.19. For an object $P \in \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_2^*W - \pi_1^*W)$, we define the *integral functor* with respect to P, denoted by Φ_P, as the following functor

$$\mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} P) : \mathrm{D^{co}Qcoh}(X_1, W_1) \to \mathrm{D^{co}Qcoh}(X_2, W_2).$$

We call the object P the *kernel* of the integral functor Φ_P.

In the following, we recall that the composition of integral functors is also an integral functor. Let X_3 be another Gorenstein quasi-projective scheme, and let $W_3 : X_3 \to \mathbb{A}^1$ be a regular function. Consider the following diagram



where all morphisms are projections. For two objects

$$P \in \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1),$$
$$Q \in \mathrm{D^{co}Qcoh}(X_2 \times X_3, p_3^*W_3 - p_2^*W_2),$$

we define another object

$$P \star Q := \mathbf{R}\pi_{13*}(\pi_{12}^*P \otimes^{\mathbf{L}} \pi_{23}^*Q) \in \mathrm{D^{co}Qcoh}(X_1 \times X_3, q_3^*W_3 - q_1^*W_1).$$

Proposition 2.20. *The notation is the same as above. The composition of integral functors*

$$\mathrm{D^{co}Qcoh}(X_1, W_1) \xrightarrow{\Phi_P} \mathrm{D^{co}Qcoh}(X_2, W_2) \xrightarrow{\Phi_Q} \mathrm{D^{co}Qcoh}(X_3, W_3)$$

is isomorphic to the following integral functor

$$\mathrm{D^{co}Qcoh}(X_1, W_1) \xrightarrow{\Phi_{P\star Q}} \mathrm{D^{co}Qcoh}(X_3, W_3).$$

Proof. The result follows from the projection formula and base change formula for derived factorization categories. □

2.4 Support properties of factorizations

Following [EP15, § 1.10], we consider set-theoretic supports of factorizations. In this section, X is a Noetherian scheme.

Definition 2.21. Let $(X, \chi, W)^G$ be a gauged LG model, and let $Z \subset X$ be a G-invariant closed subset of X. We say that a factorization $F \in \mathrm{Qcoh}_G(X, \chi, W)$ is *set-theoretically supported on* Z if the supports $\mathrm{Supp}(F_i)$ of components of F are contained in Z.

Denote by

$$\mathrm{Qcoh}_G(X, \chi, W)_Z$$

the dg subcategory of $\mathrm{Qcoh}_G(X, \chi, W)$ consisting of factorizations set-theoretically supported on Z. $H^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ is a full triangulated subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W))$. Denote by $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ the smallest thick subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ which is closed under small direct sums and contains all totalizations of short exact sequences in $Z^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$. Set

$$\mathrm{D^{co}Qcoh}_G(X, \chi, W)_Z := H^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)/\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X, \chi, W)_Z).$$

Similarly, denote by

$$\mathrm{coh}_G(X, \chi, W)_Z \quad \text{and} \quad \mathrm{Inj}_G(X, \chi, W)_Z$$

the dg subcategory of $\mathrm{coh}_G(X, \chi, W)$ and $\mathrm{Inj}_G(X, \chi, W)$, respectively, consisting of factorizations set-theoretically supported on Z. Let $\mathrm{Acycl}(\mathrm{coh}_G(X, \chi, W)_Z)$ be the smallest thick subcategory of $H^0(\mathrm{coh}_G(X, \chi, W)_Z)$ containing all totalizations of short exact sequences in $Z^0(\mathrm{coh}_G(X, \chi, W)_Z)$, and consider the Verdier quotient

$$\mathrm{Dcoh}_G(X, \chi, W)_Z := H^0(\mathrm{coh}_G(X, \chi, W)_Z)/\mathrm{Acycl}(\mathrm{coh}_G(X, \chi, W)_Z).$$

Lemma 2.22. *Let $A \in \mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ and $I \in H^0(\mathrm{Inj}_G(X, \chi, W)_Z)$. Then we have*

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(A, I) = 0.$$

Proof. Since arbitrary direct sums of short exact sequences are exact and the totalization functor commutes with arbitrary direct sums, it is enough to show that for a short exact sequence $A^\bullet : 0 \to A^1 \to A^2 \to A^3 \to 0$ in $Z^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$, we have $\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(\mathrm{Tot}(A^\bullet), I) = 0$. This follows from a similar argument as in the proof of [LS16, Lemma 2.13]. □

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By the above lemma, we see that every morphism from $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ to $\mathrm{Inj}_G(X,\chi,W)_Z$ factors through the zero object in $H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$. Hence, by [LS16, Proposition B.2], we have the following lemma.

Lemma 2.23. *Let $F \in H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ and $I \in H^0(\mathrm{Inj}_G(X,\chi,W)_Z)$. Then the natural map*

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(F,I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z}(F,I)$$

is an isomorphism.

Furthermore, we have the following result.

Lemma 2.24. *The natural functor*

$$H^0(\mathrm{Inj}_G(X,\chi,W)_Z) \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z$$

is an equivalence.

Proof. This follows from [BDFIK16, Cororally 2.25]. □

The following two propositions are G-equivariant versions of results in [EP15, § 1.10].

Proposition 2.25 (Cf. [EP15, Proposition 1.10]). (1) *The natural functor*

$$\mathrm{Dcoh}_G(X,\chi,W)_Z \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z$$

is fully faithful, and its image is a set of compact generators.

(2) *The natural functor*

$$\iota_Z : \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)$$

induced by the embedding of dg categories $\iota_Z : \mathrm{Qcoh}_G(X,\chi,W)_Z \to \mathrm{Qcoh}_G(X,\chi,W)$ is fully faithful.

(3) *The functor*

$$\iota_Z : \mathrm{Dcoh}_G(X,\chi,W)_Z \to \mathrm{Dcoh}_G(X,\chi,W)$$

induced by the embedding of dg categories $\iota_Z : \mathrm{coh}_G(X,\chi,W)_Z \to \mathrm{coh}_G(X,\chi,W)$ is fully faithful.

Proof. (1) It is enough to prove that any morphism $F \to A$ in $H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ from $F \in H^0(\mathrm{coh}_G(X,\chi,W)_Z)$ to $A \in \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ factors through some object in $\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W)_Z)$. This follows from a similar argument as in the proof of [LS16, Lemma 2.15].

We show that $\mathrm{Dcoh}_G(X,\chi,W)_Z$ generates $\mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z$ by using a similar discussion in the proof of [Pos11, Theorem 3.11.2]. By Lemmas 2.23 and 2.24, it is enough to show that for an object $I \in H^0(\mathrm{Inj}_G(X,\chi,W)_Z)$ if

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(F,I) = 0$$

for any $F \in \mathrm{coh}_G(X,\chi,W)_Z$, then $\mathrm{id}_I : I \to I$ is homotopic to zero. Consider the partially ordered set of pairs (M,h), where $M \subset I$ is a subfactorization of I and $h : M \to I$ is a contracting homotopy of the embedding $i : M \hookrightarrow I$, i.e. $d(h) = i$. By Zorn's lemma, the partially ordered set contains a maximal element. Hence, it suffices to show that given (M,h) with $M \neq I$, there exists (M', h') with $M \subsetneq M'$ and $h'|_M = h$. Take a subfactorization $M' \subset I$ such that $M \subsetneq M'$ and $M'/M \in \mathrm{coh}_G(X, \chi, W)_Z$. Since the components of I are injective sheaves, the morphism $h : M \to I$ of degree -1 can be extended to a morphism $h'' : M' \to I$. Denote by $i : M \hookrightarrow I$ and $i' : M' \hookrightarrow I$ the embeddings. Since the map $i' - d(h'')$ is a closed degree-zero morphism and vanishes on M, it induces a closed degree-zero morphism $g : M'/M \to I$. By the assumption, g is homotopic to zero, i.e. there exists a homotopy $c : M'/M \to I$ such that $d(c) = g$. Then $h' = h'' + c \circ p : M' \to I$ is a contracting homotopy for i' extending h, where $p : M' \to M'/M$ is the natural projection.

The compactness of objects in $\mathrm{Dcoh}_G(X, \chi, W)_Z$ follows from Lemmas 2.23 and 2.24. Parts (2) and (3) follows from Lemma 2.24 and part (1). □

Proposition 2.26 (Cf. [EP15, Theorem 1.10]). *Let $U := X \backslash Z$ be the complement of $Z \subset X$, and let $j : U \to X$ be the open immersion.*

(1) *The restriction*

$$j^* : \mathrm{D^{co}Qcoh}_G(X, \chi, W) \to \mathrm{D^{co}Qcoh}_G(U, \chi, W|_U)$$

is the Verdier localization by the thick subcategory $\mathrm{D^{co}Qcoh}_G(X, \chi, W)_Z$.

(2) *The restriction*

$$j^* : \mathrm{Dcoh}_G(X, \chi, W) \to \mathrm{Dcoh}_G(U, \chi, W|_U)$$

is the Verdier localization by the triangulated subcategory $\mathrm{Dcoh}_G(X, \chi, W)_Z$. The kernel of j^* is the thick envelope of $\mathrm{Dcoh}_G(X, \chi, W)_Z$ in $\mathrm{Dcoh}_G(X, \chi, W)$.

Proof. We can prove this by a standard discussion as in the proof of [EP15, Theorem 1.10].

(1) Since j^* has a right adjoint $\mathbf{R}j_* : \mathrm{D^{co}Qcoh}_G(U, \chi, W|_U) \to \mathrm{D^{co}Qcoh}_G(X, \chi, W)$ which is fully faithful, we see that j^* is the Verdier (Bousfield) localization by its kernel which is generated by cones of adjunctions $F \to \mathbf{R}j_* j^* F$ for any $F \in \mathrm{D^{co}Qcoh}_G(X, \chi, W)$.

We show that $\mathrm{Ker}(j^*) = \mathrm{D^{co}Qcoh}_G(X, \chi, W)_Z$. Since the inclusion $\mathrm{D^{co}Qcoh}_G(X, \chi, W)_Z \subset \mathrm{Ker}(j^*)$ is trivial, it is enough to show that the cone of the adjunction $F \to \mathbf{R}j_* j^* F$, for any $F \in \mathrm{D^{co}Qcoh}_G(X, \chi, W)$, is contained in $\mathrm{D^{co}Qcoh}_G(X, \chi, W)_Z$. By Lemma 2.12, we may take F as an factorization whose components are injective quasi-coherent sheaves. Then the adjunction comes from a closed morphism $F \to j_* j^* F$ in $Z^0(\mathrm{Qcoh}_G(X, \chi, W))$. Since its kernel and cokernel are objects in $\mathrm{Qcoh}_G(X, \chi, W)_Z$, so is the cone of the adjunction by an equivariant version of [LS16, Lemma 2.7.c].

(2) By Proposition 2.25(1) and [Nee92], we have a fully faithful functor

$$\overline{\mathrm{Dcoh}_G(X, \chi, W)}/\overline{\mathrm{Dcoh}_G(X, \chi, W)_Z} \longrightarrow \overline{\mathrm{Dcoh}_G(U, \chi, W|_U)},$$

where $\overline{(-)}$ denotes the idempotent completion of $(-)$. Since every morphism $D \to E$ from $D \in \mathrm{Dcoh}_G(X, \chi, W)$ to $E \in \overline{\mathrm{Dcoh}_G(X, \chi, W)_Z}$ factors through an object in $\mathrm{Dcoh}_G(X, \chi, W)_Z$, the natural functor

$$\mathrm{Dcoh}_G(X, \chi, W)/\mathrm{Dcoh}_G(X, \chi, W)_Z \to \overline{\mathrm{Dcoh}_G(X, \chi, W)}/\overline{\mathrm{Dcoh}_G(X, \chi, W)_Z}$$

is fully faithful. Hence, we see that the natural functor

$$\mathrm{Dcoh}_G(X, \chi, W)/\mathrm{Dcoh}_G(X, \chi, W)_Z \to \mathrm{Dcoh}_G(U, \chi, W|_U)$$

is also fully faithful. This functor is essentially surjective since for every G-equivariant coherent $F \in \mathrm{coh}_G U$ there exists a G-equivariant coherent sheaf $\overline{F} \in \mathrm{coh}_G X$ such that $j^* \overline{F} \cong F$ and the coherent sheaves generate $\mathrm{Dcoh}_G(U, \chi, W|_U)$ by [BDFIK16, Corollary 2.29]. □

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2.5 Koszul factorizations

Let $(X, \chi, W)^G$ be a gauged LG model such that X is a smooth variety. Let $\mathcal{E}$ be a G-equivariant locally free sheaf on X of rank r, and let

$$s : \mathcal{E} \to \mathcal{O}_X \quad \text{and} \quad t : \mathcal{O}_X \to \mathcal{E}(\chi)$$

be morphisms in $\mathrm{coh}_G X$ such that $t \circ s = W \cdot \mathrm{id}_{\mathcal{E}}$ and $s(\chi) \circ t = W$. Let $Z_s \subset X$ be the zero scheme of the section $s \in \Gamma(X, \mathcal{E}^\vee)^G$. We say that s is *regular* if the codimension of Z_s in X equals the rank r.

Definition 2.27. We define an object $K(s,t) \in \mathrm{lfr}_G(X, \chi, W)$ as

$$K(s,t) := (K_1 \xrightarrow{k_1} K_0 \xrightarrow{k_0} K_1(\chi))$$

where

$$K_1 := \bigoplus_{n=0}^{\lceil r/2 \rceil - 1} \Big(\bigwedge^{2n+1} \mathcal{E} \Big)(\chi^n), \quad K_0 := \bigoplus_{n=0}^{\lfloor r/2 \rfloor} \Big(\bigwedge^{2n} \mathcal{E} \Big)(\chi^n)$$

and

$$k_i := t \wedge (-) \oplus s \vee (-).$$

The following property will be necessary in § 4.1.

Lemma 2.28 [BFK14, Lemma 3.21 and Proposition 3.20]. (1) *have a natural isomorphism*

$$K(s,t)^\vee \cong K(t^\vee, s^\vee).$$

(2) *If s is regular, we have natural isomorphisms in* $\mathrm{Dcoh}_G(X, \chi, W)$,

$$\mathcal{O}_{Z_s} \cong K(s,t) \quad \text{and} \quad \mathcal{O}_{Z_s} \otimes \bigwedge^r \mathcal{E}^\vee(\chi^{-1})[-r] \cong K(s,t)^\vee,$$

where $\mathcal{O}_{Z_s} := (0 \to \mathcal{O}_{Z_s} \to 0)$ and $\bigwedge^r \mathcal{E}^\vee(\chi^{-1})[-r]$ is a complex in $\mathrm{coh}_G X$.

2.6 Restriction and induction functors

We construct restriction and induction functors. Let G be an affine algebraic group acting on a scheme X. Let $G \times^l X$ and $G \times^d X$ be schemes $G \times X$ with different G-actions which are defined as follows:

$$\begin{array}{ccc} G \times G \times^l X & \longrightarrow & G \times^l X \\ \Cup & & \Cup \\ (g, g', x) & \longmapsto & (gg', x) \end{array}$$

and

$$\begin{array}{ccc} G \times G \times^d X & \longrightarrow & G \times^d X \\ \Cup & & \Cup \\ (g, g', x) & \longmapsto & (gg', gx). \end{array}$$

Let $\iota : X \to G \times X$ be a morphism defined by

$$X \ni x \longmapsto (1_G, x) \in G \times X.$$

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We define an exact functor $\iota^* : \mathrm{Qcoh}_G(G \times^l X) \to \mathrm{Qcoh}X$ as

$$\begin{array}{ccc} \mathrm{Qcoh}_G(G \times^l X) & \longrightarrow & \mathrm{Qcoh}X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\mathcal{F}, \theta) & \longmapsto & \iota^*\mathcal{F}. \end{array}$$

Since $G \times^l X$ is a principal G-bundle over X, the above functor ι^* is an equivalence.

The following morphisms

$$\begin{array}{ccc} \varphi : G \times^l X & \longrightarrow & G \times^d X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, x) & \longmapsto & (g, gx) \end{array}$$

and

$$\begin{array}{ccc} \pi : G \times^d X & \longrightarrow & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, x) & \longmapsto & x \end{array}$$

are G-equivariant. The G-action $\sigma : G \times X \to X$ on X is the composition $\pi \circ \varphi$. Since the morphism φ is an isomorphism, the following functors are equivalences:

$$\varphi^* : \mathrm{Qcoh}_G(G \times^d X) \xrightarrow{\sim} \mathrm{Qcoh}_G(G \times^l X),$$
$$\varphi_* : \mathrm{Qcoh}_G(G \times^l X) \xrightarrow{\sim} \mathrm{Qcoh}_G(G \times^d X).$$

Since the morphism π is flat and affine, the following functors are exact:

$$\pi^* : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(G \times^d X),$$
$$\pi_* : \mathrm{Qcoh}_G(G \times^d X) \to \mathrm{Qcoh}_G(X).$$

Definition 2.29. We define the *restriction functor* $\mathrm{Res}_G : \mathrm{Qcoh}_G X \to \mathrm{Qcoh}X$ and the *induction functor* $\mathrm{Ind}_G : \mathrm{Qcoh}X \to \mathrm{Qcoh}_G X$ as

$$\mathrm{Res}_G := \iota^* \circ \sigma^* \quad \text{and} \quad \mathrm{Ind}_G := \sigma_* \circ (\iota^*)^{-1}.$$

Remark 2.30. (1) The restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G X \to \mathrm{Qcoh}X$ is isomorphic to the forgetful functor, i.e. $\mathrm{Res}_G(\mathcal{F}, \theta) \cong \mathcal{F}$.

(2) Although the functor Res_G sends equivariant coherent sheaves to coherent sheaves for arbitrary group G, the functor Ind_G does not preserve coherent sheaves if G is not a finite group.

Since ι^* is an equivalence, the adjoint pair $\sigma^* \dashv \sigma_*$ induces the adjoint pair

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

For a χ-semi-invariant regular function $W : X \to \mathbb{A}^1$, the restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G X \to \mathrm{Qcoh}X$ and the induction functor $\mathrm{Ind}_G : \mathrm{Qcoh}X \to \mathrm{Qcoh}_G X$ induce dg functors between factorization categories, which we denote by the same notation:

$$\mathrm{Res}_G : \mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{Qcoh}(X, W),$$
$$\mathrm{Ind}_G : \mathrm{Qcoh}(X, W) \to \mathrm{Qcoh}_G(X, \chi, W).$$

These functors are also adjoint to each other:

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

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Since the restriction and the induction functors are exact, we obtain the following induced functors between bounded derived categories

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh} X),$$
$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh} X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)$$

and the functors between coderived factorization categories

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(X, W),$$
$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(X, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W).$$

The adjunction of the restriction and the induction functors between $\mathrm{Qcoh}_G X$ and $\mathrm{Qcoh} X$ induces the adjunction of the above induced functors between triangulated categories:

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

We recall the definition of (linearly) reductiveness of algebraic groups.

Definition 2.31. Let H be an affine algebraic group over a field K:

(1) H is called *reductive* if the radical of H is a torus;
(2) H is called *linearly reductive* if every rational representations of H over K is completely reducible.

The following says that the above two properties are equivalent if the characteristic of the base field is zero.

Proposition 2.32 [MFK94, Appendix A]. *Let H be an affine algebraic group over a field K of characteristic zero. Then H is reductive if and only if it is linearly reductive.*

Lemma 2.33. *Assume that G is linearly reductive.*

(1) *The adjunction morphism*

$$\varepsilon : \mathrm{id}_{\mathrm{Qcoh}_G X} \to \mathrm{Ind}_G \circ \mathrm{Res}_G$$

is a split mono, i.e. there exists a functor morphism $\eta : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{Qcoh}_G X}$ such that $\eta \circ \varepsilon = \mathrm{id}$. The adjunction morphism

$$\mathrm{id}_{\mathrm{Qcoh}_G(X,\chi,W)} \to \mathrm{Ind}_G \circ \mathrm{Res}_G$$

is also a split mono.

(2) *The restriction functors*

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh} X),$$
$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(X, W)$$

are faithful.

Proof. (1) This follows from the argument in the proof of [Hir17, Lemma 4.56].

(2) We will prove that the upper functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh} X)$ is faithful; the proof of the faithfulness of the lower functor is similar. The functor morphism $\eta : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{Qcoh}_G X}$ constructed in part (1) naturally induces the functor morphism $\overline{\eta} : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)}$ such that the composition with the adjunction morphism

$$\mathrm{id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)} \to \mathrm{Ind}_G \circ \mathrm{Res}_G \xrightarrow{\overline{\eta}} \mathrm{id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)}$$

is the identity. Hence, any morphism f in $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)$ factors through $\mathrm{Ind}_G \circ \mathrm{Res}_G(f)$, and so $f = 0$ if $\mathrm{Res}(f) = 0$. □

3. Relative singularity categories

Relative singularity categories are introduced in [EP15], and it is shown that derived factorization categories (with some conditions on regular functions) are equivalent to relative singularity categories. In this section, we recall the definition and properties of relative singularity categories.

3.1 Triangulated categories of relative singularities

Let X be a quasi-projective scheme, and let G be an affine algebraic group acting on X. Throughout this section, we assume that X *has a G-equivariant ample line bundle*. If X is normal, this condition is satisfied by [Tho87, Lemma 2.10]. The equivariant triangulated category of singularities $\mathrm{D}^{\mathrm{sg}}_G(X)$ of X is defined as the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X)$ by the thick subcategory $\mathrm{Perf}_G(X)$ of equivariant perfect complexes. Following [Orl04], we consider a larger category $\mathrm{D}^{\mathrm{cosg}}_G(X)$ defined as the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)$ by the thick subcategory $\mathrm{Lfr}_G(X)$ of complexes which is quasi-isomorphic to bounded complexes of equivariant locally free sheaves (not necessarily of finite ranks). If G is trivial, we denote the singularity categories by $\mathrm{D}^{\mathrm{cosg}}(X)$ or $\mathrm{D}^{\mathrm{sg}}(X)$.

We recall relative singularity categories following [EP15]. Let $Z \subset X$ be a G-invariant closed subscheme of X such that $\mathcal{O}_Z$ *has finite G-flat dimension as an $\mathcal{O}_X$-module*, i.e. the G-equivariant sheaf $\mathcal{O}_Z \in \mathrm{coh}_G(X)$ has a finite resolution $F^\bullet \to \mathcal{O}_Z$ of G-equivariant flat sheaves on X. Under the assumption, we have the derived inverse image $\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$ between bounded derived categories for the closed immersion $i : Z \to X$. This functor preserves complexes of coherent sheaves: $\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh} X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh} Z)$.

Definition 3.1 [EP15, § 2.1]. We define the following Verdier quotients

$$\mathrm{D}^{\mathrm{cosg}}_G(Z/X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)/\langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z))\rangle^{\oplus},$$
$$\mathrm{D}^{\mathrm{sg}}_G(Z/X) := \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z)/\langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z))\rangle,$$

where $\langle - \rangle$ (respectively $\langle - \rangle^{\oplus}$) denotes the smallest thick subcategory containing objects in $(-)$ (respectively and closed under infinite direct sums which exist in $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$). The quotient category $\mathrm{D}^{\mathrm{sg}}_G(Z/X)$ is called the *equivariant triangulated category of singularities of Z relative to X*. If G is trivial, we denote the categories defined above by $\mathrm{D}^{\mathrm{cosg}}(Z/X)$ or $\mathrm{D}^{\mathrm{sg}}(Z/X)$.

Proposition 3.2. *Assume that G is reductive. We have natural Verdier localizations by thick subcategories*

$$\pi^{\mathrm{co}} : \mathrm{D}^{\mathrm{cosg}}_G(Z) \to \mathrm{D}^{\mathrm{cosg}}_G(Z/X),$$
$$\pi : \mathrm{D}^{\mathrm{sg}}_G(Z) \to \mathrm{D}^{\mathrm{sg}}_G(Z/X).$$

Proof. It is enough to show that $\mathrm{Lfr}_G(Z) \subset \langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z))\rangle^{\oplus}$ and $\mathrm{Perf}_G(Z) \subset \langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z))\rangle$. These inclusions follow from the assumption that X has a G-equivariant ample line bundle L. The proofs of the inclusions are similar, and we prove the only former inclusion. It is enough to show that any G-equivariant locally free sheaf E on Z is a direct summand of a bounded complex whose terms are direct sums of invertible sheaves of the form $i^*L^{\otimes n}$. By [Tho87, Lemma 1.4], there is a bounded above locally free resolution $E^\bullet \xrightarrow{\sim} E$ whose terms are as above. For any $n > 0$, we have the following triangle in $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$

$$\sigma^{\geqslant -n}E^\bullet \to E \to H^{-n}(\sigma^{\geqslant -n}E^\bullet)[n+1] \to \sigma^{\geqslant -n}E^\bullet[1],$$

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where $\sigma^{\geqslant -n}$ denotes the brutal truncation. If we choose a sufficiently large $n \gg 0$, we have

$$\mathrm{Hom}_{\mathrm{D^b(Qcoh}_G Z)}(E, H^{-n}(\sigma^{\geqslant -n} E^\bullet)[n+1]) = 0$$

by [Orl04, Lemma 1.12], since the restriction functor $\mathrm{Res}_G : \mathrm{D^b}(\mathrm{Qcoh}_G Z) \to \mathrm{D^b}(\mathrm{Qcoh} Z)$ is faithful by Lemma 2.33(2). Hence, the above triangle splits, and E is a direct summand of the complex $\sigma^{\geqslant -n} E^\bullet$. □

Remark 3.3. Note that, if X is regular, then the thick subcategory $\langle \mathrm{Im}(\mathbf{L}i^*) \rangle \subset \mathrm{D^b}(\mathrm{coh}_G Z)$ coincides with its thick subcategory $\mathrm{Perf}_G(Z)$ of equivariant perfect complexes of Z. Hence, the quotient category $\mathrm{D}^{\mathrm{sg}}_G(Z/X)$ is the same as $\mathrm{D}^{\mathrm{sg}}_G(Z)$. Similarly, $\mathrm{D}^{\mathrm{cosg}}_G(Z/X)$ is also same as $\mathrm{D}^{\mathrm{cosg}}_G(Z)$ when X is regular.

The exact functors $\mathrm{Res}_G : \mathrm{Qcoh}_G Z \to \mathrm{Qcoh} Z$ and $\mathrm{Ind}_G : \mathrm{Qcoh} Z \to \mathrm{Qcoh}_G Z$, defined in Definition 2.29, induce functors between relative singularity categories

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}(Z/X),$$
$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{cosg}}(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_G(Z/X).$$

We need the following lemma in the proof of the main result.

Lemma 3.4. *Assume that G is reductive. Then the restriction functor*

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}(Z/X)$$

is faithful.

Proof. This follows from a similar argument as in the proof of Lemma 2.33(2). □

3.2 Direct images and inverse images in relative singularity categories

Let X_1 and X_2 be quasi-projective schemes with actions of an affine algebraic group G. Assume that X_1 and X_2 have G-equivariant ample line bundles. Let $\tilde{f} : X_2 \to X_1$ be a G-equivariant morphism. Let Z_1 be a G-invariant closed subscheme of X_1 such that $\mathcal{O}_{Z_1}$ has finite G-flat dimension as a $\mathcal{O}_{X_1}$-module, and let Z_2 be the fiber product $Z_1 \times_{X_1} X_2$. Denote by f the restriction $\tilde{f}|_{Z_2} : Z_2 \to Z_1$ of $\tilde{f}$ to Z_2. We assume that the cartesian square

$$\begin{array}{ccc} Z_2 & \xrightarrow{f} & Z_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{\tilde{f}} & X_1 \end{array}$$

is *exact* in the sense of [Kuz06]. Then, $\mathcal{O}_{Z_2}$ also has finite G-flat dimension as a $\mathcal{O}_{X_2}$-module. Furthermore, we assume that $\tilde{f}$ *has finite G-flat dimension*, i.e. the derived inverse image $\mathbf{L}\tilde{f}^* : \mathrm{D}^-(\mathrm{Qcoh}_G X_1) \to \mathrm{D}^-(\mathrm{Qcoh}_G X_2)$ maps $\mathrm{D^b}(\mathrm{Qcoh}_G X_1)$ to $\mathrm{D^b}(\mathrm{Qcoh}_G X_2)$. Then f also has finite G-flat dimension.

In the above setting, the derived inverse image $\mathbf{L}f^* : \mathrm{D^b}(\mathrm{Qcoh}_G Z_1) \to \mathrm{D^b}(\mathrm{Qcoh}_G Z_2)$ induces exact functors

$$f^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2),$$
$$f^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2),$$

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and the derived direct image $\mathbf{R}f_* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_2) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_1)$ induces a right adjoint functor of $f^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2)$

$$f_\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1).$$

If f is a proper morphism, the direct image $\mathbf{R}f_* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z_2) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z_1)$ between bounded complexes of coherent sheaves induces a right adjoint functor

$$f_\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2) \to \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1)$$

of $f^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2)$.

Let X be a quasi-projective scheme with an action of an affine algebraic group G, and let $U \subset X$ be a G-invariant open subscheme. Let $Z \subset X$ be a G-invariant closed subscheme such that $\mathcal{O}_Z$ has finite G-flat dimension, and consider the fiber product $U_Z := Z \times_X U$. Denote by $\tilde{l} : U \to X$ and $l : U_Z \to Z$ the open immersions. Then we have the following exact cartesian square.

$$\begin{array}{ccc} U_Z & \xrightarrow{\ l\ } & Z \\ \downarrow & & \downarrow \\ U & \xrightarrow{\ \tilde{l}\ } & X \end{array}$$

Lemma 3.5. *The inverse image*

$$l^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_G(U_Z/U)$$

is a Verdier localization by the kernel of l°.

Proof. The direct image $\mathbf{R}l_* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}U_Z) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}Z)$ is fully faithful and right adjoint to the inverse image $l^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}Z) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}U_Z)$. By [Orl06, Lemma 1.1], the direct image functor $l_\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_G(U_Z/U)$ is fully faithful. Hence, l° admits a right adjoint functor which is fully faithful, and this implies the result. □

3.3 Relative singularity categories and derived factorization categories

In this section, X and G are the same as in § 3.1, and we assume that G is *reductive*. Let $\chi : G \to \mathbb{G}_m$ be a character of G, and let $W : X \to \mathbb{A}^1$ be a χ-semi-invariant regular function. In this section, we assume that the corresponding G-invariant section $W : \mathcal{O}_X \to \mathcal{O}(\chi)$ is injective. For example, if W is flat, this condition is satisfied. Denote by X_0 the fiber of W over $0 \in \mathbb{A}^1$, and let $i : X_0 \to X$ be the closed immersion. We have an exact functor $\tau : \mathrm{Qcoh}_G X_0 \to Z^0(\mathrm{Qcoh}_G(X, \chi, W))$ defined by

$$\tau(F) := (0 \to i_*(F) \to 0).$$

We define a natural functor

$$\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X_0) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$$

as the composition of functors

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X_0) \xrightarrow{\tau} \mathrm{D}^{\mathrm{b}}(Z^0(\mathrm{Qcoh}_G(X, \chi, W))) \xrightarrow{\mathrm{Tot}} \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W).$$

The functor Υ annihilates the thick category $\langle \mathrm{Im}(\mathbf{L}i^*) \rangle^\oplus \subset \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X_0)$, since its non-equivariant functor $\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}X_0) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(X, W)$ annihilates $\mathrm{Res}_G(\langle \mathrm{Im}(\mathbf{L}i^*) \rangle^\oplus)$

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(see [EP15, proofs of Theorems 2.7 and 2.8]) and the restriction functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(X, W)$ is faithful. Hence, it induces an exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W).$$

Similarly, we have the following exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W),$$

and the following diagram is commutative,

$$\begin{CD}
\mathrm{D}^{\mathrm{cosg}}_G(X_0/X) @>{\Upsilon}>> \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)\\
@AAA @AAA\\
\mathrm{D}^{\mathrm{sg}}_G(X_0/X) @>{\Upsilon}>> \mathrm{Dcoh}_G(X, \chi, W)
\end{CD}$$

where the vertical arrows are natural inclusion functors (which are fully faithful).

THEOREM 3.6 (Cf. [EP15, Theorems 2.7 and 2.8]). *The functors*

$$\Upsilon : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W),$$
$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W)$$

are equivalences.

In order to prove the above theorem, we need to construct the quasi-inverse of Υ. We say that a G-equivariant quasi-coherent sheaf $F \in \mathrm{Qcoh}_G X$ is **W**-*flat*, if the morphism of sheaves $W : F \to F \otimes L$ is injective. Denote by $\mathrm{Flat}^W_G(X, \chi, W)$ the dg full subcategory of $\mathrm{Qcoh}_G(X, \chi, W)$ consisting of factorizations whose components are W-flat. Then $H^0(\mathrm{Flat}^W_G(X, \chi, W))$ is a full triangulated subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W))$. Denote by $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W_G(X, \chi, W))$ the smallest thick subcategory of $H^0(\mathrm{Flat}^W_G(X, \chi, W))$ containing all totalizations of short exact sequences in the exact category $Z^0(\mathrm{Flat}^W_G(X, \chi, W))$. Consider the corresponding Verdier quotients

$$\mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X, \chi, W) := H^0(\mathrm{Flat}^W_G(X, \chi, W))/\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W_G(X, \chi, W)).$$

The restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{Qcoh}(X, W)$ and the induction functor $\mathrm{Ind}_G : \mathrm{Qcoh}(X, W) \to \mathrm{Qcoh}_G(X, \chi, W)$ preserve factorizations whose components are W-flat sheaves since $\mathrm{Res}_G : \mathrm{Qcoh}_G X \to \mathrm{Qcoh} X$ and $\mathrm{Ind}_G : \mathrm{Qcoh} X \to \mathrm{Qcoh}_G X$ are exact functors. Hence, the restriction and the induction functors induce the following functors

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W(X, W),$$
$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W(X, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X, \chi, W),$$

and these functors are adjoint to each other:

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

LEMMA 3.7. *The natural functor*

$$\mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$$

is an equivalence.

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Proof. First, we prove that the functor is essentially surjective. Let $F \in \mathrm{D^{co}Qcoh}_G(X,\chi,W)$ be an object. Since X has a G-equivariant ample line bundle, there are G-equivariant locally free sheaf E_i and a surjective morphism $p_i : E_i \to F_i$ in $\mathrm{Qcoh}_G X$ for each $i = 0, 1$. Let $E \in \mathrm{Qcoh}_G(X,\chi,W)$ be the factorization of the following form

$$E := (E_1 \oplus E_0 \xrightarrow{W \oplus \mathrm{id}_{E_0}} E_1(\chi) \oplus E_0 \xrightarrow{\mathrm{id}_{E_1(\chi)} \oplus W} E_1(\chi) \oplus E_0(\chi)).$$

Then p_1 and p_0 define a natural surjective morphism $p : E \to F$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. The kernel $K := \mathrm{Ker}(p)$ of p is in $Z^0(\mathrm{Flat}^W_G(X,\chi,W))$ since the components of K are subsheaves of W-flat sheaves. Hence, the totalization $\mathrm{Tot}(C^\bullet)$ of the complex

$$C^\bullet : \cdots \to 0 \to K \hookrightarrow E \to 0 \to \cdots$$

with the cohomological degree of E zero is in $\mathrm{D^{co}Flat}^W_G(X,\chi,W)$, and we see that the natural morphism $\mathrm{Tot}(C^\bullet) \to F$ induced by p is an isomorphism in $\mathrm{D^{co}Qcoh}_G(X,\chi,W)$.

To show the functor $\mathrm{D^{co}Flat}^W_G(X,\chi,W) \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)$ is fully faithful, it suffices to prove that for any morphism $f : E \to F$ in $H^0(\mathrm{Qcoh}_G(X,\chi,W))$ with $F \in H^0(\mathrm{Flat}^W_G(X,\chi,W))$ and the cone of f in $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W))$, there exists a morphism $g : F' \to E$ with $F' \in H^0(\mathrm{Flat}^W_G(X,\chi,W))$ such that the cone of $f \circ g$ is in $\mathrm{Acycl^{co}}(\mathrm{Flat}^W_G(X,\chi,W))$ (see [LS16, Proposition B.2.(ff1)$^{\mathrm{op}}$]). By the above argument in the previous paragraph, we can find a morphism $g : F' \to E$ with $F' \in H^0(\mathrm{Flat}^W_G(X,\chi,W))$ such that the cone of g is in $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W))$, and then the cone of $f \circ g$ is in $H^0(\mathrm{Flat}^W_G(X,\chi,W)) \cap \mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W))$. Hence, it is enough to show that

$$H^0(\mathrm{Flat}^W_G(X,\chi,W)) \cap \mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W)) \subseteq \mathrm{Acycl^{co}}(\mathrm{Flat}^W_G(X,\chi,W)).$$

For this, let $A \in H^0(\mathrm{Flat}^W_G(X,\chi,W)) \cap \mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W))$ be an object. We already know that $\mathrm{Res}_G(A) \in \mathrm{Acycl^{co}}(\mathrm{Flat}^W(X,W))$ by [EP15, Corollary 2.6(a)]. Note that the restriction functor $\mathrm{Res}_G : \mathrm{D^{co}Flat}^W_G(X,\chi,W) \to \mathrm{D^{co}Flat}^W(X,W)$ is faithful by a similar argument as in the proof of Lemma 2.33(2). Hence, the fact that $\mathrm{Res}_G(A) \in \mathrm{Acycl^{co}}(\mathrm{Flat}^W(X,W))$ implies that $A \in \mathrm{Acycl^{co}}(\mathrm{Flat}^W_G(X,\chi,W))$. □

For an object $F = (F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1(\chi)) \in Z^0(\mathrm{Flat}^W_G(X,\chi,W))$, define an object $\Xi(F) \in \mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$ by $\Xi(F) := \mathrm{Cok}(\varphi_1^F)$. It is easy to see that this defines the following exact functor

$$\Xi : H^0(\mathrm{Flat}^W_G(X,\chi,W)) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X).$$

If G is trivial, this exact functor annihilates $\mathrm{Acycl^{co}}(\mathrm{Flat}^W(X,W))$ by [EP15, Theorems 2.7 and 2.8]. Hence, since $\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D}^{\mathrm{cosg}}(X_0/X)$ is faithful, we obtain the exact functor $\Xi : \mathrm{D^{co}Flat}^W_G(X,\chi,W) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$. By Lemma 3.7, we have the left derived functor of Ξ;

$$\mathbf{L}\Xi : \mathrm{D^{co}Qcoh}_G(X,\chi,W) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X).$$

Proof of Theorem 3.6. We will show that the functors Υ and $\mathbf{L}\Xi$ are mutually inverse. Let $E \in \mathrm{D^{co}Qcoh}_G(X,\chi,W)$ be an object. By Lemma 3.7 we may assume that $E \in \mathrm{D^{co}Flat}^W_G(X,\chi,W)$. Then

$$\Upsilon\mathbf{L}\Xi(E) \cong \Upsilon\Xi(E) = (0 \to \mathrm{Cok}(\varphi_1^E) \to 0),$$

and the surjective morphism $E_0 \twoheadrightarrow \mathrm{Cok}(\varphi_1^E)$ induces the natural surjective morphism $\phi_E : E \to \Upsilon\Xi(E)$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. Since the kernel of ϕ_E is the factorization $(E_1 = E_1 \xrightarrow{W} E_1(\chi))$

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and it is isomorphic to the zero object in $H^0(\mathrm{Flat}_G^W(X,\chi,W))$, the morphism $\phi_E : E \to \Upsilon\Xi(E)$ is an isomorphism in $\mathrm{D^{co}Qcoh}_G(X,\chi,W)$. It is easy to see that the isomorphisms $\phi_{(-)}$ define an isomorphism of functors

$$\phi : \mathrm{id}_{\mathrm{D^{co}Qcoh}_G(X,\chi,W)} \xrightarrow{\sim} \Upsilon\mathbf{L}\Xi.$$

Let $F \in \mathrm{D}_G^{\mathrm{cosg}}(X_0/X)$ be an object. Then we may assume that $F \in \mathrm{Qcoh}_G X_0$. Take a surjective morphism $p : P \twoheadrightarrow i_*F$ with P locally free. Set $K := \mathrm{Ker}(p) \in \mathrm{Qcoh}_G X$ and $Q := (K \xrightarrow{i} P \xrightarrow{W} K(\chi)) \in \mathrm{Qcoh}_G(X\chi, W)$, where $i : K \to P$ is the natural inclusion. Consider the natural surjective morphism $\pi : Q \to (0 \to i_*F \to 0)$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. Then the kernel of π is the factorization $(K = K \xrightarrow{W} K(\chi))$, and it is isomorphic to the zero object in $H^0(\mathrm{Qcoh}_G(X,\chi,W))$. Hence, π is an isomorphism in $\mathrm{D^{co}Qcoh}_G(X,\chi,W)$, and so we have a natural isomorphism $\psi_F : \mathbf{L}\Xi\Upsilon(F) \xrightarrow{\sim} F$ in $\mathrm{D}_G^{\mathrm{cosg}}(X_0/X)$ defined as the composition $\mathbf{L}\Xi\Upsilon(F) \xrightarrow{\sim} \Xi\Upsilon(Q) = \mathrm{Cok}(i) = F$. We need to show that the isomorphisms $\psi_{(-)}$ are functorial in $(-)$. Since the restriction functor Res_G is isomorphic to the forgetful functor Forg_G, we have a natural isomorphism of functors $\sigma : \mathrm{Res}_G\mathbf{L}\Xi\Upsilon \xrightarrow{\sim} \mathbf{L}\Xi\Upsilon\mathrm{Res}_G$ defined by the composition

$$\mathrm{Res}_G\mathbf{L}\Xi\Upsilon \xrightarrow{\sim} \mathrm{Forg}_G\mathbf{L}\Xi\Upsilon = \mathbf{L}\Xi\Upsilon\mathrm{Forg}_G \xrightarrow{\sim} \mathbf{L}\Xi\Upsilon\mathrm{Res}_G,$$

and the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Res}_G\mathbf{L}\Xi\Upsilon(F) & \xrightarrow{\mathrm{Res}_G(\psi_F)} & \mathrm{Res}_G(F) \\ & {\scriptstyle \sigma_F}\searrow \quad \nearrow{\scriptstyle \psi_{\mathrm{Res}_G(F)}} & \\ & \mathbf{L}\Xi\Upsilon\mathrm{Res}_G(F) & \end{array}$$

Hence, we see that the isomorphisms $\psi_{(-)}$ are functorial by the fact that the isomorphisms $\psi_{(-)}$ are functorial if G is trivial and that the functor Res_G is faithful. This completes the proof of the former equivalence.

The latter equivalence follows from [EP15, Remark 2.7], which is a generalized result of [EP15, Theorem 2.7]. □

4. Derived Knörrer periodicity

Let X be a smooth quasi-projective variety, and let G be a reductive affine algebraic group acting on X. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of rank r, and let $s \in \Gamma(X, \mathcal{E}^\vee)^G$ be a G-invariant section of $\mathcal{E}^\vee$. Denote by $Z \subset X$ the zero scheme of s. We assume that s is *regular*, i.e. the codimension of Z in X is r. Let

$$\mathrm{V}(\mathcal{E}(\chi)) := \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{E}(\chi)^\vee))$$

be a vector bundle over X with the G-action induced by the equivariant structure of the locally free sheaf $\mathcal{E}(\chi)$. Denote by $\mathrm{V}(\mathcal{E}(\chi))|_Z$ the restriction of the vector bundle $\mathrm{V}(\mathcal{E}(\chi))$ to Z. Let $j : Z \hookrightarrow X$ and $i : \mathrm{V}(\mathcal{E}(\chi))|_Z \hookrightarrow \mathrm{V}(\mathcal{E}(\chi))$ be the closed immersions, and let $q : \mathrm{V}(\mathcal{E}(\chi)) \to X$ and $p : \mathrm{V}(\mathcal{E}(\chi))|_Z \to Z$ be the projections. Now we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{V}(\mathcal{E}(\chi))|_Z & \xrightarrow{i} & \mathrm{V}(\mathcal{E}(\chi)) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow{j} & X \end{array}$$

The invariant section s induces a χ-semi-invariant regular function

$$Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1.$$

Let $W : X \to \mathbb{A}^1$ be a χ-semi-invariant regular function on X. The function W induces χ-semi-invariant functions on Z, $\mathrm{V}(\mathcal{E}(\chi))$ and $\mathrm{V}(\mathcal{E}(\chi))|_Z$, which we denote by the same notation W (by abuse of notation). Since the inverse image p^* and the direct image i_* are exact and commutative with arbitrary direct sums as functors between categories of quasi-coherent sheaves, these induce (underived) functors

$$p^* : \mathrm{D^{co}Qcoh}_G(Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z, \chi, W),$$
$$i_* : \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s).$$

Restricting the composition $i_*p^* : \mathrm{D^{co}Qcoh}_G(Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s)$ to $\mathrm{Dcoh}_G(Z, \chi, W)$, we obtain an exact functor

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s).$$

Shipman proved that the above functor i_*p^* is an equivalence when $G = \mathbb{G}_m$ trivially acts on X and $W = 0$ (see also [Isi13]).

THEOREM 4.1 [Shi12, Theorem 3.4]. *The composition*

$$i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$$

is an equivalence.

The goal of this section is to show the following main result which is an analogy of the above theorem.

THEOREM 4.2. *Assume that $W|_Z : Z \to \mathbb{A}^1$ is flat. The functor*

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s)$$

is an equivalence.

Remark 4.3. Let S be a smooth quasi-projective variety, and let G be an affine reductive group acting on S. Let $W : S \to \mathbb{A}^1$ be a $\chi := \chi_1 + \chi_2$-semi-invariant non-constant regular function for some characters $\chi_i : G \to \mathbb{G}_m$. Let $X := \mathrm{V}(\mathcal{O}(\chi_1)) \cong S \times \mathbb{A}^1_{x_1}$ be the G-vector bundle over S, and let $s \in \Gamma(X, \mathcal{O}(\chi_1))^G$ be the section corresponding to the χ_1-semi-invariant function $S \times \mathbb{A}^1_{x_1} \to \mathbb{A}^1$ which is defined as the projection $(s, x_1) \mapsto x_1$. Then, S is isomorphic to the zero scheme of s, and the G-vector bundle $\mathrm{V}(\mathcal{O}(-\chi_1)(\chi))$ over X is isomorphic to the G-variety $S \times \mathbb{A}^2_{x_1,x_2}$, where the G-weights of x_i are given by χ_i. By Theorem 4.2, we have the following equivalence

$$\mathrm{Dcoh}_G(S, \chi, W) \simeq \mathrm{Dcoh}_G(S \times \mathbb{A}^2_{x_1,x_2}, \chi, W + x_1x_2).$$

This kind of equivalence is known as *Knörrer periodicity*, so the above theorem is considered as a generalization of the original Knörrer periodicity [Knö87, Theorem 3.1].

4.1 Lemmas for the main theorem

In this section, we provide some lemmas for the main result. Throughout this section, we consider the case when G is trivial.

Set

$$\omega_j := \bigwedge^r (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee \quad \text{and} \quad \omega_i := p^*\omega_j,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in X. These are invertible sheaves on Z and $\mathrm{V}(\mathcal{E})|_Z$, respectively. We define an exact functor

$$i^! : \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E})|_Z, W)$$

as $i^!(-) := \mathbf{L}i^*(-) \otimes \omega_i[-r]$. By [EP15, Theorem 3.8], the above functor $i^!$ is right adjoint to $i_* : \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E})|_Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$. Let

$$K := K(q^*s, t) \in \mathrm{lfr}(\mathrm{V}(\mathcal{E}), Q_s)$$

be the Koszul factorization of $q^*s \in \Gamma(\mathrm{V}(\mathcal{E}), q^*\mathcal{E}^\vee)$ and $t \in \Gamma(\mathrm{V}(\mathcal{E}), q^*\mathcal{E})$, where t is the tautological section. By abuse of notation, we denote by $\mathcal{O}_Z$ the object in $\mathrm{coh}(Z, 0)$ of the following form

$$(0 \to \mathcal{O}_Z \to 0).$$

Lemma 4.4. *Consider the case when $W = 0$. We have isomorphisms*

$$i_*p^*(\mathcal{O}_Z) \cong K \quad \text{and} \quad p_*i^!(K) \cong \mathcal{O}_Z$$

in $\mathrm{Dcoh}(\mathrm{V}(\mathcal{E}), Q_s)$ *and in* $\mathrm{D^{co}Qcoh}(Z, 0)$, *respectively.*

Proof. These isomorphisms follow from Lemma 2.28. In particular, the former isomorphism is an immediate consequence. Note that $\omega_i \cong i^* \bigwedge^r q^*\mathcal{E}^\vee$. We obtain the latter isomorphism as follows:

$$p_*i^!(K) \cong p_*\mathbf{L}i^*\left(\mathcal{O}_Z \otimes \bigwedge^r q^*\mathcal{E}^\vee[-r]\right) \cong p_*\mathbf{L}i^*(K^\vee) \cong p_*\mathbf{L}i^*(\mathcal{O}_{Z_{t^\vee}}) \cong \mathcal{O}_Z,$$

where the last isomorphism follows from the fact that the zero section $Z \subset \mathrm{V}(\mathcal{E})$ is isomorphic to the fiber product of closed subschemes $\mathrm{V}(\mathcal{E})|_Z \hookrightarrow \mathrm{V}(\mathcal{E})$ and $Z_{t^\vee} \hookrightarrow \mathrm{V}(\mathcal{E})$. □

Lemma 4.5. *The functor*

$$i_*p^* : \mathrm{D^{co}Qcoh}(Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$$

is fully faithful.

Proof. The functors i_*p^* and $p_*i^!$ can be represented as integral functors

$$i_*p^* \cong \Phi_{k_*\mathcal{O}_{\mathrm{V}(\mathcal{E})|_Z}} \quad \text{and} \quad p_*i^! \cong \Phi_{k_*\omega_i[-r]},$$

where $k := p \times i : \mathrm{V}(\mathcal{E})|_Z \to Z \times \mathrm{V}(\mathcal{E})$ and kernels $\mathcal{O}_{\mathrm{V}(\mathcal{E})|_Z}$ and $\omega_i[-r]$ are objects in $\mathrm{Dcoh}(\mathrm{V}(\mathcal{E})|_Z, 0)$. By easy computation, we see that there exists an object $P \in \mathrm{D^{co}Qcoh}(Z, 0)$ such that $p_*i^! \circ i_*p^* \cong \Phi_{\Delta_*P} \cong (-) \otimes P$, where $\Delta : Z \to Z \times Z$ is the diagonal embedding. Substituting $W = 0$, by Lemma 4.4, we have an isomorphism $P \cong \mathcal{O}_Z$. But P does not depend on the function W. Hence, for any W, we have an isomorphism of functors $p_*i^! \circ i_*p^* \cong \Phi_{\Delta_*P} \cong \mathrm{id}_{\mathrm{D^{co}Qcoh}(Z,W)}$. By the following lemma, this implies that the functor $i_*p^* : \mathrm{D^{co}Qcoh}(Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$ is fully faithful. □

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The following lemma is an opposite version of [Joh02, Lemma 1.1.1].[1] We give a proof for the reader's convenience.

Lemma 4.6 (Cf. [Joh02, Lemma 1.1.1]). *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between (arbitrary) categories, and suppose that F admits a right adjoint functor $G : \mathcal{B} \to \mathcal{A}$. Then, if there exists an isomorphism of functors $\alpha : \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} GF$ (α is not necessarily the adjunction morphism), then F is fully faithful.*

Proof. The isomorphism α implies that the following composition of maps is bijective:

$$\mathrm{Hom}(A, A') \xrightarrow{F} \mathrm{Hom}(F(A), F(A')) \xrightarrow{G} \mathrm{Hom}(GF(A), GF(A')).$$

Hence, it is enough to show that G is fully faithful on the image of F. Since the above composition is bijective, G is full on the image of F. Let $\varepsilon : \mathrm{id}_{\mathcal{A}} \to GF$ and $\delta : FG \to \mathrm{id}_{\mathcal{B}}$ be the adjunction morphisms. For any $f \in \mathrm{Hom}(F(A), F(A'))$ we have

$$\delta_{F(A')} \circ FG(f) \circ F(\varepsilon_A) = f \circ \delta_{F(A)} \circ F(\varepsilon_A) = f,$$

where the first equation follows from the functoriality of δ and the second equation follows from the property of the adjunction morphisms. Hence, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}(F(A), F(A')) & = & \mathrm{Hom}(F(A), F(A')) \\ {\scriptstyle G}\downarrow & & \uparrow{\scriptstyle \delta_{F(A')} \circ (-) \circ F(\varepsilon_A)} \\ \mathrm{Hom}(GF(A), GF(A')) & \xrightarrow{F} & \mathrm{Hom}(FGF(A), FGF(A')) \end{array}$$

and hence G is faithful on the image of F. □

4.2 Proof of the main theorem

In this section, we prove the main theorem. Recall that G is a reductive affine algebraic group acting on a smooth quasi-projective variety X. Since X is smooth, there is a G-equivariant ample line bundle on X. In what follows, we assume that $W|_Z : Z \to \mathbb{A}^1$ is flat.

First, we consider relative singularity categories. Let Z_0, $V|_{Z_0}$ and V_0 be the fibers of $W : Z \to \mathbb{A}^1$, $W : \mathrm{V}(\mathcal{E}(\chi))|_Z \to \mathbb{A}^1$ and $W + Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1$ over $0 \in \mathbb{A}^1$, respectively. Denote by $p_0 : V|_{Z_0} \to Z_0$ and $i_0 : V|_{Z_0} \to V_0$ the restrictions of p and i, respectively. By [Kuz06, Corollary 2.27], the following cartesian squares are exact.

$$\begin{array}{ccc} V|_{Z_0} & \xrightarrow{p_0} & Z_0 \\ \downarrow & & \downarrow \\ \mathrm{V}(\mathcal{E})|_Z & \xrightarrow{p} & Z \end{array} \qquad \begin{array}{ccc} V|_{Z_0} & \xrightarrow{i_0} & V_0 \\ \downarrow & & \downarrow \\ \mathrm{V}(\mathcal{E})|_Z & \xrightarrow{i} & \mathrm{V}(\mathcal{E}) \end{array}$$

Since p and i have finite flat dimensions, we have exact functors of relative singularity categories

$$p_0{}^{\circ} : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E})|_Z),$$
$$i_{0\circ} : \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E})|_Z) \to \mathrm{D}^{\mathrm{cosg}}(V_0/\mathrm{V}(\mathcal{E})) = \mathrm{D}^{\mathrm{cosg}}_G(V_0).$$

[1] I would like to thank Timothy Logvinenko for informing me about this lemma.

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Then the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) & \xrightarrow{i_{0\circ}p_0{}^{\circ}} & \mathrm{D}^{\mathrm{cosg}}_G(V_0) \\ \Upsilon\downarrow & & \downarrow\Upsilon \\ \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(Z,W) & \xrightarrow{i_*p^*} & \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),W+Q_s) \end{array}$$

Furthermore, we compactify V_0 and $V|_{Z_0}$. The compactifying technique appeared in [Orl06]. Let

$$P:=\mathbb{P}(\mathcal{E}(\chi)\oplus\mathcal{O}_X)=\underline{\mathrm{Proj}}(\mathrm{Sym}(\mathcal{E}(\chi)\oplus\mathcal{O}_X)^{\vee})$$

be the projective space bundle over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi)\oplus\mathcal{O}_X$. Then we have a natural equivariant open immersion

$$l:\mathrm{V}(\mathcal{E}(\chi))\to P.$$

Denote by $l|_Z:\mathrm{V}(\mathcal{E}(\chi))|_Z\to P|_Z$ the pull-back of l by the closed immersion $\overline{i}:P|_Z\to P$. Now we have the following cartesian square.

$$\begin{array}{ccc} \mathrm{V}(\mathcal{E}(\chi))|_Z & \xrightarrow{l|_Z} & P|_Z \\ i\downarrow & & \downarrow\overline{i} \\ \mathrm{V}(\mathcal{E}(\chi)) & \xrightarrow{l} & P \end{array}$$

Denote by $\overline{q}:P\to X$ the natural projection, and let $\overline{p}:=\overline{q}|_Z:P|_Z\to Z$ be the pull-back of $\overline{q}$ by the closed immersion $j:Z\to X$. Let P_0 be the G-invariant subscheme of P defined by the G-invariant section $\overline{s\oplus W}\in\Gamma(P,\mathcal{O}(1)(\chi))^G$ which is corresponding to the composition

$$\mathcal{O}_P\xrightarrow{\overline{q}^*(s\oplus W)}\overline{q}^*(\mathcal{E}\oplus\mathcal{O}(\chi^{-1}))^{\vee}\xrightarrow{\sigma}\mathcal{O}_P(1)(\chi),$$

where σ is the canonical surjection, and let $P|_{Z_0}$ be the zero scheme defined by the invariant section $\overline{i}^*(\overline{s\oplus W})\in\Gamma(P|_Z,\mathcal{O}(1)(\chi))^G$. Since the pull-back of $\overline{s\oplus W}$ (respectively $\overline{i}^*(\overline{s\oplus W})$) by the open immersion l (respectively $l|_Z$) is equal to $W+Q_s$ (respectively W), we have the following exact cartesian square.

$$\begin{array}{ccc} V|_{Z_0} & \xrightarrow{l|_{Z_0}} & P|_{Z_0} \\ i_0\downarrow & & \downarrow\overline{i_0} \\ V_0 & \xrightarrow{l_0} & P_0 \end{array}$$

Denote by $\overline{p_0}:P|_{Z_0}\to Z_0$ the pull-back of $\overline{p}:P|_Z\to Z$ by the closed immersion $Z_0\to Z$. Since the morphisms $\overline{i_0}:P|_{Z_0}\to P_0$ and $\overline{p_0}:P|_{Z_0}\to Z_0$ have finite Tor dimensions, the direct images $\mathbf{R}\overline{i_0}_*:\mathrm{D}^{\mathrm{b}}(\mathrm{coh}P|_{Z_0})\to\mathrm{D}^{\mathrm{b}}(\mathrm{coh}P_0)$ and $\mathbf{R}\overline{p_0}_*:\mathrm{D}^{\mathrm{b}}(\mathrm{coh}P|_{Z_0})\to\mathrm{D}^{\mathrm{b}}(\mathrm{coh}Z_0)$ induce the following exact functors (cf. [TT90, Proposition 2.7]),

$$\overline{i_0}_{\circ}:\mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0})\to\mathrm{D}^{\mathrm{sg}}_G(P_0),$$
$$\overline{p_0}_{\circ}:\mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0})\to\mathrm{D}^{\mathrm{sg}}_G(Z_0).$$

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Now we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{sg}}_G(Z_0) & \xrightarrow{\overline{i_0}_\circ \overline{p_0}^\circ} & \mathrm{D}^{\mathrm{sg}}_G(P_0) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle l_0{}^\circ} \\ \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) & \xrightarrow{i_{0\circ} p_0{}^\circ} & \mathrm{D}^{\mathrm{sg}}_G(V_0) \end{array}$$

where the vertical arrow on the left-hand side is a Verdier localization by Proposition 3.2.

Remark 4.7. If Z is smooth, the above vertical arrows are equivalences. Indeed, in that case, the singular locus $\mathrm{Sing}(P_0)$ is contained in V_0, whence $l_0{}^\circ$ is an equivalence by a similar argument in the proof of [Orl04, Proposition 1.14]. The equivalence of π follows from Remark 3.3.

Let $\overline{i_0}^! : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G P_0) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G P|_{Z_0})$ be the functor defined by

$$\overline{i_0}^! := \mathbf{L}\overline{i_0}^*(-) \otimes \bigwedge^r (\mathcal{I}/\mathcal{I}^2)^\vee[-r],$$

where $\mathcal{I}$ is the ideal sheaf of $\overline{i_0} : P|_{Z_0} \hookrightarrow P_0$. The functor $\overline{i_0}^!$ is a right adjoint functor of $\overline{i_0}_* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G P|_{Z_0}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G P_0)$. Indeed, these functors are adjoint when G is trivial by [Har66, III Theorem 6.7, Corollary 7.3], and the isomorphism

$$\mathrm{Hom}(\overline{i_0}_*(A), B) \cong \mathrm{Hom}(A, \overline{i_0}^!(B)),$$

where $A \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh} P|_{Z_0})$ and $B \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh} P_0)$, commutes with G-actions on each vector space of morphisms by the property in [Har66, III Proposition 6.9.c]. Hence we see that $\overline{i_0}^!$ is right adjoint to $\overline{i_0}_*$ by [BFK12, Lemma 2.2.8]. Denote by

$$\overline{i_0}^\flat : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0})$$

the functor induced by $\overline{i_0}^!$. By the above argument, we have the following adjoint pair

$$\overline{i_0}_\circ \dashv \overline{i_0}^\flat.$$

Similarly, we have a right adjoint functor

$$i_0{}^\flat : \mathrm{D}^{\mathrm{cosg}}_G(V_0) \to \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E}(\chi))|_Z)$$

of $i_{0\circ} : \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E}(\chi))|_Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$.

Proof of Theorem 4.2. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) & \xrightarrow{i_{0\circ} p_0{}^\circ} & \mathrm{D}^{\mathrm{sg}}_G(V_0) \\ {\scriptstyle \Upsilon}\downarrow & & \downarrow{\scriptstyle \Upsilon} \\ \mathrm{Dcoh}_G(Z, \chi, W) & \xrightarrow{i_* p^*} & \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s) \end{array}$$

where the vertical arrows are equivalences by Theorem 3.6. Hence, it suffices to show that the functor $i_{0\circ} p_0{}^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is an equivalence.

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First, we prove that the functor $i_{0\circ}{p_0}^{\circ} : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ is fully faithful. Let

$$\varepsilon_G : \mathrm{id}_{\mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)} \to p_{0\circ}{i_0}^{\flat} \circ i_{0\circ}{p_0}^{\circ}$$

be the adjunction morphism of the adjoint pair $i_{0\circ}{p_0}^{\circ} \dashv p_{0\circ}{i_0}^{\flat}$. It is enough to show that for any object $A \in \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)$, the cone $C_G(A)$ of the morphism $\varepsilon_G(A) : A \to p_{0\circ}{i_0}^{\flat} \circ i_{0\circ}{p_0}^{\circ}(A)$ is the zero object. But the object $\mathrm{Res}_G(C_G(A))$ is isomorphic to the cone $C(A)$ of the adjunction morphism of $\varepsilon(\mathrm{Res}_G(A)) : \mathrm{Res}_G(A) \to p_{0\circ}{i_0}^{\flat} \circ i_{0\circ}{p_0}^{\circ}(\mathrm{Res}_G(A))$ of the adjoint pair of functors between $\mathrm{D}^{\mathrm{cosg}}(Z_0/Z)$ and $\mathrm{D}^{\mathrm{cosg}}(V_0)$. Since we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{cosg}}(Z_0/Z) & \xrightarrow{\;i_{0\circ}{p_0}^{\circ}\;} & \mathrm{D}^{\mathrm{cosg}}(V_0) \\
\Upsilon\downarrow & & \downarrow\Upsilon \\
\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(Z, W) & \xrightarrow{\;i_*p^*\;} & \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)
\end{array}$$

where the vertical arrows are equivalences by Theorem 3.6, the functor $i_{0\circ}{p_0}^{\circ}$ is fully faithful by Lemma 4.5. This implies that the object $C(A)$ is the zero object. Hence, $C_G(A)$ is also the zero object since the restriction functor Res_G is faithful by Lemma 3.4. Hence, $i_{0\circ}{p_0}^{\circ} : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ is fully faithful. This implies that $i_{0\circ}{p_0}^{\circ} : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is also fully faithful, since the natural inclusions $\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)$ and $\mathrm{D}^{\mathrm{sg}}_G(V_0) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ are fully faithful by Theorem 3.6 and Proposition2.25(1).

It only remains to show that the functor $i_{0\circ}{p_0}^{\circ} : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is essentially surjective. Consider the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{sg}}_G(Z_0) & \xrightarrow{\;\overline{i_0}_{\circ}\overline{p_0}^{\circ}\;} & \mathrm{D}^{\mathrm{sg}}_G(P_0) \\
\pi\downarrow & & \downarrow{l_0}^{\circ} \\
\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) & \xrightarrow{\;i_{0\circ}{p_0}^{\circ}\;} & \mathrm{D}^{\mathrm{sg}}_G(V_0)
\end{array}$$

By a similar argument as in the proof of [Orl04, Lemma 1.11], we see that every object in $\mathrm{D}^{\mathrm{sg}}_G(V_0)$ is isomorphic to an object $F[k]$ for some G-equivariant coherent sheaf F and for some integer $k \in \mathbb{Z}$. Hence the vertical arrow on the right-hand side in the above diagram is essentially surjective, since for every object E in $\mathrm{coh}_G V_0$ there exists an object $\overline{E}$ in $\mathrm{coh}_G P_0$ such that ${l_0}^*(\overline{E}) \cong E$. Thus, we only need to prove that $\overline{i_0}_{\circ}\overline{p_0}^{\circ} : \mathrm{D}^{\mathrm{sg}}_G(Z_0) \to \mathrm{D}^{\mathrm{sg}}_G(P_0)$ is essentially surjective. To prove that, it is enough to show that the right adjoint functor $\overline{p_0}_{\circ}\overline{i_0}^{\flat} : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(Z_0)$ is fully faithful. Since the restriction functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}(P_0)$ is faithful by Lemma 3.4 and [PV11, Proposition 3.8], it follows from [Orl06, Theorem 2.1] that the adjunction $\overline{i_0}_{\circ}\overline{p_0}^{\circ} \circ \overline{p_0}_{\circ}\overline{i_0}^{\flat} \to \mathrm{id}_{\mathrm{D}^{\mathrm{sg}}_G(P_0)}$ is an isomorphism of functors by a similar argument as in the proof of the fully faithfulness of $i_{0\circ}{p_0}^{\circ} : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ in the previous paragraph. □

4.3 Cases when $W = 0$

In the previous section, we prove the main result assuming that $W|_Z : Z \to \mathbb{A}^1$ is flat. In this section, we consider the cases when $W = 0$. In these cases, using results in [Shi12], we can show the following.

With notation as above, consider $\mathbb{G}_m \times G$-action on X induced by the projection $\mathbb{G}_m \times G \to G$. Let $\theta : \mathbb{G}_m \times G \to \mathbb{G}_m$ be the character defined as the projection. Since the first factor of $\mathbb{G}_m \times G$ trivially acts on X, the G-equivariant locally free sheaf $\mathcal{E}$ has a natural $\mathbb{G}_m \times G$-equivariant structure.

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Proposition 4.8. *We have an equivalence*

$$\mathrm{D^b}(\mathrm{coh}_G Z) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s).$$

Proof. By a similar argument as in § 2.2, we obtain an equivalence

$$\mathrm{D^b}(\mathrm{coh}_G Z) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0).$$

Hence, it is enough to show the functor

$$i_* p^* : \mathrm{Dcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{Dcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence.

By Lemma 4.5, it follows that

$$i_* p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$$

is fully faithful since the forgetful functor $\mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}(Z, 0)$ is faithful. Furthermore, the above functor $i_* p^*$ is an equivalence since the right orthogonal of the image of the restricted functor $i_* p^* : \mathrm{Dcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$ vanishes by the argument in [Shi12, Theorem 3.4]. In particular, the right adjoint functor

$$p_* i^! : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0)$$

of $i_* p^*$ is also fully faithful.

Next we will show that the functor

$$i_* p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence. Let

$$\varepsilon_{\mathbb{G}_m \times G} : \mathrm{id}_{\mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z,\theta,0)} \to p_* i^! \circ i_* p^*$$

be the adjunction morphism. To show that the functor $i_* p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$ is fully faithful, we will prove that the adjunction morphism $\varepsilon_{\mathbb{G}_m \times G}$ is an isomorphism of functors. For this, it suffices to show that for any object $F \in \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0)$ the cone $C_{\mathbb{G}_m \times G}(F)$ of the morphism $\varepsilon_{\mathbb{G}_m \times G}(F) : F \to p_* i^! \circ i_* p^*(F)$ is the zero object. Recall that the categories $\mathrm{Qcoh}_{\mathbb{G}_m} Z$ and $\mathrm{Qcoh}_{\mathbb{G}_m \times G} Z$ are equivalent to the categories $\mathrm{Qcoh}[Z/\mathbb{G}_m]$ and $\mathrm{Qcoh}_G[Z/\mathbb{G}_m]$, respectively, where $[Z/\mathbb{G}_m]$ denotes the quotient stack, and we can consider the restriction and the induction functors for algebraic stacks as in § 2.6. Let $\pi_G : \mathrm{Qcoh}_{\mathbb{G}_m \times G} Z \to \mathrm{Qcoh}_{\mathbb{G}_m} Z$ be the functor corresponding to the restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G[Z/\mathbb{G}_m] \to \mathrm{Qcoh}[Z/\mathbb{G}_m]$ via the equivalences $\mathrm{Qcoh}_{\mathbb{G}_m} Z \cong \mathrm{Qcoh}[Z/\mathbb{G}_m]$ and $\mathrm{Qcoh}_{\mathbb{G}_m \times G} Z \cong \mathrm{Qcoh}_G[Z/\mathbb{G}_m]$. Then π_G naturally induces the following exact functor

$$\pi_G : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0),$$

and π_G has the right adjoint functor $\sigma_G : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0)$ induced by the induction functor. Since the argument in the proof of Lemma 2.33 works for algebraic stacks, the adjunction morphism $\mathrm{id} \to \sigma_G \circ \pi_G$ is a split mono. Hence, π_G is faithful. The object $\pi_G(C_{\mathbb{G}_m \times G}(F))$ is isomorphic to the cone $C_{\mathbb{G}_m}(F)$ of the adjunction morphism $\varepsilon_{\mathbb{G}_m}(\pi_G(F)) : \pi_G(F) \to p_* i^! \circ i_* p^*(\pi_G(F))$, and $C_{\mathbb{G}_m}(F)$ is the zero object since the functor $i_* p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$ is fully faithful. Hence we see that the object $C_{\mathbb{G}_m \times G}(F)$

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is also the zero object since π_G is faithful. By an identical argument, we see that the right adjoint functor

$$p_*i^! : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0)$$

is also fully faithful. Hence, the functor

$$i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence.

By Proposition 2.25(1), we see that the equivalence $i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$ induces an equivalence of the compact objects

$$i_*p^* : \overline{\mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0)} \to \overline{\mathrm{Dcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)},$$

where $\overline{(-)}$ denotes the idempotent completion of $(-)$. But $\mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0)$ on the left-hand side is already idempotent complete since it is equivalent to $\mathrm{D^b}(\mathrm{coh}_G Z)$. Hence, the functor

$$i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z, \theta, 0) \to \mathrm{Dcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence. □

5. Orlov's theorem for gauged LG models

In this section, we obtain a gauged LG version of the following theorem of Orlov.

Theorem 5.1 [Orl09, Theorem 40]. *Let $X \subset \mathbb{P}_k^{N-1}$ be the hypersurface defined by a section $f \in \Gamma(\mathbb{P}_k^{N-1}, \mathcal{O}(d))$. Denote by F the corresponding homogeneous polynomial.*

(1) *If $d < N$, there is a semi-orthogonal decomposition*

$$\mathrm{D^b}(\mathrm{coh}X) = \langle \mathcal{O}_X(d-N+1), \dots, \mathcal{O}_X, \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F)\rangle.$$

(2) *If $d = N$, there is an equivalence*

$$\mathrm{D^b}(\mathrm{coh}X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F).$$

(3) *If $d > N$, there is a semi-orthogonal decomposition*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F) = \langle k, \dots, k(N-d+1), \mathrm{D^b}(\mathrm{coh}X)\rangle.$$

We combine the main result with the theory of variations of GIT quotients to obtain a gauged LG version of the above theorem. For the theory of variations of GIT quotients, see [BFK12] or [BDFIK14, § 2]. This kind of approach to Orlov's theorem appeared in [Shi12, BFK12], and [BDFIK14], and our argument is similar to that in [BDFIK14, § 3].

Let S be a smooth quasi-projective variety with $\mathbb{G}_m$-action, and set

$$Q := S \times \mathbb{A}^N \times \mathbb{A}^1.$$

For $i = 1, 2$, set $G_i := \mathbb{G}_m$, and let $G := G_1 \times G_2$. For a positive integer $d > 1$, we define a G-action on Q as follows:

$$G \times Q \ni (g_1, g_2) \times (s, v_1, \dots, v_N, u) \mapsto (g_2 \cdot s, g_1v_1, \dots, g_1v_N, g_1^{-d}g_2u) \in Q,$$

where the action $\cdot$ is the original $\mathbb{G}_m$-action on S. Let $\lambda : \mathbb{G}_m \to G$ be the character defined by $\lambda(a) := (a, 1)$. Denote by Z_λ the fixed locus of λ-action on Q. Then Z_λ coincides with the zero section $S \times 0 \times 0 \subset Q$. Furthermore, set $S_+ := \{q \in Q \mid \lim_{a\to 0}\lambda(a)q \in Z_\lambda\}$ and $S_- := \{q \in Q \mid \lim_{a\to 0}\lambda(a)^{-1}q \in Z_\lambda\}$. Then

$$S_+ = S \times \mathbb{A}^N \times 0 \quad \text{and} \quad S_- = S \times 0 \times \mathbb{A}^1.$$

Denote by Q_+ (respectively Q_-) be the complement of S_+ (respectively S_-) in Q. Then the stratifications

$$Q = Q_+ \sqcup S_+ \quad \text{and} \quad Q = Q_- \sqcup S_-$$

are elementary wall crossings in the sense of [BFK12].

Let $W : S \to \mathbb{A}^1$ be a χ_1-semi-invariant function which is flat. Let $f \in \Gamma(\mathbb{P}_S^{N-1}, \mathcal{O}(d))^{\mathbb{G}_m}$ be a non-zero $\mathbb{G}_m$-invariant section, and denote by $F : \mathbb{A}_S^N \to \mathbb{A}^1$ the corresponding regular function. Since Q is the trivial line bundle over $\mathbb{A}_S^N$, the function F induces a regular function $\widetilde{F} : Q \to \mathbb{A}^1$. Then the function

$$W + \widetilde{F} : Q \to \mathbb{A}^1$$

is a $\chi_{0,1}$-semi-invariant regular function, where W is the pull-back of $W : S \to \mathbb{A}^1$ by the projection $Q \to S$, and $\chi_{0,1} : G \to \mathbb{G}_m$ is the character defined by $\chi_{0,1}(g_1, g_2) := g_2$. By [BFK12, Lemma 3.4.4] and [BFK12, Theorem 3.5.2], we have the following proposition.

Proposition 5.2. *Let $t_\pm$ be the λ-weight of the restriction of relative canonical bundle $\omega_{S_\pm/Q}$ to Z_λ, and set $\mu := -t_+ + t_-$. Let $\chi : G \to \mathbb{G}_m$ be the character defined by $\chi(g_1, g_2) := g_1 g_2$.*

(1) *If $\mu < 0$, there exist fully faithful functors*

$$\Upsilon_- : \mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}),$$
$$\Phi_- : \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}),$$

and we have the following semi-orthogonal decomposition

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) = \langle \Upsilon_-(\mu + 1), \ldots, \Upsilon_-, \Phi_-(\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}))\rangle,$$

where we denote by $\Upsilon_-(n)$ the essential image of the composition $(-) \otimes \mathcal{O}(\chi^n) \circ \Upsilon_-$.

(2) *If $\mu = 0$, we have an equivalence*

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}).$$

(3) *If $\mu > 0$, there exist fully faithful functors*

$$\Upsilon_+ : \mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}),$$
$$\Phi_+ : \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}),$$

and we have the following semi-orthogonal decomposition

$$\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) = \langle \Upsilon_+, \ldots, \Upsilon_+(-\mu + 1), \Phi_+(\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}))\rangle,$$

where we denote by $\Upsilon_+(n)$ the essential image of the composition $(-) \otimes \mathcal{O}(\chi^n) \circ \Upsilon_+$.

Since $Z_\lambda = S \times 0 \times 0$, the function $\widetilde{F}$ vanishes on $Z_\lambda \subset Q$. Hence, we have

$$\mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W).$$

Next, we have

$$Q_- = S \times \mathbb{A}^N \backslash 0 \times \mathbb{A}^1.$$

Since $F|_{S\times\mathbb{A}^N\backslash 0} \in \Gamma(S \times \mathbb{A}^N\backslash 0, \mathcal{O}(\chi_{-d,0})^\vee)^G$ and $Q_- = \mathrm{V}(\mathcal{O}(\chi_{-d,1}))$, Theorem 4.2 implies the following equivalence;

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_G(Z, \chi_{0,1}, W),$$

where $Z \subset S \times \mathbb{A}^N\backslash 0$ is the zero scheme of F. Moreover, the quotient stack $[Z/G_1]$ is isomorphic to the hypersurface X in the projective space bundle $\mathbb{P}^{N-1}_S$ over S defined by the invariant section $f \in \Gamma(\mathbb{P}^{N-1}_S, \mathcal{O}(d))^{G_2}$. Hence, we have an equivalence

$$\mathrm{Dcoh}_G(Z, \chi_{0,1}, W) \cong \mathrm{Dcoh}_{G_2}(X, \chi_1, W).$$

On the other hand, we have

$$Q_+ = S \times \mathbb{A}^N \times \mathbb{A}^1\backslash 0.$$

We consider another action of G on Q_+ as follows:

$$G \times Q_+ \ni (g_1, g_2) \times (s, v, u) \mapsto (g_1^d \cdot s, g_1 v, g_1^{-d} g_2 u) \in Q_+.$$

We denote by $\widetilde{Q_+}$ the new G-variety. Then we have a G-equivariant isomorphism

$$\varphi : \widetilde{Q_+} \xrightarrow{\sim} Q_+,$$

given by $\varphi(s, v, u) := (u \cdot s, v, u)$, where $u \in \mathbb{A}^1\backslash 0$ is considered as a point in $\mathbb{G}_m$. Since G_2 trivially acts on the first two components $S \times \mathbb{A}^N$ of $\widetilde{Q_+}$, we have

$$[\widetilde{Q_+}/G_2] \cong S \times \mathbb{A}^N \times [\mathbb{A}^1\backslash 0/G_2] \cong \mathbb{A}^N_S.$$

Hence, we have the following equivalence

$$\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_{G_1}(\mathbb{A}^N_S, \chi_d, W + F),$$

where, on the right-hand side, the G_1-action is given by the following

$$G_1 \times S \times \mathbb{A}^N \ni g_1 \times (s, v) \mapsto (g_1^d \cdot s, g_1 v).$$

Finally, note that $\mu = d - N$ and that the twisting by the G-equivariant invertible sheaf $\mathcal{O}(\chi)$ corresponds to the twisting, in $\mathrm{Dcoh}_{G_2}(X, \chi_1, W)$, by the G_2-equivariant invertible sheaf $\mathcal{O}(1)$ on X which is the pull-back of the tautological G_2-equivariant invertible sheaf on $\mathbb{P}^{N-1}_S$. Combining Proposition 5.2 and the above argument, we obtain the following gauged LG version of Orlov's theorem.

Let S be a smooth quasi-projective variety with a $\mathbb{G}_m$-action, and let $W : S \to \mathbb{A}^1$ be a χ_1-semi-invariant regular function which is flat. Consider $\mathbb{G}_m$-actions on $\mathbb{A}^N_S$ and on $\mathbb{P}^{N-1}_S$ given by

$$\mathbb{G}_m \times \mathbb{A}^N_S \ni t \times (s, v_1, \ldots, v_N) \mapsto (t^d \cdot s, tv_1, \ldots, tv_N) \in \mathbb{A}^N_S,$$
$$\mathbb{G}_m \times \mathbb{P}^{N-1}_S \ni t \times (s, v_1 : \cdots : v_N) \mapsto (t \cdot s, v_1 : \cdots : v_N) \in \mathbb{P}^{N-1}_S.$$

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Theorem 5.3. *For $d > 1$, let $f \in \Gamma(\mathbb{P}_S^{N-1}, \mathcal{O}(d))^{\mathbb{G}_m}$ be a non-zero invariant section, and let $F : \mathbb{A}_S^N \to \mathbb{A}^1$ be the corresponding χ_d-semi-invariant regular function. Let $X \subset \mathbb{P}_S^{N-1}$ be the hypersurface defined by f, and assume that the morphism $W|_X$ is flat.*

(1) *If $d < N$, there are fully faithful functors*

$$\Phi : \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) = \langle \Upsilon_{d-N+1}, \ldots, \Upsilon_0, \Phi(\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F)) \rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(i) \circ \Upsilon$.

(2) *If $d = N$, we have an equivalence*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F).$$

(3) *If $d > N$, there are fully faithful functors*

$$\Psi : \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F),$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_S^N, \chi_d, W + F) = \langle \Upsilon_0, \ldots, \Upsilon_{N-d+1}, \Psi(\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W)) \rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(\chi_i) \circ \Upsilon$.

Remark 5.4. (1) We can view Orlov's theorem (Theorem 5.1) as the case when $S = \mathrm{Spec}\, k$ and $W = 0$ in the above theorem.

(2) If $N > 1$, the assumption that $W|_X$ is flat is satisfied whenever $W : S \to \mathbb{A}^1$ is flat.

(3) For positive integers $a_1, \ldots, a_N$, applying the similar argument to the G-action on Q defined by

$$G \times Q \ni (g_1, g_2) \times (s, v_1, \ldots, v_N, u) \mapsto (g_2 \cdot s, g_1^{a_1} v_1, \ldots, g_1^{a_N} v_N, g_1^{-d} g_2 u) \in Q,$$

we can obtain the similar result for the hypersurface X in weighted projective stack bundle $\mathbb{P}_S^{N-1}(a_1, \ldots, a_N) := [S \times \mathbb{A}^N \backslash 0 / G_1]$ over S defined by the section corresponding to a G_1-invariant section $F \in \Gamma(\mathbb{A}_S^N, \mathcal{O}(\chi_d))^{G_1}$.

(4) Of course, Orlov's theorem in [Orl09] is much more general. It covers non-commutative situations unlike our setting.

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References

ADS15 N. Addington, W. Donovan and E. Segal, *The Pfaffian–Grassmannian equivalence revisited*, Algebr. Geom. **2** (2015), 332–364.

BDFIK13 M. Ballard, D. Deliu, D. Favero, M. U. Isik and L. Katzarkov, *Homological projective duality via variation of geometric invariant theory quotients*, J. Eur. Math. Soc. (JEMS), to appear. Preprint (2013), arXiv:1306.3957.

BDFIK14 M. Ballard, D. Deliu, D. Favero, M. U. Isik and L. Katzarkov, *On the derived categories of degree d hypersurface fibrations*, Preprint (2014), arXiv:1409.5568.

BDFIK16 M. Ballard, D. Deliu, D. Favero, M. U. Isik and L. Katzarkov, *Resolutions in factorization categories*, Adv. Math. **295** (2016), 195–249.

BFK12 M. Ballard, D. Favero and L. Katzarkov, *Variation of geometric invariant theory quotients and derived categories*, J. Reine Angew. Math., to appear. Preprint (2012), arXiv:1203.6643.

BFK14 M. Ballard, D. Favero and L. Katzarkov, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 1–111.

EP15 A. I. Efimov and L. Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), 1159–1292.

Hal15 D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), 871–912.

Har66 R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, vol. 20 (Springer, Heidelberg, 1966).

Hir17 Y. Hirano, *Equivalences of derived factorization categories of gauged Landau–Ginzburg models*, Adv. Math. **306** (2017), 200–278.

Isi13 M. U. Isik, *Equivalence of the derived category of a variety with a singularity category*, Int. Math. Res. Not. IMRN **2013** (2013), 2787–2808.

Joh02 P. T. Johnstone, *Sketches of an elephant: a topos theory compendium, Vol. 1*, Oxford Logic Guides, vol. 43 (The Clarendon Press, Oxford University Press, New York, 2002).

Knö87 H. Knörrer, *Cohen–Macaulay modules on hypersurface singularities I*, Invent. Math. **88** (1987), 153–164.

Kuz06 A. G. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), 23–128 (Russian); translation in Izv. Math. **70** (2006), 447–547.

LS16 V. A. Lunts and O. M. Schnürer, *Matrix factorizations and semi-orthogonal decompositions for blowing-ups*, J. Noncommut. Geom. **10** (2016), 907–979.

MFK94 D. Mumford, J. Forgarty and F. Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, third edition (Springer, Berlin, 1994).

Nee92 A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. Éc. Norm. Supér. (4) **25** (1992), 547–566.

Orl04 D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau–Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), 240–262 (Russian); translation in Proc. Steklov Math. Inst. **246**, (2004), 227–248.

Orl06 D. O. Orlov, *Triangulated categories of singularities and equivalences between Landau–Ginzburg models*, Mat. Sb. **197** (2006), 117–132; translation in Sb. Math. **197** (2006), 1827–1840.

Orl09 D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Vol. II*, Progress in Mathematics, vol. 270 (Birkhäuser, Boston, MA, 2009), 503–531.

PV11 A. Polishchuk and A. Vaintrob, *Matrix factorizations and singularity categories for stacks*, Ann. Inst. Fourier (Grenoble) **61** (2011), 2609–2642.

Pos11 L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Memoirs of the American Mathematical Society, vol. 212 (American Mathematical Society, Providence, RI, 2011).

Pos12 L. Positselski, *Contraherent cosheaves*, Preprint (2012), arXiv:1209:2995.

Ren15 J. V. Rennemo, *The homological projective dual of* $\mathrm{Sym}^2\mathbb{P}(V)$, Preprint (2015), arXiv:1509.04107.

Seg11 E. Segal, *Equivalence between GIT quotients of Landau–Ginzburg B-models*, Comm. Math. Phys. **304** (2011), 411–432.

ST14 E. Segal and R. Thomas, *Quintic threefolds and Fano elevenfolds*, J. Reine Angew. Math., to appear. Preprint (2014), arXiv:1410.6829.

Shi12 I. Shipman, *A geometric approach to Orlov's theorem*, Compositio Math. **148** (2012), 1365–1389.

Tho87 R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*, Adv. Math. **65** (1987), 16–34.

TT90 R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, in *The Grothendieck Festschrift, Vol. II*, Progress in Mathematics, vol. 88 (Birkhäuser, Boston, MA, 1990), 247–435.

TY03 M. Tousi and S. Yassemi, *Tensor products of some special rings*, J. Algebra **268** (2003), 672–676.

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