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SOME FINITENESS CONDITIONS IN LATTICES—USING NONSTANDARD PROOF METHODS

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Abstract

We discuss the application of nonstandard methods to local versions of certain lattice notions. In a particular case, we find that imposition of certain local conditions imply a surprising global one, namely boundedness of the given lattice.

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0. Introduction

Applications of nonstandard methods to distributive lattices have been explored in [5]. As noted in [5], many such applications depend on the existence of a hyperfinite extension D_ν of a given distributive lattice D. The existence of such an extension was in fact shown to follow from the fact that D is locally finite, and in [11] an equivalence for algebraic structures was established, of which the following is a specific case:

THEOREM 0.1. *A lattice L is locally finite if and only if it has a hyperfinite extension L_ν.*

In the present work, we will actually apply a more general theorem and more general methods from [11], still restricting our attention to lattices. Thus we review some pertinent definitions:

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DEFINITIONS. Let $\mathscr{P}$ be a lattice property.

(1) Any lattice with property $\mathscr{P}$ is called a *$\mathscr{P}$-lattice.*

(2) A lattice L is a *local-$\mathscr{P}$-lattice* if and only if each finitely generated sublattice of L is a $\mathscr{P}$-lattice.

(3) The property $\mathscr{P}$ is *finite generation-hereditary* if and only if every $\mathscr{P}$-lattice is a local-$\mathscr{P}$-lattice.

(4) If L is a lattice and K is an internal sublattice of ${}^*L$, then K is a *hyper-$\mathscr{P}$-lattice* if it satisfies ${}^*\mathscr{P}$.

The properties which we shall consider will be finite generation-hereditary ones, for, as was shown in [11], only such a property lends itself to a nonstandard characterization of the following form:

THEOREM 0.2. *Let $\mathscr{P}$ be a finite generation-hereditary property of lattices, and let L be a lattice. Then L is a local-$\mathscr{P}$-;lattice if and only if it has a hyper-$\mathscr{P}$ extension in ${}^*L$.*

We will not prove Theorems 0.1 and 0.2 here, because the proof of 0.2, form which 0.1 follows, is given in [11], and the methods employed there are similar to those used in the current work. Yet, unlike in [11], here we will have the added assumption that the structures we work with are (not necessarily distributive) lattices.

DEFINITIONS. Let $L = (L, \wedge, \vee)$ be a lattice and let $f, g: L \to L$.

(1) The ordered pair (f, g) is a *polarity* of L if and only if f is a decreasing join-endomorphism of L and g is an increasing meet-endomorphism of L, such that if $x \in L$, then we have that $f(g(x)) \leq x \leq g(f(x))$.

(2) The ordered pair (f, g) is a *local polarity* of L if and only if each finite subset R of L is contained in a finitely generated sublattice K of L for which $(f|K, g|K)$ is a polarity of K.

(3) A *tolerance* of L is a reflexive, symmetric sublattice of L^2.

(4) The function $f: L \to L$ is a *local join* (*meet*) *endomorphism* of L if and only if each finite subset F of L is contained in a finitely generated sublattice K of L on which the restriction of f is a join (meet) endomorphism.

Let us review certain results about tolerances and polarities on finite lattices, such as can be found in David Hobby and Ralph McKenzie [10], and others (e.g. [1]). Many of our results will have analogues in the finite case. In fact, our goal here is to obtain results for the localized notions of tolerances and polarities, and to use nonstandard methods for these results when it seems natural to do so. In the case of locally finite lattices, of which distributive lattices are a prime example, we often apply nonstandard techniques to results from the finite case, as in [5], [11], and others. For the fundamen-

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tal nonstandard techniques and notation, the reader is referred to texts on nonstandard analysis, such as [9], [12], [13].

1. Some standard results about polarities and tolerances

In [10], [1], others, standard results are given for polarities and tolerances of finite lattices, and some of the proofs given do not depend on the finiteness assumption. In such cases, we shall remove this assumption in what follows, but we will not reproduce the proofs here. The reader is referred to the bibliography for current research (e.g. [3], [4], [16], [17], etc.) on tolerances of lattices in the nonlocalized form.

LEMMA 1.1 (cf. [10, 1.2]). *Let L be a lattice, and let $f, g: L \to L$. Then (f, g) is a polarity of L if and only if both of the following hold.*

(i) *Either f is decreasing or g is increasing.*

(ii) *For all $x, y \in L$, $f(x) \leq y$ if and only if $x \leq g(y)$.*

LEMMA 1.2 (cf. [10, 1.2]). *Let L be a finite lattice.*

(i) *The set of polarities of L defines a one-to-one correspondence between the set of all decreasing join endomorphisms of L and the set of all increasing meet endomorphisms of L.*

(ii) *If ρ is a tolerance of L, then there is a (unique) polarity (f, g) of L for which $\rho = \{(x, y) \in L^2 | f(x \vee y) \leq x \wedge y\}$; in fact f and g are given by*

$$f(x) = \min\{(y \in L | (x, y) \in \rho\}, \quad \textit{and} \quad g(x) = \max\{y \in L | (x, y) \in \rho\}.$$

(iii) *If (f, g) is a polarity of L, then there is a (unique) tolerance ρ of L for which*

$$f(x) = \min\{y \in L | (x, y) \in \rho\}, \quad \textit{and} \quad g(x) = \max\{y \in L | (x, y) \in \rho\}.$$

In fact ρ is defined by $\rho = \{(x, y) \in L^2 | f(x \vee y) \leq x \wedge y\}$.

We shall investigate properties of connected tolerances in locally finite lattices, so let us review the definitions and some results (from the finite case) which are pertinent to such an investigation:

DEFINITIONS. Let L be a lattice.

(1) A tolerance ρ of L is *connected* if its transitive closure (the congruence it generates) is L^2.

(2) A tolerance ρ of L is *locally connected* if its restriction $\rho|K = \rho \cap K^2$ to any finitely generated sublattice K of L is connected as a tolerance of K.

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(3) A tolerance ρ of L is *locally subconnected* if each finite subset F of L is contained in some finitely generated sublattice K of L for which the restriction $\rho|K = \rho \cap K^2$ is connected as a tolerance of K.

(4) Let $x, y \in L$ and suppose $x < y$. If for all $z \in L$, $x \leq z \leq y$ implies that $z \in \{x, y\}$, then we say y *covers* x, and we denote this relationship by $x \prec y$.

(5) The *covering relation* of L is the set

$$C_L = \{(x, y) \in L^2 | x \prec y \text{ or } y \prec x\}.$$

(6) The covering relation of L is *locally realized* if every finite subset F of L is contained in a finitely generated sublattice K of L for which we have that $C_K = C_L|K = C_L \cap K^2$.

The following lemma will generalize (in some sense naturally) to locally finite lattices, by application of its nonstandard version to hyperfinite extensions:

LEMMA 1.3 (cf. [10, 1.3]). *Let ρ be a tolerance of a finite lattice L, and let (f, g) be the polarity defined by ρ, that is, let*

$$f(x) = \min\{y \in L | (x, y) \in \rho\}, \quad \textit{and} \quad g(x) = \max\{y \in L | (x, y) \in \rho\}.$$

Then the following are equivalent.

(i) *The tolerance ρ is connected.*

(ii) *There is a strictly increasing sequence $(x_0, \dots, x_{n-1}, x_n)$ in L with $x_0 = 0$, $x_n = 1$, and $(x_i, x_{i=1}) \in \rho$ for all $i < n$. (Note that we denote* $\max(L)$ *and* $\min(L)$ *by* 1 *and* 0, *respectively.)*

(iii) *The map f is strictly decreasing on $L\backslash\{0\}$.*

(iv) *The map g is strictly increasing on $L\backslash\{1\}$.*

Note that when Lemma 1.3 is generalized to the locally finite case, we have no real hope of including condition (ii) in the list, since it implies boundedness. Yet of course the $*$-transform of all of Lemma 1.3 will hold on any hyperfinite extension of a locally finite lattice, and this is in fact our tool in investigating locally subconnected tolerances of locally finite lattices.

The covering relation C_L on a lattice L is important in the study of connected tolerances in the finite case, and of course we shall explore its role in the locally finite case. We denote the *diagonal*, $\{(x, x) | x \in L\}$, of a lattice L by Δ_L, and the assumption of finiteness for L yields

LEMMA 1.4 (cf. [10, 1.4]). *Let L be a finite lattice.*

(i) *If $\rho = \langle \Delta_L \cup C_L \rangle$ is the sublattice of L^2 generated by $\Delta_L \cup C_L$, then ρ is the smallest connected tolerance of L.*

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(ii) *A meet endomorphism* g *(or a join endomorphism* f*) of* L *is strictly increasing (or strictly decreasing) if and only if* $x \prec y$ *in* L *implies that* $g(x) \geq y$ *(or* $x \prec y$ *in* L *implies* $f(x) \leq y$*).*

Two particular classes of finite lattices, the *tight* ones and the $0, 1$-*simple* ones, form an integral part of tame congruence theory for finite algebras, as presented in [10]. If these are expanded to include some infinite lattices, then we find uses of nonstandard methods in generalizing results about such lattices to localized versions of various properties. Consider the following standard definitions:

DEFINITIONS. Let L be a bounded lattice.

(1) If $\varphi: L \to M$ is a lattice homomorphism, then it is $0, 1$-*separating* if $\varphi^{-1}(\varphi(0)) = \{0\}$ and $\varphi^{-1}(\varphi(1)) = \{1\}$.

(2) The lattice L is $0, 1$-*simple* if it is nontrivial and every nonconstant lattice homomorphism $\varphi: L \to M$ is $0, 1$-separating.

(3) The lattice L is *tight* if it is nontrivial and every proper tolerance ρ of L is such that $(0, a) \in \rho$ implies $a = 0$, and dually, $(b, 1) \in \rho$ implies that $b = 1$.

Note that these definitions generalize those given in [10] for finite lattices. Now let us define localizations of some properties which we shall study. (Note that we use the fact that any finitely generated lattice is bounded: the supremum and infimum of a finite generating set are the bounds for the given finitely generated lattice.)

DEFINITIONS. Let L be any lattice.

(1) We say that L is *locally subtight* if each finite subset F of L is contained in some tight finitely generated sublattice of L.

(2) If L is locally subbounded then it is *locally sub-*$0, 1$-*simple* if each finite subset F of L is contained in some $0, 1$-simple finitely generated sublattice of L.

(3) We say that L is *locally tight* if each finitely generated sublattice of L is tight.

(4) If $\varphi: L \to M$ is a lattice homomorphism, then it is *locally* $0, 1$-*separating* if its restriction to any finitely generated sublattice is $0, 1$-separating.

In [10], various lemmas and exercises are presented regarding (finite) 0,1-simple and tight lattices. We will generalize some of these results to their corresponding localizations. At least one of these, however, can be used to find an interesting global result about certain locally tight lattices which are locally finite (with some additional local properties to be defined later), namely that they are actually bounded. To this end, let us review the results of [10] which are pertinent to this investigation.

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LEMMA 1.5 (cf. [10, 1.7]). *A finite lattice* L *is tight if and only if it is* $0, 1$*-simple and every strictly increasing meet endomorphism of* L *is constant* (*that is,* L^2 *is the only connected tolerance of* L).

LEMMA 1.6 (cf. [10, 1.8]). *Let* L *be a nontrivial bounded lattice. Then the following are equivalent*:

(i) *the lattice* L *is* $0, 1$*-simple*;

(ii) *there is a largest nontrivial* (*i.e.* $\neq L^2$) *congruence* θ *of* L, *and* θ *satisfies both* $[1]_\theta = \{1\}$ *and* $[0]_\theta = \{0\}$.

PROPOSITION 1.7 (cf. [10, 1.9(2)]). *Let* $\varphi: L \to M$ *be a surjective* $0, 1$*-separating lattice homomorphism, where* L *and* M *are finite lattices. If* L *is* $0, 1$*-simple, then so is* M.

The final result of [10] which we wish to consider is the following expression of the relationship between $0, 1$-separating homomorphisms and tightness for finite lattices, which we shall, of course generalize to the locally finite case. Other generalizations may be obtained, but those we shall give seem to elucidate the nonstandard proof methods in these lattices quite well.

LEMMA 1.8 (cf. [10, 1.10]). *Let* $\varphi: L \to M$ *be a surjective* $0, 1$*-separating homomorphism between finite lattices* L *and* M. *Then* L *is tight if and only if* M *is tight.*

2. Localizations and nonstandard methods

We shall consider the results of the previous section essentially in order of their appearance, always keeping in mind that the lattices and maps we consider are included among the objects of a suitable superstructure, $\mathfrak{S}$, and therefore have nonstandard extensions in ${}^*\mathfrak{S}$ (see, for example, [5], [9], [11], [12], [13], [14], [15]). Our first proposition provides us with our basic nonstandard tool in working with local polarities of any lattice whatsoever:

PROPOSITION 2.1. *Let* L *be a lattice with* $f, g: L \to L$. *Then the pair* (f, g) *is a local polarity of* L *if and only if* L *has a hyperfinitely generated extension* L_ν *for which* $(f_\nu, g_\nu) = ({}^*f|L_\nu, {}^*g|L_\nu)$ *is a* (*hyper*)*polarity of* L_ν.

PROOF. If (f, g) is a local polarity of L, then let $H \subseteq {}^*L$ be a hyperfinite set containing L. By transfer, ${}^*L$ has a hyperfinitely generated sublattice

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L_ν containing H for which (f_ν, g_ν) is a hyperpolarity. This is the desired extension L_ν.

Now suppose (f_ν, g_ν) is a hyperpolarity of a hyperfinitely generated extension L_ν of L, and let $F \subseteq L$ be finite. Then the set F is contained in some hyperfinitely generated sublattice K of ${}^*L$ (namely $K = L_\nu$) for which $({}^*f|K, {}^*g|K)$ is a hyperpolarity of K. Downward transfer yields existence of a finitely generated sublattice K of L which contains F and for which the pair $(f|K, g|K)$ is a polarity of K. Thus (f, g) is a local polarity of L, as desired.

Since any hyperfinitely generated extension of a locally finite lattice (in its nonstandard extension) is actually hyperfinite, we get the following corollary, which is useful in dealing with local polarities of locally finite lattices:

COROLLARY 2.2. *Let L be a locally finite lattice with maps $f, g.L \to L$. Then (f, g) is a local polarity of L if and only if L has a hyperfinite extension L_ν for which $(f_\nu, g_\nu) = ({}^*f|L_\nu, {}^*g|L_\nu)$ is a (hyper)polarity of L_ν.*

Recall that Lemma 1.2 established that the set of polarities defines a one-to-one correspondence between decreasing join endomorphisms and increasing meet endomorphisms on a finite lattice. Now we shall show a similar result for the set of local polarities of a locally finite lattice, by a tidy application of Corollary 2.2.

PROPOSITION 2.3. *Let L be a locally finite lattice. Then the set of local polarities of L defines a one-to-one correspondence between its domain and range.*

PROOF. Let (f, g), (f, h) and (e, h) be local polarities of L, with the intention of proving that $f = e$ and $g = h$. Let L_ν, L_μ and L_γ be hyperfinite extensions of L for which (f_ν, g_ν), (f_μ, h_μ) and (e_γ, h_γ) are hyperpolarities, respectively. Let $x \in L$. Then, as in Lemma 1.2, we get $g_\nu(x) = \max\{y \in L_\nu | f_\nu(y) \leq x\}$. Thus, since $g = g_\nu|L$ and $f = f_\nu|L$, we get $g(x) \geq y$ for all $y \in L$ with $f(y) \leq x$. By transfer, if $y \in {}^*L$ is such that ${}^*f(y) \leq x$, then ${}^*g(x) \geq y$. Therefore, since $g(x) = {}^*g(x)$ and $h = h_\mu|L$, we have

$$\begin{aligned} h(x) = h_\mu(x) &= \max\{y \in L_\mu | f_\mu(y) \leq x\} \\ &= \max\{y \in L_\mu | {}^*f(y) \leq x\} \leq g(x). \end{aligned}$$

A similar argument yields $g(x) \leq h(x)$, so $g = h$. Dually, $f = e$, and we are done.

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Since it is possible that a given decreasing join endomorphism of an infinite locally finite lattice might not be paired with any increasing meet endomorphism as a polarity, it is merely a convenience to make the following definitions.

DEFINITIONS. Let L be a lattice with $f: L \to L$.

(1) We call f a *locally polarizable join endomorphism* of L if there is some $g: L \to L$ such that the pair (f, g) is a local polarity of L.

(2) We call f a *locally polarizable meet endomorphism* of L if there is some $g: L \to L$ such that the pair (g, f) is a local polarity of L.

Note then that Proposition 2.3 merely states that on a locally finite lattice, the locally polarizable join endomorphisms and the locally polarizable meet endomorphisms are in a one-to-one correspondence, and the correspondence is defined, of course, by the set of local polarities of the given lattice.

Let us now look at tolerances of locally finite lattices. Here we shall use relationships between tolerances and polarities, as is done in the finite case. The following lemma is the result of a simple application of the transfer principle to the fact that restrictions of tolerances to sublattices are tolerances.

LEMMA 2.4. *Let L be a locally finite lattice with hyperfinite extension L_ν. If ρ is a tolerance of L, then $\rho_\nu = {}^*\rho|L_\nu$ is a (hyper)tolerance of L_ν.*

Use of this fact is made in the following proposition, which is the nonstandard version of Lemma 1.2(ii) for hyperfinite extensions of locally finite lattices. It will play an important part in our investigation of tolerances.

PROPOSITION 2.5. *Let L be a locally finite lattice and let ρ be a tolerance of L. Then for any hyperfinite extension L_ν of L, there is a unique (hyper)polarity (φ, γ) on L_ν for which*

$$\rho_\nu = \{(x, y) \in L_\nu^2 | \varphi(x \vee y) \leq x \wedge y\}.$$

In fact, φ and γ are given by

$$\varphi(x) = \min\{y \in L_\nu | (x, y) \in \rho_\nu\},$$

and

$$\gamma(x) = \max\{y \in L_\nu | (x, y) \in \rho_\nu\}.$$

PROOF. By transfer of the finite case (Lemma 1.2) we have that (φ, γ), as defined, is a hyperpolarity on L_ν and we have that

$$\rho_\nu = \{(x, y) \in L_\nu^2 | \varphi(x \vee y) \leq x \wedge y\}.$$

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Now to show uniqueness, note first that, also by transfer of the finite case, if (σ, μ) is a hyperpolarity of L_ν, then there is a *unique* hypertolerance τ of L_ν for which σ and μ are defined on L_ν by the following formulas:

$$\sigma(x) = \min\{y \in L_\nu | (x, y) \in \tau\} \quad \text{and} \quad \mu(x) = \max\{y \in L_\nu | (x, y) \in \rho_\nu\}.$$

In particular, if (σ, μ) satisfies $\rho_\nu = \{(x, y) \in L_\nu^2 | \sigma(x \vee y) \leq x \wedge y\}$, then $\rho_\nu = \tau$. Hence $\sigma = \varphi$ and $\mu = \gamma$, so we are done.

Our next proposition is a standard result, but we provide a nonstandard proof, for no other reason than its brevity and its aesthetic appeal to the (author's) intuition in thinking about lattices:

PROPOSITION 2.6. *Let L be a locally finite lattice and let (f, g) be a local polarity of L. Then the relation ρ defined by*

$$\rho = \{(x, y) \in L^2 | f(x \vee y) \leq x \wedge y\}$$

is a tolerance of L.

PROOF. Let L_ν be a hyperfinite extension of L for which (f_ν, g_ν) is a hyperpolarity. Note that we have

$$\rho_\nu = \{(x, y) \in L_\nu^2 | f_\nu(x \vee y) \leq x \wedge y\},$$

and transfer of the finite case yields that this is a hypertolerance of L_ν, that is, ρ_ν is a reflexive and symmetric internal sublattice of L_ν^2. In particular, ρ_ν is a tolerance of L_ν. As we remarked before, restrictions of tolerances to sublattices are tolerances, so since $\rho = \rho_\nu | L$, ρ is a tolerance of L, as desired.

We shall now characterize locally subconnected tolerances of locally finite lattices, as is done for connected tolerances of finite lattices in Lemma 1.3. Our attack on the locally finite case requires the following two observations. We leave their proofs to the reader.

PROPOSITION 2.7. *Let ρ be a tolerance of a locally finite lattice L. Then ρ is locally subconnected if and only if on some hyperfinite extension L_ν of L, ρ_ν is hyperconnected.*

COROLLARY 2.8. *Let ρ be a tolerance of a locally finite lattice L. Then the following are equivalent.*

(i) *The tolerance ρ is locally subconnected.*

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(ii) *There is a hyperfinite extension* L_ν *of* L *for which if* (φ, γ) *is the unique hyperpolarity of* L_ν *which satisfies*

$$\rho_\nu = \{(x, y) \in L_\nu^2 | \varphi(x \vee y) \leq x \wedge y\},$$

then φ *is strictly decreasing on* $L \backslash \{0\}$.

(iii) *There is a hyperfinite extension* L_ν *of* L *for which if* (φ, γ) *is the unique hyperpolarity of* L_ν *which satisfies*

$$\rho_\nu = \{(x, y) \in L_\nu^2 | \varphi(x \vee y) \leq x \wedge y\},$$

then γ *is strictly increasing on* $L \backslash \{1\}$.

From the above, we also get the following corollary, which is a standard result for locally subconnected tolerances of locally finite lattices, and is a natural analogue of Lemma 1.3:

COROLLARY 2.9. *Let* (f, g) *be a polarity on a locally finite lattice* L. *Then the following are equivalent.*

(i) *The tolerance* $\rho = \{(x, y) \in L^2 | f(x \vee y) \leq x \wedge y\}$ *is locally subconnected.*

(ii) *The map* f *is strictly decreasing on* $L \backslash \{0\}$.

(iii) *The map* g *is strictly increasing on* $L \backslash \{0\}$.

PROOF. All that is needed to show (i) $\Leftrightarrow$ (ii) is to note that f is strictly decreasing on $L \backslash \{0\}$ if and only if its extension f_ν to some (and in fact to every) hyperfinite L_ν containing L is itself strictly decreasing on $L_\nu \backslash \{0\}$. A similar (dual) observation yields the equivalence (i) $\Leftrightarrow$ (iii).

A direct argument can be given for the following obvious proposition, but our argument avoids "localization" within L. We note that the finite analogue of this result is part (i) of Lemma 1.4.

PROPOSITION 2.10. *Let* L *be a locally finite lattice in which the covering relation is locally realized, and let* ρ *be the sublattice of* L^2 *which is generated by* $\Delta_L \cup C_L$. *Then* ρ *is a connected tolerance of* L, *and it is in fact the smallest locally sub-connected (and locally connected) tolerance of* L.

PROOF. Let L_ν be a hyperfinite extension of L, so that ρ_ν is a hypertolerance of L_ν. Now, by transfer of the assumption that the covering relation is locally realized in L, we may assume that ρ_ν contains $\Delta_{L_\nu} \cup C_{L_\nu}$, because $\Delta_{L_\nu} \cup C_{L_\nu} = (\Delta_{*_L} \cup C_{*_L}) | L_\nu$. Thus ρ_ν is hyperconnected as a hypertolerance of L_ν (by transfer of the finite theory, of course), so ρ is locally connected and a standard argument yields connectedness of ρ: yet we will still give the

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nonstandard argument here. Let (x, y) be in L^2. Then $(x, y) \in L_\nu^2$, so that for some hyperfinite μ, the pair (x, y) is in ρ_ν^μ, since the (internal) transitive closure of ρ_ν is all of L_ν^2. Thus we get that $(x, y) \in {}^*\rho^\mu$, and so by transfer, there is a finite n for which we get $(x, y) \in \rho^n$. It follows that the transitive closure of ρ is all of L^2, that is, ρ is a connected tolerance of L. If τ is any locally connected tolerance of L (or equivalently, if τ is a locally subconnected tolerance of L), then τ_ν is a hypertolerance of L_ν. Transfer of the finite theory yields that τ_ν contains ρ_ν. But then τ contains ρ, and we are done.

Now recall that we defined $f: L \to L$ to be a local join (meet) endomorphism of L if each finite subset F of L is contained in a sublattice K of L on which the restriction of f is a join (meet) endomorphism. The following facts, stated without proof because their proofs follow exactly the same lines as those of Theorems 0.1 and 0.2, and Proposition 2.1, are the basis for the next theorem.

Facts (cf. [11]). (1) A lattice L is locally subbounded if and only if it has a bounded hyperfinitely generated extension in ${}^*L$.

(2) A locally subbounded lattice L is locally subtight if and only if it has a hyperfinitely generated internally tight extension in ${}^*L$.

(3) A locally bounded lattice L is locally tight if and only if every one of its hyperfinitely generated extensions in ${}^*L$ is internally tight.

(4) A function $f: L \to L$ is a local join (meet) endomorphism of L if and only if on some hyperfinitely generated extension L_ν of L, f_ν is a join (meet) endomorphism.

Theorem 2.11. *Let L be a locally finite lattice with at least one strictly increasing (on $L\backslash\{1\}$) local meet endomorphism and at least one strictly decreasing (on $L\backslash\{0\}$) local join endomorphism. If L is locally tight, then it is bounded.*

Proof. We only show that L has a maximum element, 1. The existence of a minimum, 0, in L, follows dually. Let $f: L \to L$ be a strictly increasing (on $L\backslash\{1\}$) meet endomorphism and let L_ν be a hyperfinite extension of L on which f_ν is a meet endomorphism. Note that local tightness of L implies internal tightness of L_ν. Since f is strictly increasing on $L\backslash\{1\}$, so is f_ν; hence by transfer of the finite case, f_ν is constant. Let $1 = \max(L_\nu)$, so that $f_\nu(x) = 1$ for all $x \in L_\nu$. In particular, if $x \in L$, then $f(x) = 1$, so in fact $1 \in L$. But then $1 = \max(L)$, so we are done.

Another use of fact 3 from above is the following local version of Lemma

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1.6, for the case of locally subbounded lattices.

PROPOSITION 2.12. *Let L be any lattice. Then L is locally sub-$0, 1$-simple if and only if it has a bounded hyperfinitely generated extension L_ν with a largest proper internal congruence θ. Said congruence θ satisfies $[1]_\theta = \{1\}$ and $[0]_\theta = \{0\}$, where we have $1 = \max(L_\nu)$ and $0 = \min(L_\nu)$.*

Note that Proposition 2.12 is of course the very tool one would need to pursue a nonstandard study of locally sub-$0, 1$-simple lattices in general. We do not continue in this course, but will point out two other (obvious) tools developed in a similar way for other finite generation-hereditary properties of lattices. The first we will state without proof, leaving its verification for the reader (see its finite analogue). For the second, we shall provide the proof, merely for aesthetic reasons.

PROPOSITION 2.13. *Let L be a locally finite lattice, and let M be any lattice. Suppose $\varphi: L \to M$ is a surjective and locally $0, 1$-separating lattice homomorphism. If L is locally sub-$0, 1$-simple, then so is M.*

PROPOSITION 2.14. *Let L be a locally finite lattice and let M be any lattice. Suppose $\varphi: L \to M$ is a surjective and locally $0, 1$-separating lattice homomorphism. Then L is locally subtight if and only if M is.*

PROOF. For hyperfinite extensions L_ν and M_ν of L and M, respectively (note that M is locally finite since L is), with $\varphi_\nu(L_\nu) = M_\nu$, we have that L_ν is internally tight if and only if M_ν is internally tight, by transfer of the finite case. The proposition follows.

3. The extension monad

In [11], the extension monad of an arbitrary algebra was introduced, and its properties in that general setting were explored. Here, we wish to investigate the relationship which the extension monad of a lattice bears to the local properties discussed herein. Let us first define this object, whose construction from the original lattice resembles the construction of a monad for an arbitrary filter. As was pointed out in [11], the notion of such monads in general is due to Luxemburg [12].

DEFINITION. Let $L = (L, \wedge, \vee)$ be a lattice with enlargement ${}^*L$. Then the *extension monad* of L is the following sublattice of ${}^*L$:

$$\widehat{L} = \bigcap \{K \leq {}^*L | K \text{ is an internal extension of } L\}.$$

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We will need certain facts proven in [11] about the extension monad which are among the following:

FACTS (cf. [11]). (1) For any lattice L, the extension monad $\widehat{L}$ is given by the following formulas:

$$\widehat{L} = \bigcap\{H \leq {}^*L | H \text{ is an internal hyperfinitely generated extension of } L\}$$
$$= \bigcup\{\langle F\rangle^{(\text{int})} \leq {}^*L | F \text{ is a finite subset of } L\}.$$

(Note. Here we have introduced the notation of [11] for the internal sublattice $\langle F\rangle^{(\text{int})}$ of ${}^*L$ which is generated by F.)

(2) The lattice L is locally finite if and only if $\widehat{L} = L$.

(3) If $f: L \to M$ is a function between the underlying sets of two lattices L and M, then f is a lattice homomorphism if and only if the restriction $\hat{f} = {}^*f|\widehat{L}$ is a lattice homomorphism, and in this case, we have that $\hat{f}: \widehat{L} \to \widehat{M}$.

Using the above facts, we may prove the following two theorems, which simultaneously provide for and limit the use of the extension monad in the study of local polarities.

THEOREM 3.1. *Let L be a lattice with $f, g: L \to L$. If the pair (f, g) is a local polarity of L, then $(\hat{f}, \hat{g})$ is a polarity of the lattice $\widehat{L}$.*

PROOF. Suppose (f, g) is a local polarity of L. Then (f, g) is a polarity of L, trivially, so $({}^*f, {}^*g)$ is a (hyper)polarity of ${}^*L$. (Note that a hyperpolarity is also a polarity, but the converse does not hold.) Thus, for $x \in \widehat{L}$, we have:

$$\hat{f}(\hat{g}(x)) = {}^*f({}^*g(x)) \leq x \leq {}^*g({}^*f(x)) = \hat{g}(\hat{f}(x)),$$

all by transfer and restriction. Similarly, $\hat{f}$ is a decreasing join-homomorphism of $\widehat{L}$ into ${}^*L$ and $\hat{g}$ is an increasing meet-homomorphism of $\widehat{L}$ into ${}^*L$. Let F be a finite subset of L with $x \in \langle F\rangle^{(\text{int})}$. Then for some finitely generated sublattice K of L which contains F, we have that the pair $(f|K, g|K)$ is a polarity of K. But then $f(K)$ and $g(K)$ are both contained in K, so by transfer, ${}^*f({}^*K)$ and ${}^*g({}^*K)$ are both contained in ${}^*K$; hence we get

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$$\hat{f}(x) = {}^*f(x) \in {}^*f(\langle F\rangle^{(\mathrm{int})}) \subset {}^*f({}^*K) \subseteq {}^*K \subseteq \widehat{L},$$

and similarly, $\hat{g}(x)$ is in $\widehat{L}$. Thus $\hat{f}(\widehat{L})$ and $\hat{g}(\widehat{L})$ are both contained in $\widehat{L}$, which makes $(\hat{f}, \hat{g})$ a polarity of $\widehat{L}$.

Theorem 3.2. *Let L be a lattice with lattice endomorphisms f and g, and suppose $(\hat{f}, \hat{g})$ is a local polarity of $\widehat{L}$. Then (f, g) is a local polarity of L.*

Proof. First note that (f, g) is a polarity of L, so that $({}^*f, {}^*g)$ is a (hyper)polarity of ${}^*L$. Let F be a finite subset of L and since $(\hat{f}, \hat{g})$ is a local polarity of $\widehat{L}$, let E be a finite subset of $\widehat{L}$ with F contained in $\langle E\rangle$, such that $(\hat{f}|\langle E\rangle, \hat{g}|\langle E\rangle)$ is a polarity of $\langle E\rangle$. Then $\hat{f}(E) \cup \hat{g}(E)$ is contained in $\langle E\rangle$ and so in $\langle E\rangle^{(\mathrm{int})}$, so that since $\hat{f}|\langle E\rangle^{(\mathrm{int})}$ and $\hat{g}|\langle E\rangle^{(\mathrm{int})}$ are internal lattice homomorphisms from $\langle E\rangle^{(\mathrm{int})}$ into ${}^*L$ with $\hat{f}(E)$ and $\hat{g}(E)$ contained in $\langle E\rangle^{(\mathrm{int})}$, we get that $\hat{f}(\langle E\rangle^{(\mathrm{int})})$ and $\hat{g}(\langle E\rangle^{(\mathrm{int})})$ are both contained in $\langle E\rangle^{(\mathrm{int})}$. Thus $(\hat{f}|\langle E\rangle^{(\mathrm{int})}, \hat{g}|\langle E\rangle^{(\mathrm{int})})$ is a hyperpolarity of $\langle E\rangle^{(\mathrm{int})}$. Since F is contained in $\langle E\rangle^{(\mathrm{int})}$, we have established existence of a hyperfinitely generated sublattice K of ${}^*L$, namely $K = \langle E\rangle^{(\mathrm{int})}$, which contains F and for which $(\hat{f}|K, \hat{g}|K) = ({}^*f|K, {}^*g|K)$ is a hyperpolarity of K. Since F is a *finite* subset of L, we may apply downward transfer to obtain existence of a finitely generated sublattice K of L which contains F such that $(f|K, g|K)$ is a polarity of K. Since F was chosen arbitrarily, (f, g) is a local polarity of L, and we are done.

Note that Theorem 3.1 and 3.2 do not require local finiteness of the given lattice L. In light of fact 2 in this section, we see that attempts to prove any assertion about a locally finite lattice L, via its extension monad $\widehat{L}$ are no 'easier' than direct attempts, since in this case, $\widehat{L} = L$. In particular, since the results involving tolerances which were considered in the previous section all included this assumption of local finiteness, we have no applications of the notion of the extension monad to those results.

Another fact from [11] which limits application of this notion of the extension monad is the following: *a lattice L is finitely generated if and only if we have that $\widehat{L} = {}^*L$.* Thus assertions involving the extension monad of a finitely generated lattice L are merely assertions about the nonstandard extension ${}^*L$, and again $\widehat{L}$ contributes nothing in this case. The applications of the extension monad will always be in the realm 'between' finite generation and local finiteness, when $\widehat{L}$ is neither L nor ${}^*L$, as is possible in 3.1 and 3.2.

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