J. Aust. Math. Soc. **101** (2016), 1–28
doi:10.1017/S1446788715000774

GLOBALIZATION OF TWISTED PARTIAL HOPF ACTIONS

MARCELO M. S. ALVES, ELIEZER BATISTA,
MICHAEL DOKUCHAEV and ANTONIO PAQUES✉

(Received 24 March 2015; accepted 4 August 2015; first published online 26 February 2016)

Communicated by A. Sims

Abstract

In this work, we review some properties of twisted partial actions of Hopf algebras on unital algebras and give necessary and sufficient conditions for a twisted partial action to have a globalization. We also elaborate a series of examples.

2010 *Mathematics subject classification*: primary 16T05; secondary 16S40.

Keywords and phrases: partial group action, partial Hopf action, crossed product, twisted partial action.

1. Introduction

Partial group actions arose in operator algebra theory in [25–27, 35], in order to characterize C^*-algebras generated by partial isometries as more general crossed products. The algebraic study of partial actions and related notions was initiated in [19, 20, 27, 33, 39, 40], motivating further investigations. In particular, the Galois theory of partial group actions developed in [23] inspired further Galois theoretic results in [11, 15, 30, 34, 38], as well as the introduction and study of partial Hopf actions and coactions in [16], which, in turn, became the starting point for further research in partial Hopf (co)actions in [4–9].

The concepts of a twisted partial action and the corresponding crossed product, introduced for C^*-algebras in [26] and adapted for abstract rings in [21], suggested the idea of extending these notions for Hopf algebras, unifying twisted partial group actions, partial Hopf actions and twisted actions of Hopf algebras. This was done in [8], where such crossed products were related to the so-called partially cleft extensions of algebras and examples were elaborated using algebraic groups.

One source of examples of partial actions of groups on algebras is by restricting a global action to an ideal. More precisely, if a group G acts on a unital algebra B and

The third and the fourth authors were partially supported by Fapesp of Brazil. The first and third authors were also partially supported by CNPq of Brazil.

$e \in B$ is a central idempotent, there is a canonical partial action of the same group G over the unital ideal $A = eB$ viewed as a unital algebra. The same occurs in the theory of partial actions of Hopf algebras: given a Hopf algebra (or simply a bialgebra) H and a left H-module algebra B, if $e \in B$ is a central idempotent, then the unital ideal $A = eB$ becomes a left partial H-module algebra. The globalization problem consists in determining whether a given partial action originates from a global action by means of a restriction.

The study of the globalization problem of partial group actions was initiated in the PhD thesis of Abadie [1] (see also [2]), and independently by Steinberg in [40] and Kellendonk and Lawson in [33]. Other globalization results were obtained in [6, 7, 10–14, 17–19, 22, 24, 28, 29, 31, 36]. In particular, in [5] a globalization theorem for the case of partial actions of Hopf algebras on unital algebras was proved, whereas in [22] the globalization of twisted partial group actions on rings was investigated. The importance of the globalization problem lies in the possibility of relating partial actions with global ones and trying to generalize, for the partial case, results valid for global actions.

As a first example of how the technique of globalization can be used to prove results we mention [23], in which the globalization theorem was used to study the Galois theory for partial actions of groups on commutative algebras. More precisely, the authors prove an isomorphism between the invariant subalgebra with relation to the partial action and the invariant subalgebra with respect to the global action. This helped to generalize the Chase–Harrison–Rosenberg theory of Galois extensions of commutative algebras to the case of partial actions. The second example of the usefulness of the globalization theorem is given in [7], in which the analogue of the version of the Blattner–Montgomery theorem for the partial smash product was done. In the classical case, given a finite-dimensional Hopf algebra H and a left H-module algebra A, we have the isomorphism $(A\#H)\#H^* \cong A \otimes \mathrm{End}(H)$. In the partial case, the isomorphism is not valid anymore, but with the aid of the globalization theorem, we can calculate the kernel and the image of the morphism $\Phi : \underline{A\#H}\#H^* \to A \otimes \mathrm{End}(H)$.

The aim of this article is to go further in the investigation initiated in [8] by considering the globalization of twisted partial Hopf actions. In Section 2 we recall some basic concepts about twisted partial actions of Hopf algebras, which we formalize now in the form which is most convenient for our purpose. We start with the concept of a partial measuring map of a bialgebra (or Hopf algebra) H on a unital algebra A, giving, in our examples, a complete characterization of these maps in the case of H being the group algebra κG, the dual $(\kappa G)^*$ of the group algebra κG of a finite group G, and the Sweedler Hopf algebra H_4 and A coinciding (in all three cases) with the base field κ. Then we proceed with the concept of a twisted partial action of a Hopf algebra (or, more generally, a bialgebra), in which we include all properties needed to guarantee that the partial crossed product is associative and unital. Next we consider the symmetric twisted partial Hopf actions and describe them in detail for specific Hopf algebras. In particular, the case of a group algebra κG recovers the theory of twisted partial actions of groups as developed in [21, 22]. For the

Sweedler Hopf algebra H_4, the only symmetric twisted partial actions are the global ones. Furthermore, for the case of the dual of the group algebra $(\kappa G)^*$ of a finite group G, the partial cocycles appearing have remarkable symmetries and we relate them to global cocycles of dual group algebras of quotient groups. We present in detail the specific example for the Klein four-group K_4, acting on the base field κ, and show that the symmetric twisted partial actions of $(\kappa K_4)^*$ on κ are parameterized by the zeros $(x, y) \in \kappa^2$ of a polynomial in x, y of degree 2.

Section 3 is dedicated to the globalization theorem itself. The globalization of a symmetric twisted partial action of a Hopf algebra H on a unital algebra A given by a symmetric pair of partial cocycles ω and ω' consists of the following data: (1) an algebra B (not necessarily unital); (2) a twisted (global) action of H on B with convolution invertible cocycle $v : H \otimes H \to B$; (3) an algebra monomorphism $\varphi : A \to B$ such that its image $\varphi(A)$ is a unital ideal in B and the restriction of this global twisted action is isomorphic to the original twisted partial action on A. The main result of this paper can be formulated as follows: a symmetric twisted partial action of a Hopf algebra H on a unital algebra A associated to the symmetric pair of partial cocycles ω and ω' is globalizable if and only if there exists a normalized convolution invertible linear map $\tilde{\omega} : H \otimes H \to A$ satisfying certain compatibility conditions for intertwining the partial action of H on A and the restriction of the twisted action of H on B. The question of the uniqueness of a globalization is not so straightforward to address in this context as it is for partial actions of groups [19] and partial actions of Hopf algebras [6]. Even the form of the algebra B cannot be given in a simple way for the general case. We conclude the section with two examples. The first shows that the example of a twisted partial Hopf action, constructed in [8] using the relation between algebraic groups and Hopf algebras, is globalizable. The second is an explicit partial cocycle for $(\kappa K_4)^*$ which leads to a globalizable symmetric twisted partial action.

2. Symmetric twisted partial actions

In this paper, unless otherwise explicitly stated, κ will denote an arbitrary (associative) unital commutative ring and unadorned $\otimes$ will mean $\otimes_\kappa$. We also omit the sum in the Sweedler notation for the comultiplication of a Hopf algebra. In what follows the symbol 1 will stand for the unit element 1_κ of κ.

Definition 2.1. Let H be a κ-bialgebra and A a unital κ-algebra with unit element $\mathbf{1}_A$. We say that H partially measures A if there is a linear map

$$\begin{aligned} \cdot : H \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a, \end{aligned}$$

satisfying the following conditions:

(PM1) $1_H \cdot a = a$, for every $a \in A$;

(PM2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for every $h \in H$ and $a, b \in A$;

(PM3) $h \cdot (k \cdot \mathbf{1}_A) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)} k \cdot \mathbf{1}_A)$, for every $h, k \in H$.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

The map $\cdot : H \otimes A \to A$ is called a measuring map of H on A.

Note that if $h \cdot \mathbf{1}_A = \epsilon(h)\mathbf{1}_A$, then we have the usual notion that H measures A.

Example 2.2. Note that every partial Hopf action in the sense defined by Caenepeel and Janssen [16] is an example of a measuring map of a Hopf algebra H on a unital algebra A. Indeed, if H is a Hopf algebra then a partial action of H on a unital κ-algebra A is a linear map

$$\begin{aligned} H \otimes A &\to A, \\ h \otimes a &\mapsto h \cdot a, \end{aligned}$$

which satisfies (PM1), (PM2) and

$$h \cdot (k \cdot a) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)}k \cdot a) \quad \text{for every } h, k \in H, a \in A. \tag{2.1}$$

It is readily seen that a measuring of a Hopf algebra on the base field is automatically a partial action. In what follows we revisit three examples of such measurings that appeared previously in [4, 9]: we provide more detailed versions of [4, Example 3.7] and [9, Proposition 6.2] in Examples 2.3 and 2.4, respectively, and in Example 2.5 we redo the calculations of [4, Example 3.8], regarding partial actions of the Sweedler algebra on κ, in a simpler and more readable manner.

Example 2.3. Let G be a group and κ a field. One can classify all the partial measuring maps of the Hopf algebra κG on the base field κ, which correspond to the partial actions of G on κ. The map $\cdot : \kappa G \otimes \kappa \to \kappa$ can be viewed as a linear functional $\lambda : \kappa G \to \kappa$ given by $\lambda_g = \lambda(g) = g \cdot 1$. Condition (PM1) implies that $\lambda(e) = 1$ (where e denotes 1_G) and condition (PM2) means that $\lambda_g = \lambda_g \lambda_g$, which implies that either $\lambda_g = 1$ or $\lambda_g = 0$. Consider now the following subset of the group G:

$$L = \{g \in G \mid \lambda_g = 1\}.$$

By (PM3), we have that $\lambda_g \lambda_h = \lambda_g \lambda_{gh}$; therefore, if $g, h \in L$ then $gh \in L$. In the same way, putting $h = g^{-1}$, one can conclude that if $g \in L$ then $g^{-1} \in L$ and it follows that L is a subgroup of G. Therefore, the partial measurings of the Hopf algebra κG on the base field κ are classified by the subgroups of G.

Example 2.4. Let G be a finite group. We can classify all the partial measuring maps of the Hopf algebra $(\kappa G)^*$ on a field κ. By [9] they correspond to the partial G-gradings of κ. Again, they can be classified by linear functionals $\lambda : (\kappa G)^* \to \kappa$, defined by $\lambda(p_g) = p_g \cdot 1$, where $\{p_g\}_{g \in G}$ is the canonical basis of $(\kappa G)^*$. We know, from (PM1), that $\sum_{g \in G} \lambda(p_g) = 1$; then there are some $g \in G$ such that $\lambda(p_g) \neq 0$. Again, consider the subset

$$L = \{g \in G \mid \lambda(p_g) \neq 0\}.$$

Condition (PM3) reads

$$\lambda(p_g)\lambda(p_h) = \lambda(p_{gh^{-1}})\lambda(p_h).$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

This implies that, if $g, h \in L$ then $gh^{-1} \in L$, therefore L is a subgroup of G. Moreover, putting $h = g$, we obtain $\lambda(p_g) = \lambda(p_e)$ for every $g \in L$. Finally, by (PM1), we have

$$\sum_{g\in G} \lambda(p_g) = \sum_{g\in L} \lambda(p_e) = |L|\lambda(p_e) = 1,$$

which implies that $\operatorname{char} \kappa \nmid |L|$ and

$$\lambda(p_g) = \lambda(p_e) = \frac{1}{|L|} \quad \forall g \in L.$$

Therefore, the partial G-gradings of the base field κ are classified by the subgroups $L \leq G$, whose order is not divisible by $\operatorname{char} \kappa$ and the linear functionals $\lambda^L : (\kappa G)^* \to \kappa$ given by

$$\lambda^L(p_g) = \begin{cases} \dfrac{1}{|L|} & \text{if } g \in L \\ 0 & \text{if } g \notin L. \end{cases}$$

EXAMPLE 2.5. Consider the four-dimensional Sweedler Hopf algebra $H_4 = H = \langle 1_H, x, g \mid x^2 = 0, g^2 = 1_H, gx = -xg\rangle$ over a field κ, whose characteristic is not 2, with comultiplication and counit given by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad \Delta(x) = x \otimes 1_H + g \otimes x, \quad \epsilon(x) = 0.$$

Let us classify the partial measuring maps of H_4 on the base field κ. Each partial measuring map is given by a functional $\lambda : H_4 \to \kappa$ defined as $\lambda_h = \lambda(h) = h \cdot 1$. By (PM1) we have that $\lambda_{1_H} = 1$, and by (PM2) we conclude that $\lambda_g = \lambda_g\lambda_g$, then $\lambda_g = 1$ or $\lambda_g = 0$. Assuming $\lambda_g = 1$, we have from (PM3) that

$$\begin{aligned} \lambda_g\lambda_x &= \lambda_g\lambda_{gx}, \\ \lambda_x\lambda_g &= \lambda_x\lambda_g + \lambda_g\lambda_{xg}, \\ \lambda_{xg}\lambda_g &= \lambda_{xg}\lambda_{1_H} + \lambda_{1_H}\lambda_x. \end{aligned}$$

The first equality implies that $\lambda_x = \lambda_{gx} = -\lambda_{xg}$ and the second leads to $\lambda_x = 0$. Therefore, for $\lambda_g = 1$, the only solution is the global case $\lambda = \epsilon$. Considering instead $\lambda_g = 0$, we obtain from the above equations again that $\lambda_x = \lambda_{gx} = -\lambda_{xg}$ and, in fact, no other constraint appears. Therefore, for $\lambda_g = 0$ one can assign any value for $\lambda_x \in \kappa$ and each such functional λ gives rise to a partial measuring of H_4 on κ.

The definition of a twisted partial action of a bialgebra (or Hopf algebra) H on a unital algebra A was given in [8, Definition 1] in a rather general form. We shall use the following version which already incorporates the normalization condition [8, (16)], the 2-cocycle equality [8, (17)] and also (iii) of [8, Definition 2].

DEFINITION 2.6. A twisted partial action of a bialgebra (or Hopf algebra) H on a unital algebra A consists of two linear maps, $\cdot : H \otimes A \to A$ and $\omega : H \otimes H \to A$, satisfying the following conditions:

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

(TPA1) H partially measures A via the linear map;

(TPA2) $(h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot a)$, for every $h, l \in H$ and $a \in A$;

(TPA3) $\omega(h, l) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot \mathbf{1}_A)$, for every $h, l \in H$;

(TPA4) $\omega(h, 1_H) = \omega(1_H, h) = h \cdot \mathbf{1}_A$, for every $h \in H$;

(TPA5) $(h_{(1)} \cdot \omega(k_{(1)}, l_{(1)}))\omega(h_{(2)}, k_{(2)}l_{(2)}) = \omega(h_{(1)}, k_{(1)})\omega(h_{(2)}k_{(2)}, l)$, for every h, k, $l \in H$.

The algebra A, in its turn, is called a twisted partial H-module algebra.

The map ω is called a partial 2-cocycle or twisting. If $h \cdot \mathbf{1}_A = \epsilon(h)\mathbf{1}_A$, then this partial measuring coincides with the usual definition as in [37] and the twisted partial action, in this case, is merely a twisted action in the usual sense. On the other hand, if the partial 2-cocycle satisfies $\omega(h, k) = h \cdot (k \cdot \mathbf{1}_A)$, then it is called a trivial partial cocycle and the resulting twisted partial action is the usual partial action, as defined in [16].

Note that by [8, Proposition 1], Definition 2.6 implies

$$\omega(h, l) = (h_{(1)} \cdot (l_{(1)} \cdot \mathbf{1}_A))\omega(h_{(2)}, l_{(2)}) = (h_{(1)} \cdot \mathbf{1}_A)\omega(h_{(2)}, l), \tag{2.2}$$

for every $h, l \in H$.

Example 2.7. A source of examples of twisted partial actions can be given by the restriction to a unital ideal of a twisted action. More precisely, let H be a Hopf algebra with invertible antipode and B a, possibly nonunital, algebra measured by H through the linear map $\triangleright : H \otimes B \to B$. In the nonunital case, we simply consider the condition

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b),$$

since there is no meaning for the expression $h \triangleright 1_B = \epsilon(h)1_B$. A twisted action of H on B is given by this measuring and a convolution invertible linear map $u : H \otimes H \to M(B)$, where $M(B)$ is the multiplier algebra of B, such that

$$u(h, 1_H) = u(1_H, h) = \epsilon(h)1_{M(B)},$$
$$(h_{(1)} \triangleright (l_{(1)} \triangleright a))u(h_{(2)}, l_{(2)}) = u(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \triangleright a),$$
$$(h_{(1)} \overline{\triangleright} u(k_{(1)}, l_{(1)}))u(h_{(2)}, k_{(2)}l_{(2)}) = u(h_{(1)}, k_{(1)})u(h_{(2)}k_{(2)}, l).$$

Note that the first and the third equalities above are identities of multipliers; $1_{M(B)}$ is simply the identity map in B. The map $\overline{\triangleright} : H \otimes M(B) \to M(B)$ which appears in the third identity requires a more detailed explanation. First, the multiplier algebra $M(B)$ can be viewed as a pullback [32]

$$\begin{array}{ccc} M(B) & \xrightarrow{\pi_1} & \mathrm{End}_B(B) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \lambda} \\ {}_B\mathrm{End}(B) & \xrightarrow[\rho]{} & {}_B\mathrm{Hom}_B(B \otimes B, B) \end{array}$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

where the map $\lambda : \mathrm{End}_B(B) \to {}_B\mathrm{Hom}_B(B \otimes B, B)$ is given by $\lambda(L)(a \otimes b) = aL(b)$, the map $\rho : {}_B\mathrm{End}(B) \to {}_B\mathrm{Hom}_B(B \otimes B, B)$ is given by $\rho(R)(a \otimes b) = R(a)b$, and π_1 and π_2 are the canonical projections. That is equivalent to the classical definition of the multiplier algebra[1] as the algebra generated by pairs $(L, R) \in \mathrm{End}_B(B) \times {}_B\mathrm{End}(B)$ satisfying the identity $aL(b) = R(a)b$. For simplicity, let us denote a multiplier $x = (L, R)$ such that $L(a) = xa$ and $R(a) = ax$. Then we can use the universal property of a pullback in order to define an action $\overline{\triangleright} : H \otimes M(B) \to M(B)$. First, define two linear maps $\Phi : H \otimes M(B) \to \mathrm{End}_B(B)$ and $\Psi : H \otimes M(B) \to {}_B\mathrm{End}(B)$ respectively by

$$\Phi(h \otimes x)(b) = h_{(1)} \triangleright (x(S(h_{(2)}) \triangleright b)),$$
$$\Psi(h \otimes x)(b) = h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright b)x).$$

It is easy to see that $\Phi(h \otimes x)$ really belongs to $\mathrm{End}_B(B)$, indeed

$$\begin{aligned}
\Phi(h \otimes x)(ab) &= h_{(1)} \triangleright (x(S(h_{(2)}) \triangleright (ab)))\\
&= h_{(1)} \triangleright (x((S(h_{(3)}) \triangleright a)(S(h_{(2)}) \triangleright b)))\\
&= h_{(1)} \triangleright ((x(S(h_{(3)}) \triangleright a))(S(h_{(2)}) \triangleright b))\\
&= (h_{(1)} \triangleright (x(S(h_{(4)}) \triangleright a)))(h_{(2)} \triangleright (S(h_{(3)}) \triangleright b))\\
&= (h_{(1)} \triangleright (x(S(h_{(4)}) \triangleright a)))(h_{(2)}S(h_{(3)}) \triangleright b)\\
&= (h_{(1)} \triangleright (x(S(h_{(2)}) \triangleright a)))b\\
&= (\Phi(h \otimes x)(a))b.
\end{aligned}$$

Similarly, one proves that $\Psi(h \otimes x) \in {}_B\mathrm{End}(B)$. Finally, $\lambda \circ \Phi = \rho \circ \Psi$. Indeed, consider $a, b \in B$, $h \in H$ and $x \in M(B)$. Then

$$\begin{aligned}
\lambda \circ \Phi(h \otimes x)(a \otimes b) &= a(h_{(1)} \triangleright (x(S(h_{(2)}) \triangleright b)))\\
&= (h_{(2)}S^{-1}(h_{(1)}) \triangleright a)(h_{(3)} \triangleright (x(S(h_{(4)}) \triangleright b)))\\
&= h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright a)(x(S(h_{(3)}) \triangleright b)))\\
&= h_{(2)} \triangleright (((S^{-1}(h_{(1)}) \triangleright a)x)(S(h_{(3)}) \triangleright b))\\
&= (h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright a)x))(h_{(3)}S(h_{(4)}) \triangleright b)\\
&= (h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright a)x))b\\
&= \rho \circ \Psi(h \otimes x)(a \otimes b).
\end{aligned}$$

Therefore, by the universal property of $M(B)$ as a pullback, there is a unique linear map $\overline{\triangleright} : H \otimes M(B) \to M(B)$ given by

$$(h \,\overline{\triangleright}\, x)b = h_{(1)} \triangleright (x(S(h_{(2)}) \triangleright b))$$
$$b(h \,\overline{\triangleright}\, x) = h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright b)x).$$

One can restrict this twisted action to an ideal $A = 1_A B$, where 1_A is a central idempotent in B. This generates a twisted partial action of H on A given by the partial measuring

$$h \cdot a = 1_A(h \triangleright a)$$

[1] The concept of a multiplier algebra first appeared in the context of C^*-algebras [3]. For abstract algebras its detailed definition can be seen, for example, in [19].

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

and the partial 2-cocycle

$$\omega(h,k) = (h_{(1)} \cdot 1_A)u(h_{(2)}, k_{(1)})(h_{(3)}k_{(2)} \cdot 1_A).$$

The proof that this indeed defines a twisted partial action is basically the same as given in [8].

We recall the definition of partial crossed product introduced in [8]. Given a twisted partial action $(H, A, \cdot, \omega)$, there is an associative product defined on the vector space $A \otimes H$:

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b)\omega(h_{(2)}, k_{(1)}) \otimes h_{(3)}k_{(2)}.$$

The algebra $A \otimes H$ is not unital in general. Nevertheless, it contains a subalgebra $A\#_\omega H = (A \otimes H)(\mathbf{1}_A \otimes 1_H)$ with unit element $(\mathbf{1}_A \otimes 1_H)$, called the partial crossed product. As a subspace of $A \otimes H$, it is generated by the elements

$$a\#h = a(h_{(1)} \cdot \mathbf{1}_A) \otimes h_{(2)}.$$

Given any global 2-cocycle $v : H \otimes H \to \kappa$, where κ has the global H-action, $h \cdot 1 = \epsilon(h)$, one usually transfers the algebra structure of the crossed product $\kappa\#_v H$ to H via the canonical isomorphism $\kappa \otimes H \to H$. This algebra is denoted by ${}_vH$ and the product is given by

$$h \bullet_v k = v(h_{(1)}, k_{(1)})h_{(2)}k_{(2)}.$$

If $(H, \kappa, \cdot, \omega)$ is a twisted partial action then the canonical isomorphism $\kappa \otimes H \simeq H$ defines on H the new product

$$h \bullet_\omega k = (h_{(1)} \cdot 1)\omega(h_{(2)}, k_{(1)})h_{(3)}k_{(2)}.$$

We will denote this algebra by ${}_\omega H$. The partial crossed product, which will be denoted by ${}_\omega \underline{H}$, can be identified with the subspace of H generated by the elements $(h_{(1)} \cdot 1)h_{(2)}$, and the product of generators is given by

$$((h_{(1)} \cdot 1)h_{(2)}) \bullet_\omega ((k_{(1)} \cdot 1)k_{(2)}) = (h_{(1)} \cdot 1)\omega(h_{(2)}, k_{(1)})h_{(3)}k_{(2)}.$$

Later on, we will describe the algebra ${}_\omega \underline{H}$ when $H = \kappa G$ and $H = (\kappa G)^*$.

Recall that a twisted partial action of a group G on a unital κ-algebra A is defined in [21] as a triple

$$(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}),$$

where, for each $g, h \in G$, D_g is an ideal of A, $\alpha_g : D_{g^{-1}} \to D_g$ is a κ-algebra isomorphism, $w_{g,h}$ is an invertible multiplier of $D_g D_{gh}$, and some properties are satisfied. If each D_g is generated by a central idempotent 1_g, then the twisted partial action is called unital and, as was pointed out in [8], this matches our concept of a partial action of the group algebra κG on A. In this case, $1_g = g \cdot 1_A$, and it is reasonable to have under consideration partial actions of a Hopf algebra H on some unital algebra A such that the map $\mathbf{e} \in \mathrm{Hom}_k(H, A)$, given by $\mathbf{e}(h) = (h \cdot \mathbf{1}_A)$, is central with respect to the convolution product. These partial actions are, in some sense, more akin to partial group actions.

Furthermore, in the case of a unital twisted partial group action the $\omega_{g,h}$ are invertible elements in $D_g D_{gh}$, for all $g, h \in G$. In particular, if the group action is global, then every element $\omega_{g,h}$ is invertible in A, which is automatically translated into the Hopf algebra setting by saying that the cocycle $\omega \in \mathrm{Hom}_k(H \otimes H, A)$ is convolution invertible. In the partial case, we have to consider more suitable conditions to replace the convolution invertibility for the cocycle, and the idea is to define a unital ideal in the convolution algebra $\mathrm{Hom}_\kappa(H \otimes H, A)$, in which the partial 2-cocycle lives and has an inverse [8].

Let $A = (A, \cdot, \omega)$ be a twisted partial H-module algebra. It is easy to see that the linear maps $f_1, f_2 : H \otimes H \to A$, defined by $f_1(h, k) = (h \cdot \mathbf{1}_A)\epsilon(k)$ and $f_2(h, k) = (hk \cdot \mathbf{1}_A)$, are both (convolution) idempotents in $\mathrm{Hom}(H \otimes H, A)$. We also have that $\mathbf{e}$ is an idempotent in $\mathrm{Hom}(H, A)$ (and $f_1(h, k) = \mathbf{e}(h)\epsilon(k)$). Notice that if f_1 and f_2 are central in the convolution algebra $\mathrm{Hom}_\kappa(H \otimes H, A)$, then the partial 2-cocycle ω lies in the unital ideal $\langle f_1 * f_2 \rangle \trianglelefteq \mathrm{Hom}_\kappa(H \otimes H, A)$, thanks to (TPA3) and (2.2).

Definition 2.8 [8]. Let H be a κ-bialgebra (or Hopf algebra) and A be a unital κ-algebra. A symmetric twisted partial action of H on A is given by the data $(H, A, \cdot, \omega, \omega')$, such that:

(STPA1) $(H, A, \cdot, \omega)$ is a twisted partial action of H on A;

(STPA2) the above defined maps f_1 and f_2 are central in the convolution algebra $\mathrm{Hom}_\kappa(H \otimes H, A)$;

(STPA3) there exists a linear map $\omega' \in \langle f_1 * f_2 \rangle$ such that $\omega * \omega' = \omega' * \omega = f_1 * f_2$.

The algebra A is called a symmetric twisted partial H-module algebra.

Observe that the unit element of the ideal $\langle f_1 * f_2 \rangle$ is the central idempotent $f_1 * f_2$ itself, which thanks to (PM3) reads $f_1 * f_2(h, k) = h \cdot (k \cdot \mathbf{1}_A)$. Note also that the centrality of f_1 in $\mathrm{Hom}_\kappa(H \otimes H, A)$, automatically implies the centrality of $\mathbf{e} \in \mathrm{Hom}_k(H, A)$. With the aid of the inverse ω' one can rewrite condition (TPA2) in Definition 2.6 as

$$h \cdot (k \cdot a) = \omega(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \cdot a)\omega'(h_{(3)}, k_{(3)}). \tag{2.3}$$

Example 2.9. Any (global) twisted action of a Hopf algebra H on a unital algebra A is automatically symmetric. Indeed, as $h \cdot \mathbf{1}_A = \epsilon(h)\mathbf{1}_A$, we have $f_1(h, k) = f_2(h, k) = \epsilon(h)\epsilon(k)\mathbf{1}_A$. Then both f_1 and f_2 are equal to the unit of the convolution algebra $\mathrm{Hom}_k(H \otimes H, A)$, which can be absorbed on the left or on the right. More generally, given any (possibly nonunital) twisted H-module algebra B, with (global) twisted action given by the measuring map $\triangleright : H \otimes B \to B$ and an invertible and normalized 2-cocyle $u : H \otimes H \to M(B)$, one can define a twisted partial action of H on any unital ideal $A \trianglelefteq B$, generated by a central idempotent $\mathbf{1}_A \in B$, according to Example 2.7. Moreover, if f_1 and f_2 are central in $\mathrm{Hom}_\kappa(H \otimes H, \kappa)$, then this twisted partial action is symmetric, with the inverse (partial) cocycle given by

$$\omega'(h, k) = (h_{(1)}k_{(1)} \cdot \mathbf{1}_A)u^{-1}(h_{(2)}, k_{(2)})(h_{(3)} \cdot \mathbf{1}_A).$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

REMARK 2.10. If H is co-commutative and A is commutative, then for every twisted partial action of H on A the functions f_1 and f_2 are clearly central, because the convolution algebra $\mathrm{Hom}_k(H \otimes H, A)$ is commutative. Moreover, if the twisted partial action is symmetric (i.e. satisfies (STPA3)), then the twisting disappears from equality (2.3), which now simply means that $h \cdot (k \cdot a) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)}k \cdot a)$, resulting in a partial action of H on A. Thus in this case a symmetric twisted partial action disassembles into a pair consisting of a partial action of H on A and a partial 2-cocycle ω with (STPA3). A partial 2-cocycle now means a κ-linear map $\omega : H \otimes H \to A$ satisfying conditions (TPA3)–(TPA5) from Definition 2.6.

EXAMPLE 2.11. Let us consider the specific case of twisted partial actions of a group algebra κG on a field κ. Then every twisted partial action of κG on κ is automatically symmetric. Indeed, as we know from Example 2.3, the measuring maps are classified by subgroups $L \leq G$ such that $\lambda_g = 1$ if $g \in L$ and $\lambda_g = 0$ otherwise. One can classify every twisted partial action of κG on κ, determining the possible values for the partial 2-cocycle ω. Equality (2.2) gives us

$$\omega(g, h) = \lambda_g \lambda_h \omega(g, h) \quad \forall g, h \in G.$$

Then automatically $\omega(g, h) = 0$ if one of its two entries does not belong to the subgroup L. The cocycle condition (TPA5) in Definition 2.6 reads

$$\lambda_g \omega(h, k)\omega(g, hk) = \omega(g, h)\omega(gh, k),$$

which, taking $g, h, k \in L$, transfers into the classical 2-cocycle condition for the group cohomology of the group L with values in the trivial L-module κ^*:

$$\omega(h, k)\omega(g, hk) = \omega(g, h)\omega(gh, k).$$

Conversely, given a classical 2-cocycle $\omega : L \times L \to \kappa^*$, one may extend ω to $\kappa G \otimes \kappa G$, by setting $\omega(g, h) = 0$ if g or h does not belong to L, and thus obtaining a partial 2-cocycle. Hence the partial 2-cocycles of κG related to the given partial action of κG on κ can be identified with the classical 2-cocycles of the subgroup L with values in κ^*. Obviously, setting $\omega'(g, h) = \omega^{-1}(g, h)$ if and only if both g and h lie in L, we see that the twisted partial action $(\kappa G, \kappa, \cdot, \omega)$ is necessarily symmetric.

In order to describe the crossed product in this case, let ν be a 2-cocycle of L and let ω be its extension to κG as above. The vector space ${}_\omega\underline{\kappa G}$ is generated by the elements $\lambda_g g$, where $\lambda_g = 1$ if $g \in L$ and zero otherwise, hence ${}_\omega\underline{\kappa G} = \kappa L$ as a vector space. It also follows that ${}_\omega\underline{\kappa G} = {}_\nu\kappa L$ as an algebra. Observe that ${}_\nu\kappa L$ is the twisted group ring of L over κ corresponding the 2-cocycle ν.

EXAMPLE 2.12. The only symmetric twisted partial actions of the Sweedler Hopf algebra $H = H_4$ on a κ-algebra A are the global ones. Indeed, for any linear function $\phi : H_4 \otimes H_4 \to A$, we have

$$f_1 * \phi(x, g) = f_1(x, g)\phi(1_H, g) + f_1(g, g)\phi(x, g) = (x \cdot \mathbf{1}_A)\phi(1_H, g) + (g \cdot \mathbf{1}_A)\phi(x, g),$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

while, on the other hand,

$$\phi * f_1(x,g) = \phi(x,g)f_1(1_H,g) + \phi(g,g)f_1(x,g) = \phi(x,g) + \phi(g,g)(x\cdot \mathbf{1}_A).$$

Taking ϕ with $(g,g)\mapsto \mathbf{1}_A$, $(x,g)\mapsto 0$, and $(1_H,g)\mapsto 0$, we obtain that $x\cdot\mathbf{1}_A = 0 = \epsilon(x)\mathbf{1}_A$, and for ϕ with $(g,g)\mapsto 0$, $(x,g)\mapsto \mathbf{1}_A$, and $(1_H,g)\mapsto 0$, we get $g\cdot\mathbf{1}_A = \mathbf{1}_A = \epsilon(g)\mathbf{1}_A$. Then by (PM3),

$$0 = g\cdot(x\cdot\mathbf{1}_A) = (g\cdot\mathbf{1}_A)(gx\cdot\mathbf{1}_A) = gx\cdot\mathbf{1}_A,$$

so that $gx\cdot\mathbf{1}_A = \epsilon(gx)\mathbf{1}_A$, and by linearity, $h\cdot\mathbf{1}_A = \epsilon(h)\mathbf{1}_A$ for all $h\in H_4$. Consequently, f_1 is central in $\mathrm{Hom}_\kappa(H\otimes H, A)$ exactly when the twisted partial action is global.

Example 2.13. The above example shows, in particular, that there is no symmetric twisted partial action of the Sweedler Hopf algebra on the base field κ which is not global. Nevertheless, we shall construct a partial cocycle defining a (nonglobal) twisted partial action of $H = H_4$ on κ.

As we have seen in Example 2.5, the truly partial measuring maps of H_4 on κ are given by functionals $\lambda : H_4 \to \kappa$ such that $\lambda_g = 0$ (recall that λ_x may have any value in κ and $\lambda_x = \lambda_{gx} = -\lambda_{xg}$). Taking such a partial measuring, the normalization condition (TPA4) for the partial cocycle gives us

$$\omega(1_H,h) = \omega(h,1_H) = \lambda_h \quad \forall h\in H_4,$$

which means that $\omega(1_H,1_H) = 1$, $\omega(1_H,g) = \omega(g,1_H) = 0$, $\omega(x,1_H) = \omega(1_H,x) = -\omega(xg,1_H) = -\omega(1_H,xg) = \lambda_x$. The axioms (TPA2) and (TPA3), combined, allow us to write the identities

$$\omega(h,k) = \lambda_{h_{(1)}}\omega(h_{(2)},k) \tag{2.4}$$

and

$$\omega(h,k) = \lambda_{h_{(1)}}\lambda_{k_{(1)}}\omega(h_{(2)},k_{(2)}). \tag{2.5}$$

Then (2.4) gives

$$\begin{aligned}
\omega(g,h) &= \lambda_g\omega(g,h) = 0 \quad \forall h\in H_4,\\
\omega(x,x) &= \lambda_x\omega(1_H,x) + \lambda_g\omega(x,x) = \lambda_x^2,\\
\omega(x,xg) &= \lambda_x\omega(1_H,xg) + \lambda_g\omega(x,xg) = -\lambda_x^2,
\end{aligned}$$

and (2.5) results in

$$\begin{aligned}
\omega(h,g) &= \lambda_{h_{(1)}}\lambda_g\omega(h_{(2)},g) = 0 \quad \forall h\in H_4,\\
\omega(xg,x) &= \lambda_{xg}\lambda_x\omega(g,1_H) + \lambda_{xg}\lambda_g\omega(g,x) + \lambda_1\lambda_x\omega(xg,1_H) + \lambda_1\lambda_g\omega(xg,x)\\
&= -\lambda_x^2.
\end{aligned}$$

Moreover, (2.4) and (2.5) do not give any restriction on the value of $\omega(xg,xg)$. Then, taking arbitrary $\omega(xg,xg)\in\kappa$, it is readily seen that the map $\omega : H_4\otimes H_4\to\kappa$ also satisfies (TPA3) and, consequently, (TPA2). A list of verifications certifies that the cocycle identity (TPA5) holds for ω.

EXAMPLE 2.14. Given a finite group G and a field κ with char $\kappa \nmid |G|$, in order to classify the symmetric twisted partial actions of $(\kappa G)^*$ on κ, one needs first to analyse the centrality of f_1 and f_2. Consider a partial measuring map $\lambda : (\kappa G)^* \to \kappa$ determined by the subgroup L, such that $\lambda(p_g) = 1/|L|$ if $g \in L$ and $\lambda(p_g) = 0$ otherwise. Take any linear function $\phi : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ and $g, h \in G$. Then

$$f_1 * \phi(p_g, p_h) = \sum_{r,s\in G} f_1(p_r, p_s)\phi(p_{r^{-1}g}, p_{s^{-1}h}) = \sum_{r\in L} \lambda(p_r)\phi(p_{r^{-1}g}, p_h)$$

and

$$\phi * f_1(p_g, p_h) = \sum_{r,s\in G} \phi(p_{gr^{-1}}, p_{hs^{-1}}) f_1(p_r, p_s) = \sum_{r\in L} \lambda(p_r)\phi(p_{gr^{-1}}, p_h).$$

Then the equality $f_1 * \phi = \phi * f_1$ is true for every ϕ if and only if the subgroup L is contained in the centre of G. It is readily seen that in this case f_2 is also central.

The second step is to define a normalized partial 2-cocycle $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ which is invertible in the unital ideal generated by $f_1 * f_2$. If $\omega * f_1 * f_2 = \omega$, then it is easy to see that $\omega = \omega * f_1 = \omega * f_2$. Take $g, h \in G$. Then

$$\omega(p_g, p_h) = f_1 * \omega(p_g, p_h) = \frac{1}{|L|} \sum_{r\in L} \omega(p_{r^{-1}g}, p_h).$$

Therefore, $\omega(p_g, p_h) = \omega(p_{kg}, p_h)$ for any $k \in L$. For the identity $\omega = \omega * f_2$, we have

$$\omega(p_g, p_h) = f_2 * \omega(p_g, p_h) = \frac{1}{|L|} \sum_{r\in L} \omega(p_{r^{-1}g}, p_{r^{-1}h}),$$

which leads to the conclusion that $\omega(p_g, p_h) = \omega(p_{kg}, p_{kh})$ for any $k \in L$. Joining these two pieces of information, we obtain

$$\omega(p_g, p_h) = \omega(p_{kg}, p_{lh}) \quad \forall k, l \in L,$$

and considering the fact that L is a subgroup of the centre of G, then also

$$\omega(p_g, p_h) = \omega(p_{gk}, p_{hl}) \quad \forall k, l \in L.$$

Let us say that a κ-linear function $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ is L-invariant if ω satisfies the above two equalities. Notice that the above computations show that a κ-linear function $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ is L-invariant exactly when ω is contained in the ideal $\langle f_1 * f_2 \rangle$ of $\mathrm{Hom}_\kappa((\kappa G)^* \otimes (\kappa G)^*, A)$.

Taking into account this L-invariance of the partial 2-cocycle, the first member of the cocycle identity

$$(m_{(1)} \cdot \omega(n_{(1)}, l_{(1)}))\omega(m_{(2)}, n_{(2)}l_{(2)}) = \omega(m_{(1)}, n_{(1)})\omega(m_{(2)}n_{(2)}, l),$$

for $m = p_g$, $n = p_h$ and $l = p_k$, reads

$$\begin{aligned}(m_{(1)} \cdot \omega(n_{(1)}, l_{(1)}))\omega(m_{(2)}, n_{(2)}l_{(2)}) &= \sum_{r,s,t\in G} \lambda(p_r)\omega(p_{hs^{-1}}, p_{kt^{-1}})\omega(p_{r^{-1}g}, p_s p_t) \\ &= \sum_{s\in G} \sum_{r\in L} \frac{1}{|L|} \omega(p_{hs^{-1}}, p_{ks^{-1}})\omega(p_{r^{-1}g}, p_s)\end{aligned}$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

$$= \sum_{s\in G} \omega(p_{hs^{-1}}, p_{ks^{-1}}) \Big(\sum_{r\in L} \frac{1}{|L|} \omega(p_g, p_s) \Big)$$
$$= \sum_{s\in G} \omega(p_{hs^{-1}}, p_{ks^{-1}}) \omega(p_g, p_s).$$

The second member, in turn, is

$$\omega(m_{(1)}, n_{(1)})\omega(m_{(2)}n_{(2)}, l) = \sum_{s,t\in G} \omega(p_{gs^{-1}}, p_{ht^{-1}})\omega(p_s p_t, p_k)$$
$$= \sum_{s\in G} \omega(p_{gs^{-1}}, p_{hs^{-1}})\omega(p_s, p_k).$$

Therefore, the cocycle identity (TPA5) transforms into

$$\sum_{s\in G} \omega(p_{hs^{-1}}, p_{ks^{-1}})\omega(p_g, p_s) = \sum_{s\in G} \omega(p_{gs^{-1}}, p_{hs^{-1}})\omega(p_s, p_k), \tag{2.6}$$

for all $g, h, k \in G$. It is readily seen that (2.6) means that ω is a global 2-cocycle of $(\kappa G)^*$ with respect to the trivial action on κ. Let $G\backslash L$ be a full set of representatives of left (= right) cosets of G by L. Then an L-invariant function $\omega \in \mathrm{Hom}_\kappa((\kappa G)^* \otimes (\kappa G)^*, A)$ satisfies (2.6) if and only if

$$\sum_{s\in G\backslash L} \omega(p_{hs^{-1}}, p_{ks^{-1}})\omega(p_g, p_s) = \sum_{s\in G\backslash L} \omega(p_{gs^{-1}}, p_{hs^{-1}})\omega(p_s, p_k), \tag{2.7}$$

for all $g, h, k \in G\backslash L$.

Now let $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ be a partial 2-cocycle, so that our partial measuring, determined by the central subgroup L of G, is a symmetric twisted partial action. Then, taking $v(p_{gL}, p_{hL}) = |L|^2\omega(p_g, p_h)$ for g, h in $G\backslash L$, we have, thanks to the L-invariance of ω, a well-defined κ-linear map $v : (\kappa\, G/L)^* \otimes (\kappa\, G/L)^* \to \kappa$, which in view of (2.7) satisfies the global 2-cocyle equality with respect to the trivial action on κ. The normalization condition (TPA4) of ω directly implies that v is normalized. Furthermore, since our twisted partial action is symmetric, there exists $\omega' : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ satisfying (STPA3). Taking $u(p_{gL}, p_{hL}) = |L|^2\omega'(p_g, p_h)$, we compute that

$$(v * u)(p_{gL}, p_{hL}) = \sum_{s,t\in G\backslash L} v(p_{gs^{-1}L}, p_{ht^{-1}L})u(p_{sL}, p_{tL})$$
$$= |L|^4 \sum_{s,t\in G\backslash L} \omega(p_{gs^{-1}}, p_{ht^{-1}})\omega'(p_s, p_t)$$
$$= \frac{|L|^4}{|L|^2} \sum_{s,t\in G\backslash L} \sum_{l,m\in L} \omega(p_{gs^{-1}l^{-1}}, p_{ht^{-1}m^{-1}})\omega'(p_{sl}, p_{tm})$$
$$= |L|^2 \sum_{x,y\in G} \omega(p_{gx^{-1}}, p_{hy^{-1}})\omega'(p_x, p_y)$$
$$= |L|^2(\omega * \omega')(p_g, p_h) = |L|^2\lambda(p_g)\lambda(p_h) = \epsilon(p_{gL})\epsilon(p_{hL}),$$

showing that u is the convolution inverse of v. Consequently, v is a normalized global convolution invertible 2-cocycle of $(\kappa\, G/L)^*$.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

Conversely, given a normalized convolution invertible global 2-cocycle v of $(\kappa G/L)^*$ with respect to the trivial action of $(\kappa G/L)^*$ on κ, define the κ-linear functions $\omega, \omega' : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ by

$$\omega(p_g, p_h) = \frac{1}{|L|^2} v(p_{gL}, p_{hL}) \quad \text{and} \quad \omega'(p_g, p_h) = \frac{1}{|L|^2} v^{-1}(p_{gL}, p_{hL}).$$

Then the above considerations imply that we obtain a symmetric twisted partial action of $(\kappa G)^*$ on κ.

Thus we have one-to-one correspondences between the following three sets:

- the partial 2-cocycles $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ with respect to the partial action of $(\kappa G)^*$ on κ, determined by the central subgroup $L \subseteq G$, which are invertible in the ideal $\langle f_1 * f_2 \rangle$;
- the L-invariant global 2-cocycles $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ with respect to the trivial action of $(\kappa G)^*$ on κ, which satisfy equalities (TPA4) and (STPA3);
- the normalized convolution invertible global 2-cocycles $v : (\kappa G/L)^* \otimes (\kappa G/L)^* \to \kappa$ with respect to the trivial action of $(\kappa G/L)^*$ on κ.

Now let $\omega : (\kappa G)^* \otimes (\kappa G)^* \to \kappa$ be a partial 2-cocycle such that we have a symmetric twisted partial action (i.e. ω is as in the first item above), and let $v : (\kappa G/L)^* \otimes (\kappa G/L)^* \to \kappa$ be the corresponding global 2-cocycle. Then the map

$$\phi : {}_v(\kappa G/L)^* \to {}_\omega\underline{(\kappa G)^*}, \qquad p_{gL} \mapsto \sum_{x \in L} p_{gx}$$

is an isomorphism of algebras. In fact, as a vector space, ${}_\omega\underline{(\kappa G)^*}$ is generated by the elements

$$\sum_{x \in G} \lambda(p_{x^{-1}}) p_{gx} = \frac{1}{|L|} \sum_{x \in L} p_{gx}.$$

If $G\backslash L$ is a transversal for L in G then $\{p_{tL} : t \in G\backslash L\}$ is a basis of ${}_v(\kappa G/L)^*$, and its image under ϕ consists of the elements

$$\sum_{x \in L} p_{tx} \quad (t \in G\backslash L),$$

which form a basis of ${}_\omega\underline{(\kappa G)^*}$; therefore ϕ is a linear isomorphism.

Using the L-invariance of ω, we see that the algebra structure on ${}_\omega(\kappa G)^*$ is defined on basis elements by

$$p_g \bullet_\omega p_h = \sum_{r,s,t \in G} \lambda(p_r) \omega(p_{r^{-1}gs^{-1}}, p_{ht^{-1}}) p_s p_t = \sum_{t \in G} \omega(p_{gt^{-1}}, p_{ht^{-1}}) p_t.$$

From this expression, the L-invariance and the definition of ω, it follows that

$$\begin{aligned} \phi(p_{tL}) \bullet_\omega \phi(p_{t'L}) &= \sum_{s \in G} \sum_{l,m \in L} \omega(p_{tls^{-1}}, p_{t'ms^{-1}}) p_s \\ &= |L|^2 \sum_{s \in G} \omega(p_{ts^{-1}}, p_{t's^{-1}}) p_s \end{aligned}$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

$$
\begin{aligned}
&= |L|^2 \sum_{s\in G\backslash L} \omega(p_{ts^{-1}}, p_{t's^{-1}}) \sum_{x\in L} p_{sx} \\
&= \phi\Big(|L|^2 \sum_{s\in G\backslash L} \omega(p_{ts^{-1}}, p_{t's^{-1}}) p_{sL}\Big) \\
&= \phi\Big(\sum_{s\in G\backslash L} \nu(p_{ts^{-1}L}, p_{t's^{-1}L}) p_{sL}\Big) \\
&= \phi(p_{tL} \bullet_\nu p_{t'L}),
\end{aligned}
$$

with $t, t' \in G\backslash L$. In addition, it is trivially seen that ϕ is unital.

Example 2.15. Consider the previous example for the specific case when the group G is the Klein 4-group $K_4 = \langle a, b \mid a^2 = b^2 = e\rangle$. Take the subgroup $L = \langle a\rangle$. Then the values of the partial measuring map λ are $\lambda(p_e) = \lambda(p_a) = \frac{1}{2}$ and $\lambda(p_b) = \lambda(p_{ab}) = 0$. As the group K_4 is abelian, the subgroup L is automatically in the centre, then we have no obstruction to the centrality of f_1 and f_2 (moreover, since K_4 is abelian, the Hopf algebra $(\kappa K_4)^*$ is co-commutative, and consequently the convolution algebra $\mathrm{Hom}_k((\kappa K_4)^* \otimes (\kappa K_4)^*, \kappa)$ is commutative). The L-invariance of the partial 2-cocycle reads

$$
\begin{aligned}
\omega(p_e, p_e) &= \omega(p_a, p_e) = \omega(p_e, p_a) = \omega(p_a, p_a),\\
\omega(p_e, p_b) &= \omega(p_a, p_b) = \omega(p_e, p_{ab}) = \omega(p_a, p_{ab}),\\
\omega(p_b, p_e) &= \omega(p_{ab}, p_e) = \omega(p_b, p_a) = \omega(p_{ab}, p_a),\\
\omega(p_b, p_b) &= \omega(p_{ab}, p_b) = \omega(p_b, p_{ab}) = \omega(p_{ab}, p_{ab}).
\end{aligned}
$$

The normalization condition, $\omega(1_H, h) = \omega(h, 1_H) = h \cdot \mathbf{1}_A$, gives us three conditions:

$$
\begin{aligned}
\omega(p_e, p_e) + \omega(p_e, p_b) &= \tfrac{1}{4},\\
\omega(p_e, p_e) + \omega(p_b, p_e) &= \tfrac{1}{4},\\
\omega(p_e, p_b) + \omega(p_b, p_b) &= 0.
\end{aligned}
$$

This leads to $\omega(p_b, p_e) = \omega(p_e, p_b) = -\omega(p_b, p_b)$ and $\omega(p_e, p_b) = \frac{1}{4} - \omega(p_e, p_e)$. Therefore, we end up with only one independent value of the partial 2-cocycle, $x = \omega(p_e, p_e)$. One directly checks that with these data, the 2-cocycle condition is satisfied for arbitrary value of x. Similarly, ω' is also L-invariant and normalized and can be determined in a similar form in terms of $y = \omega'(p_e, p_e)$. Furthermore, a direct computation shows that the condition $\omega * \omega' = f_1 * f_2$ is equivalent to the equality

$$
32xy - 6(x + y) + 1 = 0. \tag{2.8}
$$

Therefore the symmetric twisted partial actions of $(\kappa K_4)^*$ on κ are parameterized by the points $(x, y) \in \kappa^2$ satisfying (2.8).

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

3. Globalization

As already seen, taking a twisted action of a Hopf algebra H on an algebra B and then restricting it to a unital ideal A, one obtains a twisted partial Hopf action of H on A. Now, given a twisted partial action of a Hopf algebra H on a unital algebra A, we can ask what conditions must be satisfied in order to view it as a restriction of a global action. The globalization problem was solved in several contexts. In particular, for partial group actions on C^*-algebras the answer was given by Abadie (see [1] or [2]). For the case of partial group actions on unital rings a criterion for the existence of a globalization was given in [19], whereas for partial Hopf actions the globalization always exists, as was shown in [6]. As to the twisted partial group actions, the globalization is rather more involved, as can be seen in [22]. We recall briefly the meaning of a globalization of a twisted partial group action. We denote by $\mathcal{U}(B)$ the group of the invertible elements of an algebra B. Given a twisted partial action of G on an algebra A,

$$\{\{D_g\}_{g\in G}, \{\alpha_g : D_{g^{-1}} \to D_g\}_{g\in G}, \{w_{g,h} \in \mathcal{U}(D_g D_{gh})\}_{g,h\in G}\},$$

a globalization for this twisted partial action is a quadruple

$$\{B, \beta : G \to \mathrm{Aut}(B), \varphi, u : G \times G \to \mathcal{U}(M(B))\}$$

where $M(B)$ is the multiplier algebra of B, such that:

(1) (B, β, u) is a twisted action of G on B with cocycle u;
(2) $\varphi : A \to B$ is a monomorphism of algebras and $\varphi(A) \trianglelefteq B$;
(3) $\varphi(D_g) = \varphi(A) \cap \beta_g(\varphi(A))$;
(4) $B = \sum_{g\in G} \beta_g(\varphi(A))$;
(5) φ intertwines the twisted partial action of G on A with the induced twisted partial action on $\varphi(A)$ obtained by the restriction of the twisted action β on B.

In this section we will follow some ideas present in [22] to construct a globalization for a twisted partial action of a Hopf algebra H on a unital algebra A.

Definition 3.1. Let $(A, (w, w'))$, $(A', (v, v'))$ be two symmetric twisted partial H-module algebras. A map $\varphi : A \to A'$ is an isomorphism of symmetric twisted partial H-module algebras if, for all $h, k \in H$ and $a \in A$:

(i) φ is an algebra isomorphism;
(ii) $\varphi(h \cdot a) = h \cdot \varphi(a)$;
(iii) $\varphi(w(h, k)) = v(h, k)$, $\varphi(w'(h, k)) = v'(h, k)$.

Definition 3.2. Let A be a symmetric twisted partial H-module algebra with the pair (w, w'). A globalization of A is a pair (B, φ), where B is a (possibly nonunital) twisted H-module algebra with invertible cocycle $u : H \otimes H \to M(B)$ and $\varphi : A \to B$ is an algebra monomorphism such that:

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

(i) $\varphi(A)$ is an ideal in B;
(ii) $\varphi : A \to \varphi(A)$ is an isomorphism of symmetric twisted partial H-module algebras, where $\varphi(A)$ has the structure induced by B.

If the algebra B is unital then we call the globalization unital.

Our main result is as follows.

THEOREM 3.3. *Let A be a symmetric twisted partial H-module algebra with the pair (w, w').*

(1) *If this partial action has a globalization then there is a convolution invertible κ-linear map $\tilde{w} : H \otimes H \to A$ satisfying*

$$(h_{(1)} \cdot \tilde{w}(k_{(1)}, l_{(1)}))\tilde{w}(h_{(2)}, k_{(2)}l_{(2)}) = (h_{(1)} \cdot \mathbf{1}_A)\tilde{w}(h_{(2)}, k_{(1)})\tilde{w}(h_{(3)}k_{(2)}, l) \tag{3.1}$$

with $\tilde{w}(1_H, h) = \tilde{w}(h, 1_H) = \epsilon(h)\mathbf{1}_A$, such that $\omega = (f_1 * f_2) * \tilde{w}$ and $\omega' = (f_1 * f_2) * \tilde{w}^{-1}$, that is,

$$\omega(h, k) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)}k_{(1)} \cdot \mathbf{1}_A)\tilde{w}(h_{(3)}, k_{(2)}), \tag{3.2}$$
$$\omega'(h, k) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)}k_{(1)} \cdot \mathbf{1}_A)\tilde{w}^{-1}(h_{(3)}, k_{(2)}). \tag{3.3}$$

(2) *If there is a convolution invertible linear map $\tilde{w}$ as before, then A admits a unital globalization.*

PROOF. (1) Suppose that the pair (B, φ) is a globalization of the twisted partial action of H on A. Then there are a twisted action of H on B given by $\triangleright : H \otimes B \to B$ and a normalized, convolution invertible cocycle $u \in \mathrm{Hom}(H \otimes H, M(B))$, as well as an algebra monomorphism $\varphi : A \to B$ such that $\varphi(A)$ is an ideal of B. The monomorphism φ also intertwines the partial action of H on A with the induced partial action of H on $\varphi(A)$, which is given by

$$h \cdot \varphi(a) = \varphi(\mathbf{1}_A)(h \triangleright \varphi(a)),$$

and the induced partial cocycle is given by the expressions

$$\begin{aligned} v(h, k) &= (h_{(1)} \cdot (k_{(1)} \cdot \varphi(\mathbf{1}_A)))u(h_{(2)}, k_{(2)}) \\ &= (h_{(1)} \cdot \varphi(\mathbf{1}_A))u(h_{(2)}, k_{(1)})(h_{(3)}k_{(2)} \cdot \varphi(\mathbf{1}_A)), \\ v'(h, k) &= u^{-1}(h_{(1)}, k_{(1)})(h_{(2)} \cdot (k_{(2)} \cdot \varphi(\mathbf{1}_A))) \\ &= (h_{(1)}k_{(1)} \cdot \varphi(\mathbf{1}_A))u^{-1}(h_{(2)}, k_{(2)})(h_{(3)} \cdot \varphi(\mathbf{1}_A)). \end{aligned}$$

Therefore, we have $\varphi(h \cdot a) = h \cdot \varphi(a)$, $\varphi(\omega) = v$, and $\varphi(\omega') = v'$.

Define $\tilde{w} \in \mathrm{Hom}(H \otimes H, A)$ such that

$$\varphi(\tilde{w}(h, k)) = \varphi(\mathbf{1}_A)u(h, k).$$

As the map φ is a monomorphism, one can conclude that $\tilde{w}$ is well defined. First, note that $\tilde{w}$ is normalized, for

$$\varphi(\tilde{w}(h, 1_H)) = \varphi(\mathbf{1}_A)u(h, 1_H) = \varphi(\mathbf{1}_A)\epsilon(h) = \varphi(\mathbf{1}_A\epsilon(h)).$$

By the injectivity of φ, we have $\tilde{w}(h, 1_H) = \epsilon(h)\mathbf{1}_A$. Similarly, $\tilde{w}(1_H, h) = \epsilon(h)\varphi(\mathbf{1}_A)$. Also it is easy to see that $\tilde{w}$ is convolution invertible: defining $\tilde{w}^{-1}$ by $\varphi(\tilde{w}^{-1}) = \varphi(\mathbf{1}_A)u^{-1}(h, k)$, then

$$\begin{aligned}\varphi(\tilde{w}(h_{(1)}, k_{(1)})\tilde{w}^{-1}(h_{(2)}, k_{(2)})) &= \varphi(\tilde{w}(h_{(1)}, k_{(1)}))\varphi(\tilde{w}^{-1}(h_{(2)}, k_{(2)}))\\ &= \varphi(\mathbf{1}_A)u(h_{(1)}, k_{(1)})u^{-1}(h_{(2)}, k_{(2)})\\ &= \varphi(\mathbf{1}_A)\epsilon(h)\epsilon(k)\\ &= \varphi(\mathbf{1}_A\epsilon(h)\epsilon(k)).\end{aligned}$$

Again, by the injectivity of φ, we get

$$\tilde{w}(h_{(1)}, k_{(1)})\tilde{w}^{-1}(h_{(2)}, k_{(2)}) = \mathbf{1}_A\epsilon(h)\epsilon(k).$$

The other inversion formula is obtained in a similar way.

The modified cocycle expression (3.1) follows from

$$\begin{aligned}\varphi((h_{(1)} \cdot \tilde{w}(k_{(1)}, l_{(1)}))\tilde{w}(h_{(2)}, k_{(2)}l_{(2)})) &= \varphi((h_{(1)} \cdot \tilde{w}(k_{(1)}, l_{(1)})))\varphi(\tilde{w}(h_{(2)}, k_{(2)}l_{(2)}))\\ &= (h_{(1)} \cdot \varphi(\tilde{w}(k_{(1)}, l_{(1)})))\varphi(\tilde{w}(h_{(2)}, k_{(2)}l_{(2)}))\\ &= (h_{(1)} \cdot (\varphi(\mathbf{1}_A)u(k_{(1)}, l_{(1)})))\varphi(\mathbf{1}_A)u(h_{(2)}, k_{(2)}l_{(2)})\\ &= (h_{(1)} \cdot \varphi(\mathbf{1}_A))(h_{(2)} \cdot u(k_{(1)}, l_{(1)}))\varphi(\mathbf{1}_A)u(h_{(3)}, k_{(2)}l_{(2)})\\ &= (h_{(1)} \cdot \varphi(\mathbf{1}_A))(h_{(2)}\overline{\triangleright}u(k_{(1)}, l_{(1)}))u(h_{(3)}, k_{(2)}l_{(2)})\\ &\overset{(*)}{=} (h_{(1)} \cdot \varphi(\mathbf{1}_A))u(h_{(2)}, k_{(1)})u(h_{(3)}k_{(2)}, l)\\ &= (h_{(1)} \cdot \varphi(\mathbf{1}_A))\varphi(\tilde{w}(h_{(2)}, k_{(1)}))\varphi(\tilde{w}(h_{(3)}k_{(2)}, l))\\ &= \varphi((h_{(1)} \cdot \mathbf{1}_A)\tilde{w}(h_{(2)}, k_{(1)})\tilde{w}(h_{(3)}k_{(2)}, l)),\end{aligned}$$

where in the equality (*) above we used the fact that the cocycle equation is an identity between multipliers.

Finally, for expressions (3.2) and (3.3), we have

$$\begin{aligned}\varphi((h_{(1)} \cdot (k_{(1)} \cdot \mathbf{1}_A))\tilde{w}(h_{(2)}, k_{(2)})) &= (h_{(1)} \cdot (k_{(1)} \cdot \varphi(\mathbf{1}_A)))\varphi(\mathbf{1}_A)u(h_{(2)}, k_{(2)})\\ &= (h_{(1)} \cdot (k_{(1)} \cdot \varphi(\mathbf{1}_A)))u(h_{(2)}, k_{(2)})\\ &= v(h, k) = \varphi(\omega(h, k)),\end{aligned}$$

which gives

$$\omega(h, k) = (h_{(1)} \cdot (k_{(1)} \cdot \mathbf{1}_A))\tilde{w}(h_{(2)}, k_{(2)}).$$

The other equalities are obtained similarly.

(2) Consider the map

$$\begin{aligned}\varphi : A &\to \mathrm{Hom}(H, A)\\ a &\mapsto \varphi(a) : h \mapsto h \cdot a.\end{aligned}$$

From (2.1) one can easily see that φ is an algebra homomorphism. In addition, if $a \in \mathrm{Ker}(\varphi)$, then

$$a = 1_H \cdot a = \varphi(a)(1_H) = 0;$$

therefore, φ is an algebra monomorphism.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

We begin by proving that the algebra $\mathrm{Hom}(H, A)$ is a twisted H-module algebra, then we show that there is a subalgebra B of $\mathrm{Hom}(H, A)$ which inherits the structure of the twisted H-module and contains $\varphi(A)$ as an ideal.

Let us prove that $\mathrm{Hom}(H, A)$ is a twisted H-module algebra. Firstly, H measures $\mathrm{Hom}(H, A)$ by means of

$$(h \triangleright \theta)(k) = \tilde{w}(k_{(1)}, h_{(1)})\theta(k_{(2)}h_{(2)})\tilde{w}^{-1}(k_{(3)}, h_{(3)}).$$

In fact, given $h, k \in H$ and $\theta, \phi \in \mathrm{Hom}(H, A)$, we have

$$\begin{aligned}
(h \triangleright (\theta * \phi))(k) &= \tilde{w}(k_{(1)}, h_{(1)})(\theta * \phi)(k_{(2)}h_{(2)})\tilde{w}^{-1}(k_{(3)}, h_{(3)}) \\
&= \tilde{w}(k_{(1)}, h_{(1)})\theta(k_{(2)}h_{(2)})\phi(k_{(3)}h_{(3)})\tilde{w}^{-1}(k_{(4)}, h_{(4)}) \\
&= \tilde{w}(k_{(1)}, h_{(1)})\theta(k_{(2)}h_{(2)})\epsilon(k_{(3)}h_{(3)})\phi(k_{(4)}h_{(4)})\tilde{w}^{-1}(k_{(5)}, h_{(5)}) \\
&= [\tilde{w}(k_{(1)}, h_{(1)})\theta(k_{(2)}h_{(2)})\tilde{w}^{-1}(k_{(3)}, h_{(3)})] \\
&\qquad \times [\tilde{w}(k_{(4)}, h_{(4)})\phi(k_{(5)}h_{(5)})\tilde{w}^{-1}(k_{(6)}, h_{(6)})] \\
&= (h_{(1)} \triangleright \theta)(k_{(1)})(h_{(2)} \triangleright \phi)(k_{(2)}) \\
&= (h_{(1)} \triangleright \theta) * (h_{(2)} \triangleright \phi)(k).
\end{aligned}$$

It is also easy to see that $1_H \triangleright \theta = \theta$, for all $\theta \in \mathrm{Hom}(H, A)$. Indeed,

$$\begin{aligned}
(1_H \cdot \theta)(h) &= \tilde{w}(h_{(1)}, 1_H)\theta(h_{(2)}1_H)\tilde{w}^{-1}(h_{(3)}, 1_H) \\
&= \epsilon(h_{(1)})\theta(h_{(2)})\epsilon(h_{(3)}) = \theta(h).
\end{aligned}$$

The map $\eta(k) = \epsilon(k)\mathbf{1}_A$ is the unit of $\mathrm{Hom}(H, A)$; then one needs to verify that $(h \triangleright \eta)(k) = \epsilon(h)\eta(k)$:

$$\begin{aligned}
(h \triangleright \eta)(k) &= \tilde{w}(k_{(1)}, h_{(1)})\eta(k_{(2)}h_{(2)})\tilde{w}^{-1}(k_{(3)}, h_{(3)}) \\
&= \tilde{w}(k_{(1)}, h_{(1)})\epsilon(k_{(2)})\epsilon(h_{(2)})\tilde{w}^{-1}(k_{(3)}, h_{(3)}) \\
&= \tilde{w}(k_{(1)}, h_{(1)})\tilde{w}^{-1}(k_{(2)}, h_{(2)}) \\
&= \epsilon(h)\epsilon(k)\mathbf{1}_A = \epsilon(h)\eta(k).
\end{aligned}$$

The twisted H-module structure appears when one expands the expression for $(h \triangleright (k \triangleright \theta))$:

$$\begin{aligned}
&(h \triangleright (k \triangleright \theta))(l) \\
&\quad = \tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\theta(l_{(3)}h_{(3)}k_{(2)})\tilde{w}^{-1}(l_{(4)}h_{(4)}, k_{(3)}) \\
&\quad = \tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\underbrace{\epsilon(l_{(3)})\epsilon(h_{(3)})\epsilon(k_{(2)})}\theta(l_{(4)}h_{(4)}k_{(3)})\underbrace{\epsilon(l_{(5)})\epsilon(h_{(5)})\epsilon(k_{(4)})} \\
&\qquad\quad \times \tilde{w}^{-1}(l_{(6)}h_{(6)}, k_{(5)})\tilde{w}^{-1}(l_{(7)}, h_{(7)}) \\
&\quad = \tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\underbrace{\tilde{w}^{-1}(l_{(3)}, h_{(3)}k_{(2)})\tilde{w}(l_{(4)}, h_{(4)}k_{(3)})}\theta(l_{(5)}h_{(5)}k_{(4)}) \\
&\qquad\quad \times \underbrace{\tilde{w}^{-1}(l_{(6)}, h_{(6)}k_{(5)})\tilde{w}(l_{(7)}, h_{(7)}k_{(6)})}\tilde{w}^{-1}(l_{(8)}h_{(8)}, k_{(7)})\tilde{w}^{-1}(l_{(9)}, h_{(9)})
\end{aligned}$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

$$
\begin{aligned}
&= [\tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\tilde{w}^{-1}(l_{(3)}, h_{(3)}k_{(2)})] \\
&\qquad \times \tilde{w}(l_{(4)}, h_{(4)}k_{(3)})\theta(l_{(5)}h_{(5)}k_{(4)})\tilde{w}^{-1}(l_{(6)}, h_{(6)}k_{(5)}) \\
&\qquad \times [\tilde{w}(l_{(7)}, h_{(7)}k_{(6)})\tilde{w}^{-1}(l_{(8)}h_{(8)}, k_{(7)})\tilde{w}^{-1}(l_{(9)}, h_{(9)})] \\
&= [\tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\tilde{w}^{-1}(l_{(3)}, h_{(3)}k_{(2)})](h_{(4)}k_{(3)} \triangleright \theta(l_{(4)})) \\
&\qquad \times [\tilde{w}(l_{(5)}, h_{(5)}k_{(4)})\tilde{w}^{-1}(l_{(6)}h_{(6)}, k_{(5)})\tilde{w}^{-1}(l_{(7)}, h_{(7)})].
\end{aligned}
$$

If we define $u : H \otimes H \to \mathrm{Hom}(H, A)$ as the map

$$u(h,k)(l) = \tilde{w}(l_{(1)}, h_{(1)})\tilde{w}(l_{(2)}h_{(2)}, k_{(1)})\tilde{w}^{-1}(l_{(3)}, h_{(3)}k_{(2)}),$$

then its convolution inverse is

$$u^{-1}(h,k)(l) = \tilde{w}(l_{(1)}, h_{(1)}k_{(1)})\tilde{w}^{-1}(l_{(2)}h_{(2)}, k_{(2)})\tilde{w}^{-1}(l_{(3)}, h_{(3)}),$$

and we have

$$(h \triangleright (k \triangleright \theta))(l) = u(h_{(1)}, k_{(1)}) * (h_{(2)}k_{(2)} \triangleright \theta) * u^{-1}(h_{(3)}, k_{(3)})(l). \tag{3.4}$$

The map u is truly a cocycle, that is, it satisfies the cocycle identity

$$(h_{(1)} \triangleright u(k_{(1)}, l_{(1)})) * u(h_{(2)}, k_{(2)}l_{(2)}) = u(h_{(1)}, k_{(1)}) * u(h_{(2)}k_{(2)}, l). \tag{3.5}$$

In fact, beginning on the left-hand side and repeatedly using the definition of u, we get

$$
\begin{aligned}
&(h_{(1)} \triangleright u(k_{(1)}, l_{(1)})) * u(h_{(2)}, k_{(2)}l_{(2)})(m) \\
&\quad = (h_{(1)} \triangleright u(k_{(1)}, l_{(1)}))(m_{(1)})u(h_{(2)}, k_{(2)}l_{(2)})(m_{(2)}) \\
&\quad = [\tilde{w}(m_{(1)}, h_{(1)})u(k_{(1)}, l_{(1)})(m_{(2)}h_{(2)})\tilde{w}^{-1}(m_{(3)}, h_{(3)})]u(h_{(4)}, k_{(2)}l_{(2)})(m_{(4)}) \\
&\quad = \tilde{w}(m_{(1)}, h_{(1)})u(k_{(1)}, l_{(1)})(m_{(2)}h_{(2)})\underbrace{\tilde{w}^{-1}(m_{(3)}, h_{(3)})\tilde{w}(m_{(4)}, h_{(4)})} \\
&\qquad\quad \times \tilde{w}(m_{(5)}h_{(5)}, k_{(2)}l_{(2)})\tilde{w}^{-1}(m_{(6)}, h_{(6)}k_{(3)}l_{(3)}) \\
&\quad = \tilde{w}(m_{(1)}, h_{(1)})u(k_{(1)}, l_{(1)})(m_{(2)}h_{(2)})\tilde{w}(m_{(3)}h_{(3)}, k_{(2)}l_{(2)})\tilde{w}^{-1}(m_{(4)}, h_{(4)}k_{(3)}l_{(3)}) \\
&\quad = \tilde{w}(m_{(1)}, h_{(1)})\tilde{w}(m_{(2)}h_{(2)}, k_{(1)})\tilde{w}(m_{(3)}h_{(3)}k_{(2)}, l_{(1)}) \\
&\qquad\quad \times \underbrace{\tilde{w}^{-1}(m_{(4)}h_{(4)}, k_{(3)}l_{(2)})\tilde{w}(m_{(5)}h_{(5)}, k_{(4)}l_{(3)})}\tilde{w}^{-1}(m_{(6)}, h_{(6)}k_{(5)}l_{(4)}) \\
&\quad = \tilde{w}(m_{(1)}, h_{(1)})\tilde{w}(m_{(2)}h_{(2)}, k_{(1)})\tilde{w}(m_{(3)}h_{(3)}k_{(2)}, l_{(1)})\tilde{w}^{-1}(m_{(4)}, h_{(4)}k_{(3)}l_{(2)}) \\
&\quad = \tilde{w}(m_{(1)}, h_{(1)})\tilde{w}(m_{(2)}h_{(2)}, k_{(1)})\epsilon(m_{(3)}h_{(3)}k_{(2)}) \\
&\qquad\quad \times \tilde{w}(m_{(4)}h_{(4)}k_{(3)}, l_{(1)})\tilde{w}^{-1}(m_{(5)}, h_{(5)}k_{(4)}l_{(2)}) \\
&\quad = [\tilde{w}(m_{(1)}, h_{(1)})\tilde{w}(m_{(2)}h_{(2)}, k_{(1)})\tilde{w}^{-1}(m_{(3)}, h_{(3)}k_{(2)})] \\
&\qquad\quad \times [\tilde{w}(m_{(4)}, h_{(4)}k_{(3)})\tilde{w}(m_{(5)}h_{(5)}k_{(4)}, l_{(1)})\tilde{w}^{-1}(m_{(6)}, h_{(6)}k_{(5)}l_{(2)})] \\
&\quad = u(h_{(1)}, k_{(1)})(m_{(1)})u(h_{(2)}k_{(2)}, l)(m_{(2)}) \\
&\quad = u(h_{(1)}, k_{(1)}) * u(h_{(2)}k_{(2)}, l)(m).
\end{aligned}
$$

Finally, it is easily checked that u is a normalized 2-cocycle: if η denotes the unit element of $\mathrm{Hom}(H, A)$, then

$$u(h, 1_H) = u(1_H, h) = \epsilon(h)\eta.$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

We will now show that $\varphi(A)$ is an ideal in a twisted H-module subalgebra of $\mathrm{Hom}(H,A)$. Define B as the subalgebra of $\mathrm{Hom}(H,A)$ generated by the elements of the form $h \triangleright \varphi(a)$, for all $h \in H$ and $a \in A$, and by the functions $u^{\pm 1}(h,k)$, for all $h,k \in H$. Since $u(1_H,1_H)(h) = \epsilon(1_H)\eta(h) = \eta(h)$, B is a unital subalgebra of $\mathrm{Hom}(H,A)$.

It is easy to see that $H \triangleright B \subseteq B$, because of the law of composition (3.4) and the cocycle identity (3.5). Therefore, B is a twisted H-module subalgebra of $\mathrm{Hom}(H,A)$. One needs only verify that $\varphi(A)$ is an ideal in B. This is accomplished by showing the following identities:

(i) $\varphi(a) * (h \triangleright \varphi(b)) = \varphi(a(h \cdot b))$;
(ii) $(h \triangleright \varphi(b)) * \varphi(a) = \varphi((h \cdot b)a)$;
(iii) $\varphi(a) * u^{\pm 1}(h,k) = \varphi(a\tilde{w}^{\pm 1}(h,k))$;
(iv) $u^{\pm 1}(h,k) * \varphi(a) = \varphi(\tilde{w}^{\pm 1}(h,k)a)$.

For identity (i) we have, for $a,b \in A$ and $h \in H$,

$$\begin{aligned}\varphi(a) * (h \triangleright \varphi(b))(k) &= (k_{(1)} \cdot a)\tilde{w}(k_{(2)},h_{(1)})(k_{(3)}h_{(2)} \cdot b)\tilde{w}^{-1}(k_{(4)},h_{(3)})\\ &= (k_{(1)} \cdot a)(k_{(2)} \cdot \mathbf{1}_A)\tilde{w}(k_{(3)},h_{(1)})(k_{(4)}h_{(2)} \cdot \mathbf{1}_A)(k_{(5)}h_{(3)} \cdot b)\\ &\quad \times (k_{(6)}h_{(4)} \cdot \mathbf{1}_A)\tilde{w}^{-1}(k_{(7)},h_{(5)})(k_{(8)} \cdot \mathbf{1}_A),\end{aligned}$$

where the last factor appears because $e(k) = (k \cdot \mathbf{1}_A)$ commutes convolutionally. Since f_1 and f_2 are also central in $\mathrm{Hom}(H \otimes H, A)$ (by (STPA2)) and $\omega = f_1 * f_2 * \tilde{w}$, the latter expression can be rewritten as

$$\begin{aligned}&= (k_{(1)} \cdot a)\omega(k_{(2)},h_{(1)})(k_{(3)}h_{(2)} \cdot b)\omega'(k_{(4)},h_{(3)})\\ &= (k_{(1)} \cdot a)(k_{(2)} \cdot (h \cdot b))\\ &= k \cdot (a(h \cdot b)) = \varphi(a(h \cdot b))(k).\end{aligned}$$

Identity (ii) is obtained in a similar manner.

For identity (iii) we have, for $a \in A$ and $h,k,l \in H$,

$$\begin{aligned}\varphi(a) * u(h,k)(l) &= \varphi(a)(l_{(1)})u(h,k)(l_{(2)})\\ &= (l_{(1)} \cdot a)\tilde{w}(l_{(2)},h_{(1)})\tilde{w}(l_{(3)}h_{(2)},k_{(1)})\tilde{w}^{-1}(l_{(4)},h_{(3)}k_{(2)})\\ &= (l_{(1)} \cdot a)(l_{(2)} \cdot \mathbf{1}_A)\tilde{w}(l_{(3)},h_{(1)})\tilde{w}(l_{(4)}h_{(2)},k_{(1)})\tilde{w}^{-1}(l_{(5)},h_{(3)}k_{(2)})\\ &= (l_{(1)} \cdot a)(l_{(2)} \cdot \tilde{w}(h_{(1)},k_{(1)}))\tilde{w}(l_{(3)},h_{(2)}k_{(2)})\tilde{w}^{-1}(l_{(4)},h_{(3)}k_{(3)})\\ &= (l_{(1)} \cdot a)(l_{(2)} \cdot \tilde{w}(h,k))\\ &= l \cdot (a\tilde{w}(h,k)) = \varphi(a\tilde{w}(h,k))(l).\end{aligned}$$

In a similar way, we obtain the expression for $u^{-1}(h,k)$ and the identity (iv) as well. Therefore $\varphi(A) \trianglelefteq B$.

From identity (i) we conclude quickly that the map φ intertwines the partial action on A with the partial action on $\varphi(A)$, indeed

$$h \cdot \varphi(a) = \varphi(\mathbf{1}_A) * (h \triangleright \varphi(a)) = \varphi(h \cdot a).$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

The image of the partial cocycle $\varphi(\omega(h,k))$ is the induced cocycle

$$v(h,k) = (h_{(1)} \cdot \varphi(\mathbf{1}_A)) * u(h_{(2)}, k_{(1)}) * (h_{(3)}k_{(2)} \cdot \varphi(\mathbf{1}_A))$$

for the twisted partial action of H on the ideal $\varphi(A)$. In fact,

$$\begin{aligned}
\varphi(\omega(h,k))(l) &= l \cdot \omega(h,k) \\
&= l \cdot [(h_{(1)} \cdot \mathbf{1}_A)\omega(h_{(2)}, k_{(1)})(h_{(3)}k_{(2)} \cdot \mathbf{1}_A)] \\
&= (l_{(1)} \cdot (h_{(1)} \cdot \mathbf{1}_A))(l_{(2)} \cdot \omega(h_{(2)}, k_{(1)}))(l_{(3)} \cdot (h_{(3)}k_{(2)} \cdot \mathbf{1}_A)) \\
&= (l_{(1)} \cdot (h_{(1)} \cdot \mathbf{1}_A))(l_{(2)} \cdot [(h_{(2)} \cdot \mathbf{1}_A)\tilde{w}(h_{(3)}, k_{(1)})(h_{(4)}k_{(2)} \cdot \mathbf{1}_A)]) \\
&\quad \times (l_{(3)} \cdot (h_{(5)}k_{(3)} \cdot \mathbf{1}_A)) \\
&= (l_{(1)} \cdot (h_{(1)} \cdot \mathbf{1}_A))(l_{(2)} \cdot \tilde{w}(h_{(2)}, k_{(1)}))(l_{(3)} \cdot (h_{(3)}k_{(2)} \cdot \mathbf{1}_A)).
\end{aligned}$$

Using equality (3.1) for $\tilde{w}$, we obtain

$$\begin{aligned}
\varphi(\omega(h,k))(l) &= l \cdot \omega(h,k) \\
&= (l_{(1)} \cdot (h_{(1)} \cdot \mathbf{1}_A))\tilde{w}(l_{(2)}, h_{(2)})\tilde{w}(l_{(3)}h_{(3)}, k_{(1)}) \\
&\quad \times (\tilde{w}^{-1}(l_{(4)}, h_{(4)}k_{(2)}))(l_{(5)} \cdot (h_{(5)}k_{(3)} \cdot \mathbf{1}_A)) \\
&= (l_{(1)} \cdot (h_{(1)} \cdot \mathbf{1}_A))u(h_{(2)}, k_{(1)})(l_{(2)})(l_{(3)} \cdot (h_{(3)}k_{(2)} \cdot \mathbf{1}_A)) \\
&= \varphi(h_{(1)} \cdot \mathbf{1}_A) * u(h_{(2)}, k_{(1)}) * \varphi(h_{(3)}k_{(2)} \cdot \mathbf{1}_A)(l) \\
&= (h_{(1)} \cdot \varphi(\mathbf{1}_A)) * u(h_{(2)}, k_{(1)}) * (h_{(3)}k_{(2)} \cdot \varphi(\mathbf{1}_A))(l) \\
&= v(h,k)(l).
\end{aligned}$$

In an analogous manner, it can be shown that

$$\begin{aligned}
\varphi(\omega'(h,k)) &= (h_{(1)}k_{(1)} \cdot \varphi(\mathbf{1}_A)) * u^{-1}(h_{(2)}, k_{(2)}) * (h_{(3)} \cdot \varphi(\mathbf{1}_A)) \\
&= v'(h,k)(l).
\end{aligned}$$

This concludes the proof. □

For the case of H being a co-commutative Hopf algebra, the H-submodule $H \triangleright \varphi(A)$ is a nonunital subalgebra of $\mathrm{Hom}(H, A)$ and it carries a globalization of the twisted partial action on A.

This is because we can write, for $a, b \in A$ and $h, k \in H$, using the co-commutativity of H, the product $(h \triangleright \varphi(a)) * (k \triangleright \varphi(b))$ as a linear combination in $H \triangleright \varphi(A)$. Indeed,

$$\begin{aligned}
(h \triangleright \varphi(a)) * (k \triangleright \varphi(b)) &= (h_{(1)} \triangleright \varphi(a)) * (h_{(2)}S(h_{(3)}) \triangleright (k \triangleright \varphi(b))) \\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(7)})) \\
&\quad * (h_{(3)} \triangleright (S(h_{(6)}) \triangleright (k \triangleright \varphi(b)))) * u(h_{(4)}, S(h_{(5)})) \\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(9)})) * (h_{(3)} \triangleright (S(h_{(8)}) \triangleright (k \triangleright \varphi(b)))) \\
&\quad * u(h_{(4)}, S(h_{(7)})) * u(h_{(5)}S(h_{(6)}), h_{(10)}) \\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(9)})) * (h_{(3)} \triangleright (S(h_{(8)}) \triangleright (k \triangleright \varphi(b)))) \\
&\quad * (h_{(4)} \triangleright u(S(h_{(7)}), h_{(10)})) * u(h_{(5)}, S(h_{(6)})h_{(11)})
\end{aligned}$$

$$
\begin{aligned}
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(10)})) * (h_{(3)} \triangleright (S(h_{(9)}) \triangleright (k \triangleright \varphi(b))))\\
&\quad * (h_{(4)} \triangleright u(S(h_{(8)}), h_{(11)})) * u(h_{(5)}, S(h_{(6)})h_{(7)})\\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(7)})) * (h_{(3)} \triangleright (S(h_{(6)}) \triangleright (k \triangleright \varphi(b))))\\
&\quad * (h_{(4)} \triangleright u(S(h_{(5)}), h_{(8)}))\\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}S(h_{(3)}), h_{(4)}) * u^{-1}(h_{(5)}, S(h_{(9)}))\\
&\quad * (h_{(6)} \triangleright (S(h_{(6)}) \triangleright (k \triangleright \varphi(b)))) * (h_{(7)} \triangleright u(S(h_{(8)}), h_{(10)}))\\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}S(h_{(5)}), h_{(6)}) * u^{-1}(h_{(3)}, S(h_{(4)}))\\
&\quad * (h_{(7)} \triangleright (S(h_{(10)}) \triangleright (k \triangleright \varphi(b)))) * (h_{(8)} \triangleright u(S(h_{(9)}), h_{(11)}))\\
&= (h_{(1)} \triangleright \varphi(a)) * u^{-1}(h_{(2)}, S(h_{(5)})h_{(6)}) * (h_{(3)} \triangleright u^{-1}(S(h_{(4)}), h_{(7)}))\\
&\quad * (h_{(8)} \triangleright (S(h_{(11)}) \triangleright (k \triangleright \varphi(b)))) * (h_{(9)} \triangleright u(S(h_{(10)}), h_{(12)}))\\
&= (h_{(1)} \triangleright \varphi(a)) * (h_{(2)} \triangleright u^{-1}(S(h_{(6)}), h_{(7)})) * (h_{(3)} \triangleright (S(h_{(5)}) \triangleright (k \triangleright \varphi(b))))\\
&\quad * (h_{(3)} \triangleright u(S(h_{(4)}), h_{(8)}))\\
&= h_{(1)} \triangleright (\varphi(a) * u^{-1}(S(h_{(4)}), h_{(5)}) * (S(h_{(3)}) \triangleright (k \triangleright \varphi(b))) * u(S(h_{(2)}), h_{(6)})),
\end{aligned}
$$

and this last expression lies in $H \triangleright \varphi(A)$, because $\varphi(A)$ is an ideal of the algebra B obtained in the previous theorem.

And, again only for co-commutative Hopf algebras, given any $h, k, l \in H$, the following expression in B,

$$\theta(h, k, l) = u^{-1}(S(h_{(3)}), h_{(4)}) * (S(h_{(2)}) \triangleright u(k, l)) * u(S(h_{(1)}), h_{(5)}),$$

satisfies the relation

$$h_{(1)} \triangleright \theta(h_{(2)}, k, l) = \epsilon(h)u(k, l).$$

Then, since $\theta(h, k, l) \in B$ and $\varphi(A)$ is an ideal in B, we have, for any $a \in A$ and $h, k, l \in H$, that

$$\varphi(a) * \theta(h, k, l) \in \varphi(A),$$

which implies that

$$
\begin{aligned}
h_{(1)} \triangleright (\varphi(a) * \theta(h_{(2)}, k, l)) &= (h_{(1)} \triangleright \varphi(a)) * (h_{(2)} \triangleright \theta(h_{(3)}, k, l))\\
&= (h_{(1)} \triangleright \varphi(a)) * \epsilon(h_{(2)})u(k, l)\\
&= (h \triangleright \varphi(a)) * u(k, l).
\end{aligned}
$$

That is, the product $(h \triangleright \varphi(a)) * u(k, l) \in H \triangleright \varphi(A)$. Analogously, we obtain

$$h_{(1)} \triangleright (\theta(h_{(2)}, k, l) * \varphi(a)) = u(k, l) * (h \triangleright \varphi(a)).$$

Note that $H \triangleright \varphi(A)$ is a nonunital algebra. The above computation shows that for any k, l in H, the elements $u^{\pm 1}(k, l)$ can be viewed as multipliers of the algebra $H \triangleright \varphi(A)$ since any product of elements of the form $h \triangleright \varphi(a)$ and $u(k, l)$ can be written as an element of $H \triangleright \varphi(A)$. This is the case for globalization of twisted partial group actions as shown in [22]. In this case, for each $g \in G$, the algebra which carries the globalization is $B = \sum_{g \in G} \beta_g(\varphi(A))$, the subspaces $\beta_g(\varphi(A))$ are ideals of this algebra, and the 2-cocycle components $u_{g,h}$ belong to the multiplier algebra of B.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

EXAMPLE 3.4 [8]. Let κ be an isomorphic copy of the complex numbers $\mathbb{C}$ and let $\mathbb{S}^1 \subseteq \mathbb{C}$ be the circle group, that is, the group of all complex roots of 1. Furthermore, let G be an arbitrary finite group seen as a subgroup of S_n for some n. Taking the action of $G \subseteq S_n$ on $(\kappa\mathbb{S}^1)^{\otimes n}$ by permutation of roots, consider the smash product Hopf algebra

$$H_1' = (\kappa\mathbb{S}^1)^{\otimes n} \rtimes \kappa G,$$

which is co-commutative. Let $X \subseteq G$ be an arbitrary subset which is not a subgroup and consider the subalgebra $\tilde{A} = (\sum_{g\in X} p_g)(\kappa G)^* \subseteq (\kappa G)^*$, and define the commutative algebra $A' = \kappa[t, t^{-1}]^{\otimes n} \otimes \tilde{A}$.

In order to simplify the notation, write

$$t_i = 1 \otimes \cdots \otimes 1 \otimes t \otimes 1 \otimes \cdots \otimes 1, \tag{3.6}$$

where t belongs to the i-copy of $\kappa[t, t^{-1}]$. Then we have the elementary monomials in $\kappa[t, t^{-1}]$, given by

$$t_1^{k_1} \ldots t_n^{k_n} = t^{k_1} \otimes \cdots \otimes t^{k_n}.$$

In turn, the generators of $(\kappa\mathbb{S}^1)^{\otimes n}$ can be written in terms of the roots χ_θ of unity as follows:

$$\chi_{\theta_1,\ldots\theta_n} = \chi_{\theta_1} \otimes \cdots \otimes \chi_{\theta_n} \in (\kappa\mathbb{S}^1)^{\otimes n}, \tag{3.7}$$

where $\chi_{\theta_i} \in \mathbb{S}^1$ is the root of 1 whose angular coordinate is θ_i and which belongs to the i-factor of $(\kappa\mathbb{S}^1)^{\otimes n}$.

Then with the notation established in (3.6) and (3.7), the formula

$$(\chi_{\theta_1,\ldots,\theta_n} \otimes u_g) \cdot (t_1^{k_1} \ldots t_n^{k_n} \otimes p_s) = \begin{cases} \exp\Big\{i \sum_{j=1}^{n} k_j \theta_{gs^{-1}(j)}\Big\} t_1^{k_1} \ldots t_n^{k_n} \otimes p_{sg^{-1}} & \text{if } s^{-1}g \in X, \\ 0 & \text{if } s^{-1}g \notin X, \end{cases}$$

where $g \in G$ and $s \in X \subseteq G$, gives a left partial action $\cdot : H_1' \times A' \to A'$. As H_1' is a co-commutative Hopf algebra and A' is a commutative algebra, then thanks to Remark 2.10 it is enough to give a partial 2-cocycle ω attached to our partial action, which is invertible in the ideal $\langle f_1 * f_2 \rangle$, that is, a κ-linear map $\omega : H \otimes H \to A$ satisfying conditions (TPA3)–(TPA5) and (STPA3).

Assume that G is such that its Schur multiplier is nontrivial and take a normalized 2-cocycle $\gamma : G \times G \to \kappa^*$ which is not a coboundary (for a concrete such γ when G is the Klein 4-group, see [8]). For arbitrary $h = \chi_{\theta_1,\ldots,\theta_n} \otimes u_g$ and $l = \chi_{\theta_1',\ldots,\theta_n'} \otimes u_s$ in H_1', set

$$\omega(h, l) = \gamma(g, s)(h \cdot (l \cdot \mathbf{1}_{A'})).$$

Then, as explained in [8], the pair (α, ω) forms a twisted partial action of $H_1' = (\kappa\mathbb{S}^1)^{\otimes n} \rtimes \kappa G$ on A'. Moreover, taking

$$\omega'(h, l) = \gamma(g, s)^{-1} (h \cdot (l \cdot \mathbf{1}_{A'})),$$

we readily see that it is symmetric.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

We now observe that this symmetric twisted partial action is globalizable. Indeed, the auxiliary convolution invertible map $\tilde{w} : H_1' \otimes H_1' \to A'$ can be given by

$$\tilde{w}(h, l) = \gamma(g, s)\epsilon(h)\epsilon(l),$$

where $h = \chi_{\theta_1,\ldots,\theta_n} \otimes u_g$ and $l = \chi_{\theta_1',\ldots,\theta_n'} \otimes u_s$. Direct verifications show that the convolution inverse of $\tilde{w}$ is

$$\tilde{w}^{-1}(h, l) = \gamma(g, s)^{-1}\epsilon(h)\epsilon(l),$$

and $\tilde{w}$, $\tilde{w}^{-1}$ satisfy all conditions of Theorem 3.3.

Example 3.5. Consider now the symmetric twisted partial action given in Example 2.15, where $H = (\kappa K_4)^*$, the dual of the group algebra of the Klein 4-group $K_4 = \langle a, b \mid a^2 = b^2 = e\rangle$, and A is the base field κ, defined by the measuring map $\lambda(p_e) = \lambda(p_a) = \frac{1}{2}$ and $\lambda(p_b) = \lambda(p_{ab}) = 0$. As was shown in this case, the partial 2-cocyle ω and its partial inverse ω' are fully determined by a pair $(x, y) \in \kappa^2$ satisfying the quadratic equation (2.8) where $x = \omega(p_e, p_e)$ and $y = \omega'(p_e, p_e)$. One readily notes that the pairs $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{8}, \frac{1}{8})$ are solutions for (2.8). Obviously, the first one gives the trivial partial 2-cocycle $\omega(p_g, p_h) = \omega'(p_g, p_h) = f_1 * f_2\,(p_g, p_h) = p_g \cdot (p_h \cdot 1)$. So we take the second one and then we have

$$x = \omega(p_e, p_e) = \omega(p_a, p_e) = \omega(p_e, p_a) = \omega(p_a, p_a) = \tfrac{1}{8}$$

and

$$\begin{aligned}\omega(p_e, p_b) &= \omega(p_a, p_b) = \omega(p_e, p_{ab}) = \omega(p_a, p_{ab})\\ &= \omega(p_b, p_e) = \omega(p_{ab}, p_e) = \omega(p_b, p_a) = \omega(p_{ab}, p_a)\\ &= -\omega(p_b, p_b) = -\omega(p_{ab}, p_b) = -\omega(p_b, p_{ab}) = -\omega(p_{ab}, p_{ab})\\ &= -\tfrac{1}{8},\end{aligned}$$

and $\omega = \omega'$.

We shall prove that this symmetric twisted partial action is globalizable. In order to consider the auxiliary map $\tilde{w}$, let us denote $X_{g,h} = \tilde{w}(p_g, p_h)$, for $g, h \in K_4$, so that $\tilde{w}$ is determined by 16 variables. The normalization condition gives us

$$\sum_{g\in K_4} \tilde{w}(p_g, p_h) = \sum_{g\in K_4} X_{g,h} = \epsilon(p_h) = \begin{cases}1 & \text{if } h = e,\\ 0 & \text{if } h \neq e,\end{cases} \tag{3.8}$$

and

$$\sum_{h\in K_4} \tilde{w}(p_g, p_h) = \sum_{h\in K_4} X_{g,h} = \epsilon(p_g) = \begin{cases}1 & \text{if } g = e,\\ 0 & \text{if } g \neq e.\end{cases} \tag{3.9}$$

From condition (3.1), we have, for $g, h, f \in K_4$, the equations

$$\sum_{\substack{r\in\langle a\rangle\\ s\in K_4}} X_{s,fh^{-1}s} X_{r^{-1}g,s^{-1}h} = \sum_{\substack{x\in\langle a\rangle\\ y\in K_4}} X_{x^{-1}y,hg^{-1}y} X_{y^{-1}g,f} \tag{3.10}$$

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

whereas from (3.2), we obtain

$$\omega(p_g, p_h) = \sum_{r,s\in K_4} \lambda(p_r)\lambda(p_s)\tilde{w}(p_{r^{-1}g}, p_{s^{-1}h}) = \frac{1}{4}\sum_{r,s\in\langle a\rangle} X_{r^{-1}g,s^{-1}h}, \tag{3.11}$$

with $g, h \in K_4$. Next, writing $Y_{g,h} = \tilde{w}^{-1}(p_g, p_h)$, we also have the equations

$$\begin{aligned}(\tilde{w} * \tilde{w}^{-1})(p_g, p_h) &= \sum_{r,s\in K_4} X_{r,s}Y_{r^{-1}g,s^{-1}h} \\ &= \epsilon(p_g)\epsilon(p_h) = \begin{cases}1 & \text{if } (g,h) = (e,e), \\ 0 & \text{if } (g,h) \neq (e,e),\end{cases}\end{aligned} \tag{3.12}$$

as well as

$$\omega'(p_g, p_h) = \sum_{r,s\in K_4} \lambda(p_r)\lambda(p_s)\tilde{w}^{-1}(p_{r^{-1}g}, p_{s^{-1}h}) = \frac{1}{4}\sum_{r,s\in\langle a\rangle} Y_{r^{-1}g,s^{-1}h}, \tag{3.13}$$

with $g, h \in K_4$, the latter coming from (3.3). Direct computations show that

$$X_{1,1} = Y_{1,1} = X_{1,b} = Y_{1,b} = X_{b,1} = Y_{b,1} = -X_{b,b} = -Y_{b,b} = \tfrac{1}{2},$$
$$X_{ab^i,b^j} = Y_{ab^i,b^j} = X_{b^i,ab^j} = Y_{b^i,ab^j} = X_{ab^i,ab^j} = Y_{ab^i,ab^j} = 0,$$

$i, j = 0, 1$, is a solution for (3.8)–(3.13), showing that our symmetric twisted partial action is globalizable.

References

[1] F. Abadie, 'Sobre ações parciais, fibrados de Fell e grupóides', PhD Thesis, Universidade de São Paulo, 1999.

[2] F. Abadie, 'Enveloping actions and Takai duality for partial actions', *J. Funct. Anal.* **197**(1) (2003), 14–67.

[3] C. A. Akemann, G. K. Pedersen and J. Tomiyama, 'Multipliers of C^*-algebras', *J. Funct. Anal.* **13** (1973), 277–301.

[4] E. R. Alvares, M. M. S. Alves and E. Batista, 'Partial Hopf module categories', *J. Pure Appl. Algebra* **217** (2013), 1517–1534.

[5] M. M. S. Alves and E. Batista, 'Partial Hopf actions, partial invariants and a Morita context', *J. Algebra Discrete Math.* **3** (2009), 1–19.

[6] M. M. S. Alves and E. Batista, 'Enveloping actions for partial Hopf actions', *Comm. Algebra* **38** (2010), 2872–2902.

[7] M. M. S. Alves and E. Batista, 'Globalization theorems for partial Hopf (co)actions and some of their applications', *Contemp. Math.* **537** (2011), 13–30.

[8] M. M. S. Alves, E. Batista, M. Dokuchaev and A. Paques, 'Twisted partial actions of Hopf algebras', *Israel J. Math.* **197** (2013), 263–308.

[9] M. M. S. Alves, E. Batista and J. Vercruysse, 'Partial representations of Hopf algebras', *J. Algebra* **426**(15) (2015), 137–187.

[10] D. Bagio, W. Cortes, M. Ferrero and A. Paques, 'Actions of inverse semigroups on algebras', *Comm. Algebra* **35**(12) (2007), 3865–3874.

[11] D. Bagio and A. Paques, 'Partial groupoid actions: globalization, Morita theory and Galois theory', *Comm. Algebra* **40** (2012), 3658–3678.

[12] L. Bemm, 'Ações parciais de grupos sobre anéis semiprimos', PhD Thesis, Porto Alegre, 2011.

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press

[13] L. Bemm, W. Cortes, M. Ferrero and S. S. D. Flora, 'Partial crossed products and Goldie rings', *Comm. Algebra* **43**(9) (2015), 3705–3724.

[14] L. Bemm and M. Ferrero, 'Globalization of partial actions on semiprime rings', *J. Algebra Appl.* **12**(4) (2013), 1250202.

[15] S. Caenepeel and E. D. Groot, 'Galois corings applied to partial Galois theory', *Proc. Int. Conf. on Mathematics and its Applications (ICMA2004)* (Kuwait University, 2005), 117–134.

[16] S. Caenepeel and K. Janssen, 'Partial (co)actions of Hopf algebras and partial Hopf–Galois theory', *Comm. Algebra* **36** (2008), 2923–2946.

[17] W. Cortes and M. Ferrero, 'Globalizations of partial actions on semiprime rings', *Contemp. Math.* **499** (2009), 27–35.

[18] W. Cortes, M. Ferrero and E. Marcos, 'Partial actions on categories', *Comm. Algebra*, to appear, arXiv:1107.3850.

[19] M. Dokuchaev and R. Exel, 'Associativity of crossed products by partial actions, enveloping actions and partial representations', *Trans. Amer. Math. Soc.* **357**(5) (2005), 1931–1952.

[20] M. Dokuchaev, R. Exel and P. Piccione, 'Partial representations and partial group algebras', *J. Algebra* **226**(1) (2000), 251–268.

[21] M. Dokuchaev, R. Exel and J. J. Simón, 'Crossed products by twisted partial actions and graded algebras', *J. Algebra* **320** (2008), 3278–3310.

[22] M. Dokuchaev, R. Exel and J. J. Simón, 'Globalization of twisted partial actions', *Trans. Amer. Math. Soc.* **362** (2010), 4137–4160.

[23] M. Dokuchaev, M. Ferrero and A. Paques, 'Partial actions and Galois theory', *J. Pure Appl. Algebra* **208**(1) (2007), 77–87.

[24] M. Dokuchaev, A. del Río and J. J. Simón, 'Globalizations of partial actions on non-unital rings', *Proc. Amer. Math. Soc.* **135**(2) (2007), 343–352.

[25] R. Exel, 'Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner–Voiculescu exact sequences', *J. Funct. Anal.* **122** (1994), 361–401.

[26] R. Exel, 'Twisted partial actions: a classification of regular C^*-algebraic bundles', *Proc. Lond. Math. Soc.* (3) **74**(3) (1997), 417–443.

[27] R. Exel, 'Partial actions of groups and actions of inverse semigroups', *Proc. Amer. Math. Soc.* **126**(12) (1998), 3481–3494.

[28] R. Exel, T. Giordano and D. Gonçalves, 'Envelope algebras of partial actions as groupoid C^*-algebras', *J. Operator Theory* **65** (2011), 197–210.

[29] M. Ferrero, 'Partial actions of groups on semiprime rings', in: *Groups, Rings and Group Rings*, Lecture Notes in Pure and Applied Mathematics, 248 (Chapman & Hall/CRC, Boca Raton, FL, 2006), 155–162.

[30] D. Freitas and A. Paques, 'On partial Galois Azumaya extensions', *Algebra Discrete Math.* **11** (2011), 64–77.

[31] N. D. Gilbert, 'Actions and expansions of ordered groupoids', *J. Pure Appl. Algebra* **198** (2005), 175–195.

[32] K. Janssen and J. Vercruysse, 'Multiplier bi- and Hopf algebras', *J. Algebra Appl.* **9**(2) (2010), 275–303.

[33] J. Kellendonk and M. V. Lawson, 'Partial actions of groups', *Internat. J. Algebra Comput.* **14**(1) (2004), 87–114.

[34] J.-M. Kuo and G. Szeto, 'The structure of a partial Galois extension', *Monatsh Math.* **175**(4) (2014), 565–576.

[35] K. McClanahan, 'K-theory for partial crossed products by discrete groups', *J. Funct. Anal.* **130**(1) (1995), 77–117.

[36] M. G. Megrelishvili and L. Schröder, 'Globalization of confluent partial actions on topological and metric spaces', *Topology Appl.* **145** (2004), 119–145.

[37] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics, 82 (American Mathematical Society, Providence, RI, 1993).

[38] A. Paques, V. Rodrigues and A. Sant'Ana, 'Galois correspondences for partial Galois Azumaya extensions', *J. Algebra Appl.* **10**(5) (2011), 835–847.

[39] B. Steinberg, 'Inverse semigroup homomorphisms via partial group actions', *Bull. Aust. Math. Soc.* **64**(1) (2001), 157–168.

[40] B. Steinberg, 'Partial actions of groups on cell complexes', *Monatsh. Math.* **138**(2) (2003), 159–170.

MARCELO M. S. ALVES, Departamento de Matemática,
Universidade Federal do Paraná, 81531-980 Curitiba, PR, Brazil
e-mail: marcelomsa@ufpr.br

ELIEZER BATISTA, Departamento de Matemática,
Universidade Federal de Santa Catarina, 88040-900 Florianópolis, SC, Brazil
e-mail: eliezer1968@gmail.com

MICHAEL DOKUCHAEV, Instituto de Matemática e Estatística,
Universidade de São Paulo, Rua do Matão, 1010, 05508-090 São Paulo, SP, Brazil
e-mail: dokucha@gmail.com

ANTONIO PAQUES, Instituto de Matemática,
Universidade Federal do Rio Grande do Sul, 91509-900 Porto Alegre, RS, Brazil
e-mail: paques@mat.ufrgs.br

https://doi.org/10.1017/S1446788715000774 Published online by Cambridge University Press