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EXISTENCE THEOREMS FOR SOME NON-LINEAR EQUATIONS OF EVOLUTION

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1. Introduction. In recent years considerable attention has been focused on non-linear hyperbolic differential equations with the object of establishing the existence of global solutions. It is our aim here to establish the existence of weak solutions of boundary value problems for non-linear equations of the form

$$u_{tt}(t) + A(t)u(t) + du_t(t) = f(t), \tag{1–1}$$

where d is a real constant called the damping coefficient, $u(t)$ is a vector-valued function defined on a subinterval of the real line into a space of complex-valued functions $u(x)$ defined on a bounded domain Ω in the real Euclidean space E^N of N dimensions, $u_t(t) \equiv du(t)/dt$, and $A(t)$ is the family of partial differential operators of order $2m$ ($m = 1, 2, \ldots$) on Ω given in generalized divergence form by

$$A(t)u(x) = \sum_{|\alpha|\leqq m} D^\alpha A_\alpha(x, t, u(x), \ldots, D^m u(x)) \tag{1–2}$$

with

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \qquad \text{for } 1 \leqq j \leqq N$$

and for each N-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$, $D^\alpha = D_1^{\alpha_1} \ldots D_N^{\alpha_N}$, $|\alpha| = \sum_{j=1}^N \alpha_j$, and $D^m u(x) = (D^\alpha u(x))_{|\alpha|=m}$.

Particular interest has been given to the equation

$$u_{tt}(t) - \Delta_N u(t) + |u(t)|^\rho u(t) = f(t) \qquad (\rho > 0), \tag{1–3}$$

where Δ_N denotes the N-dimensional Laplace operator

$$\Delta_N \equiv -(D_1^2 + \ldots + D_N^2).$$

Browder [**1**] has obtained the existence of strict solutions of the Cauchy problem for a class of operator differential equations of the form

$$u_{tt}(t) + Au(t) + M(u(t)) = 0,$$

where A is a positive, densely defined, self-adjoint linear operator on a Hilbert space and M is a non-linear function from the domain of $A^{1/2}$ to that Hilbert

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space satisfying certain growth, Lipschitzian and definiteness conditions. His results include the existence and uniqueness of strict solutions of equation (1–3) for $f(t) = 0$ with ρ depending on the space dimension N and they specialize to give results obtained previously by Jörgens [**12**]. Sather [**24**] has employed a compactness argument to obtain the existence of weak and strict global solutions of an initial-boundary value problem for (1–3). Lions and Strauss [**17**] have established the existence of weak solutions of initial-boundary value problems involving non-linear evolution equations of the form

$$u_{tt}(t) + A(t)u(t) + \beta(t; u(t), u_t(t)) = f(t),$$

where each $A(t)$ is an unbounded formally self-adjoint linear operator and the operator $\beta(t; u(t), u_t(t))$ is non-linear in $u(t)$ and $u_t(t)$ and is, in some sense, close to a "dissipative" operator. Compactness arguments were used and monotonicity properties of the non-linearities exploited. Monotonicity arguments had been used previously for equations involving bounded operators, particularly non-linear integral equations, by Minty [**18**; **19**]. They were first applied to partial differential operators of elliptic and parabolic type by Browder [**2**; **3**; **4**].

As one might expect, fewer results have been obtained concerning the existence of periodic solutions of boundary value problems for non-linear hyperbolic equations. For some previous results concerning this problem the reader is referred to [**21**; **5**]. For non-linear wave equations with a strong damping, as in the present paper, the reader should compare the parallel results of Rabinowitz [**22**; **23**]. For periodic solutions of non-linear wave equations in E^2 without damping hypotheses the reader is referred to [**6**; **7**; **10**].

This paper was developed to study physical systems whose defining equations possess non-linear expressions in the higher order space derivatives. Such is the case, for example, for a system governed by the equation

$$u_{tt}(t) - \left(\frac{\partial u(t)}{\partial x}\right)^2 \frac{\partial u(t)}{\partial x^2} + k \sin u(t) = f(t)$$

in one space dimension. In § 3, sufficient conditions are imposed on the function $f(t)$, the functions A_α, and the coefficient d to ensure in Theorem 1 (§ 6) the existence of very weak periodic solutions $u(t)$ of the boundary value problem defined by (1–1) and given boundary conditions. In § 4, sufficient conditions on these same quantities are formulated which guarantee in Theorem 2 (§ 7) the existence of weak solutions $v(t)$ of the initial-boundary value problem defined by (1–1) and given boundary and initial conditions. In order to establish these existence results we consider a sequence of equations which are finite-dimensional approximations to (1–1) and the given boundary conditions. It then remains to pass to the limit, and this is accomplished by employing combined compactness and monotonicity arguments. Section 2 consists of the mathematical preliminaries necessary to give a precise formulation and treatment of the above problems. In § 5 we prove the preliminary functional analytic results required to obtain the existence theorems in §§ 6, 7, and in § 8 specific examples of the families $A(t)$ are considered.

2. Notation, function spaces. Ω is a fixed, bounded domain (an open, connected set) in E^N with boundary and closure written $\partial\Omega$ and $\bar{\Omega}$, respectively. A point $(x_1, \ldots, x_N)$ of E^N is denoted simply by x and the volume element $dx_1 \ldots dx_N$ by dx. Integration over subsets of E^N is taken with respect to the N-dimensional Lebesgue measure.

The Lebesgue spaces $L^p(\Omega)$ are, for $1 \leqq p \leqq \infty$, the collection of equivalence classes of (real or complex-valued) almost everywhere equal functions $u(x)$ defined on Ω and such that

$$||u||_{0,p}^p \equiv \int_\Omega |u(x)|^p \, dx < \infty \quad \text{if } 1 \leqq p \leqq \infty,$$

$$||u||_{0,\infty} \equiv \operatorname*{essup}_\Omega |u(x)| < \infty.$$

$L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_\Omega u(x)\overline{v(x)} \, dx.$$

The Sobolev spaces $W^{m,p}(\Omega)$ are, for $1 \leqq p < \infty$ and $m = 1, 2, \ldots$, the collection of functions in $L^p(\Omega)$ all derivatives in the distribution sense of order not exceeding m of which also belong to $L^p(\Omega)$. $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$||u||_{m,p} = \left(\sum_{|\alpha| \leqq m} ||D^\alpha u||_{0,p}^p \right)^{1/p}$$

and is separable and reflexive for $1 < p < \infty$. $C_0^\infty(\Omega)$ is the set of functions which are infinitely often continuously differentiable and have compact support in Ω. $W_0^{m,p}(\Omega)$ is the closure of this linear manifold in the norm of $W^{m,p}(\Omega)$ and, since it will be the framework for most of our discussions, shall be denoted simply by W. For brevity in notation, we shall always denote the $L^2(\Omega)$ norm $|| \cdot ||_{0,2}$ by $|| \cdot ||$.

If X is a Banach space, then X^* shall denote its dual or conjugate space, that is, the space of all bounded, conjugate linear functionals on X, and $\langle w, u \rangle$ the natural pairing of an element $u \in X$ and $w \in X^*$. Strong continuity and differentiability of vector-valued functions on a subinterval S of E^1 into X are defined in the usual way (see, for example, [**11**]).

$C^k(\tau; X)$, $k = 1, 2, \ldots$, is the collection containing every vector-valued function which, together with all its strong derivatives of order not exceeding k, is defined on E^1 into X, is strongly continuous, and is periodic in t of period τ. $C^k(S; X)$, $k = 1, 2, \ldots$, is defined analogously.

Let $|| \cdot ||_X$ be the norm of X. We denote by $L^p(\tau; X)$, for $1 \leqq p \leqq \infty$, the collection of equivalence classes of almost everywhere equal functions $u(t)$

defined on E^1 into X and periodic in t of period τ such that

$$||u||_{L^p(\tau;X)} \equiv \int ||u(t)||_X^p \, dt < \infty \qquad \text{if } 1 \leqq p < \infty,$$

$$||u||_{L^\infty(\tau;X)} \equiv \operatorname{essup} ||u(t)||_X < \infty,$$

where the integration with respect to t is taken over any interval of periodicity. If X is separable and reflexive, $L^p(\tau; X)$ for $1 < p < \infty$ is a reflexive Banach space with conjugate space isomorphic to $L^q(\tau; X^*)$, where $1/p + 1/q = 1$. In addition, the dual space $(L^1(\tau; X))^*$ is linearly isomorphic to the Banach space $L^\infty(\tau; X^*)$. The spaces $L^p(S; X)$ for $1 \leqq p \leqq \infty$ are defined analogously.

The linear spaces of complex-valued functions $u(x, t)$ over $\Omega \times E^1$ are defined in the usual way. The Lebesgue measure shall always be denoted simply by the symbol μ since it will be clear from the context which dimension is under consideration.

Since it shall be necessary to consider the dependence of the functions A_α in (1–2) on their several variables without considering the latter variables to be functions, we adopt the notation

$$A_\alpha(x, t, z) = A_\alpha(x, t, z_1, \ldots, z_M)$$

for any fixed α, where the complex variables z_k, $k = 1, \ldots, M$, are indexed to correspond to $D^\beta u(x)$ for all fixed $\beta = (\beta_1, \ldots, \beta_N)$, $|\beta| \leqq m$.

3. Problem I. It is our aim to prove the existence of at least one function $u(t)$, periodic in t of period τ, which satisfies

$$u_{tt}(t) + A(t)u(t) + du_t(t) = f(t), \qquad t \in E^1,$$

$$D^\alpha u(t)|_{\partial\Omega} = 0, \qquad |\alpha| \leqq m - 1, t \in E^1,$$

where the family $A(t)$, $t \in E^1$, is given by (1–2). The above boundary-value problem together with the periodicity requirement shall be called Problem I. The following conditions are imposed on this problem.

(I.1) Each $A_\alpha(x, t, z_1, \ldots, z_M)$ is a complex-valued function, measurable in x on Ω for fixed $t, z_1, \ldots, z_M$ and jointly continuous in the real variable t and the complex variables $z_1, \ldots, z_M$ for almost every fixed x in Ω. A_α is once continuously differentiable in t on E^1 and periodic in t of period τ for almost every fixed x in Ω and fixed $z_1, \ldots, z_M$. There exist a real number $p \geqq 2$ and a continuous function $g(s)$ of the real variable s such that for all $u(x)$ in W ($W \equiv W_0^{m,p}(\Omega)$), all $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $|\alpha| \leqq m$ and almost all x in Ω,

$$|A_\alpha(x, t, u(x), \ldots, D^m u(x))| \leqq g(||u||_{m,p}) \left\{ \sum_{|\beta| \leqq m} |D^\beta u(x)|^{p-1} + 1 \right\}$$

for all t in E^1.

Before imposing the additional conditions on Problem I, we require the following definition.

Definition 3.1. $B(t)$, $t \in E^1$, is said to be a family of admissible lower order operators for Problem I if

$$B(t)u(x) = \sum_{|\beta| \leqq m-1} D^\beta B_\beta(x, t, u(x), \ldots, D^{m-1}u(x))$$

for all t in E^1, each $B_\beta(x, t, z_1, \ldots, z_{M'})$ satisfies the same measurability, continuity, differentiability and periodicity requirements as the functions A_α and there exists a continuous function $g_1(s)$ such that for all $u(x)$ in W, all β with $|\beta| \leqq m - 1$ and almost all x in Ω,

$$|B_\beta(x, t, u(x), \ldots, D^{m-1}u(x))| \leqq g_1(||u||_{m,p})\left\{\sum_{|\gamma| \leqq m-1} |D^\gamma u(x)|^{p-1} + 1\right\}$$

for all t in E^1.

(I.2) The family of Dirichlet forms

$$a(t; u, v) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|}\langle A_\alpha(x, t, u, \ldots, D^m u), D^\alpha v\rangle$$

associated with the family $A(t)$ satisfies the following:

(i) There exist a continuous real-valued function $c_0(s)$ with $\lim_{s\to\infty} c_0(s) = \infty$ and a positive constant k_0 such that

$$\operatorname{Re} a(t; u, u) \geqq c_0(||u||_{m,p}) + k_0||u||^2$$

for all $u(x)$ in W and all t in E^1.

(ii) There exists a family $B(t)$ of admissible lower order operators such that

$$\operatorname{Re}\{a(t; u, u - v) - a(t; v, u - v) + b(t; u, u - v) - b(t; v, u - v)\} \geqq 0$$

for all $u(x)$ and $v(x)$ in W and all t in E^1, where

$$b(t; u, v) = \sum_{|\beta| \leqq m-1} (-1)^{|\beta|}\langle B_\beta(x, t, u, \ldots, D^{m-1}u), D^\beta v\rangle$$

is the family of Dirichlet forms associated with $B(t)$.

(I.3) There exists a family $r(t; \cdot)$, $t \in E^1$, of real-valued functions, periodic in t of period τ, such that for all t in E^1 and all $\psi(t) \in C^1(E^1; W)$, $r(t; \psi(t))$ is once continuously differentiable in t,

$$\operatorname{Re} a(t; \psi(t), \psi_t(t)) \geqq \frac{d}{dt} r(t; \psi(t)),$$

and $r(t; \cdot)$ satisfies the following:

(i) There exists a continuous real-valued function $c_1(s)$ with $\lim_{s\to\infty} c_1(s) = \infty$ such that for all $u(x)$ in W and all t in E^1,

$$r(t; u) \geqq c_1(||u||_{m,p});$$

(ii) There exist constants k_1 and k_2 such that for all $u(x)$ in W and all t in E^1,

$$r(t; u) \leqq k_1 \operatorname{Re} a(t; u, u) + k_2.$$

(I.4) d is a positive real constant.

(I.5) $f(t)$ is a fixed element in $L^2(\tau; L^2(\Omega))$.

We now state precisely what is meant by a weak solution of Problem I. This concept is defined relative to a certain class of test functions. Let $\mathfrak{F}_1$ be the set of functions $\mathfrak{F}_1 \equiv \{\Phi(t) | \Phi(t) \in L^1(\tau; W) \text{ and } \Phi_t(t) \in L^1(\tau; L^2(\Omega))\}$, where differentiation is considered in the sense of distributions.

Definition 3.2. By a weak (or generalized) solution of Problem I we mean a function $u(t)$ with $u(t) \in L^\infty(\tau; W)$ and $u_t(t) \in L^\infty(\tau; L^2(\Omega))$ such that

$$\int \{-(u_t(t), \Phi_t(t)) + a(t; u(t), \Phi(t)) + d(u_t(t), \Phi(t))\}\, dt = \int (f(t), \Phi(t))\, dt$$

for all $\Phi(t)$ in $\mathfrak{F}_1$.

It shall be proved in § 6 that if Problem I satisfies conditions (I.1)–(I.5), then it has at least one weak solution in the sense of Definition 3.2.

Remarks. The semi-boundedness requirement or growth restriction on the functions A_α in condition (I.1) can be weakened to

$$|A_\alpha(x, t, u(x), \ldots, D^m u(x))| \leqq g(||u||_{m,p}) \left\{ \sum_{|\beta| \leqq m} |D^\beta u(x)|^{p-1+c_{\alpha\beta}} + 1 \right\}$$

where the $c_{\alpha\beta}$ satisfy

$$0 \leqq c_{\alpha\beta} \leqq (1/p - (m - |\beta|)/N)^{-1} \{(p-1)(m - |\beta|)/N + (m - |\alpha|)/N\}$$

with equality only if $1/p - (m - |\beta|)/N \geqq 0$. The boundedness condition in Definition 3.1 can be weakened analogously. However, for brevity in notation we have restricted our attention to the simpler conditions indicated.

For each fixed t in E^1, conditions (I.1)–(I.2) closely parallel those used by Browder [**2**] to prove the existence of variational solutions for a class of non-linear elliptic boundary value problems. (I.2)–(ii) is called a generalized monotonicity condition.

Condition (I.3) is the most restrictive requirement on Problem I and suggests that the family of forms $a(t; u, v)$ be close to multilinear. For example, if $A(t)$ is given by

$$A(t)u(x) = -\Delta_2 u(x) + |u(x)|^\rho u(x) \qquad (\rho \geqq 0),$$

then

$$r(t; u) = \frac{1}{2} \left|\left| \frac{\partial u}{\partial x_1} \right|\right|^2 + \frac{1}{2} \left|\left| \frac{\partial u}{\partial x_2} \right|\right|^2 + \frac{||u||_{0,\rho+2}^{\rho+2}}{\rho + 2}$$

for all t in E^1 and $u(x)$ in $W_0^{1,2}(\Omega)$ and clearly satisfies (i) and (ii). We consider as a second example one in which the functions are assumed to be real-valued. Let

$$A(t)u(x) = -\frac{\partial}{\partial x_1}\left(\frac{\partial u(x)}{\partial x_1}\right)^3 - \frac{\partial}{\partial x_2}\left(\frac{\partial u(x)}{\partial x_2}\right)^3 + k \sin u(x).$$

Then

$$r(t;u) = \frac{1}{4}\left|\left|\frac{\partial u}{\partial x_1}\right|\right|_{0,4}^4 + \frac{1}{4}\left|\left|\frac{\partial u}{\partial x_2}\right|\right|_{0,4}^4 - k\int_\Omega \cos u(x)\,dx$$

for all t in E^1 and all $u(x)$ in $W_0^{1,4}(\Omega)$.

Finally, condition (I.4) can be weakened to $d \neq 0$ provided that the A_α and $f(t)$ are even periodic functions of t.

Related problems. The existence theory for Problem I also holds if we replace the Dirichlet boundary conditions $(D^\alpha u(t)|_{\partial\Omega} = 0,\ |\alpha| \leqq m-1,\ t \in E^1)$ by more general boundary conditions which correspond to a closed linear subspace V of $W^{m,p}(\Omega)$ which contains $C_0^\infty(\Omega)$. Then W is replaced by $W^{m,p}(\Omega)$ in (I.1)–(I.3) and by V in Definition 3.2 and in the definition of $\mathfrak{F}_1$. However, for $V \neq W$, Ω must satisfy a strong cone condition (see, for example, [**20**]).

4. Problem II. It is also our aim to prove the existence of at least one function $v(t)$ which satisfies

$$\begin{aligned} v_{tt}(t) + A(t)v(t) + dv_t(t) &= f(t), \qquad t \in [0, T), \\ D^\alpha v(t)|_{\partial\Omega} &= 0, \qquad |\alpha| \leqq m-1,\ t \in [0,T), \\ v(0) = v_0, &\qquad v_t(0) = v_1, \end{aligned}$$

where $A(t)$, $t \in [0, T)$, is the family given by (1–2). The above initial-boundary value problem shall be called Problem II and the following conditions are imposed upon it.

(II.1) (This is just condition (I.1) with the restriction of t to $[0, T)$ and without the periodicity requirement.)

Families $B(t)$, $t \in [0, T)$, of admissible lower order operators for Problem II are defined analogously to Definition 3.1 as are the associated families $b(t; u, v)$, $t \in [0, T)$, of Dirichlet forms.

(II.2) The family of forms $a(t; u, v)$, $t \in [0, T)$, associated with $A(t)$, $t \in [0, T)$, satisfies for all $u(x)$ and $v(x)$ in W and all t in $[0, T)$,

$$\operatorname{Re}\{a(t;u,u-v) - a(t;v,u-v) + b(t;u,u-v) - b(t;v,u-v)\} \geqq 0.$$

(II.3) There exist two families of real-valued functions $r(t;\cdot)$, $t \in [0, T)$, and $h(t;\cdot,\cdot\cdot)$, $t \in [0, T)$, such that for all t in $[0, T)$ and all

$$\psi(t) \in C^1([0,T);W),$$

$r(t;\psi(t))$ is once continuously differentiable in t, $h(t;\psi(t),\psi_t(t))$ is continuous in t, and

$$\operatorname{Re} a(t;\psi(t),\psi_t(t)) \geqq \frac{d}{dt} r(t;\psi(t)) - h(t;\psi(t),\psi_t(t)),$$

where there exist constants k_1 and k_2 and continuous real-valued functions $g_2(s)$ and $c_2(s)$ with $\lim_{s\to\infty} c_2(s) = \infty$ such that

(i) For all $u(x)$ in W and all t in $[0, T)$,

$$g_2(||u||_{m,p}) \geqq r(t;u) \geqq c_2(||u||_{m,p})$$

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(ii) For all $u(x)$ and $v(x)$ in W and all t in $[0, T)$,

$$h(t; u, v) \leqq k_1\{c_2(||u||_{m,p}) + ||v||^2\} + k_2.$$

(II.4) d is any real constant.

(II.5) $f(t) \in L^2([0, T); L^2(\Omega))$.

(II.6) $v_0(x) \in W$ and $v_1(x) \in L^2(\Omega)$.

Let $\mathfrak{G}_1$ be the set

$$\mathfrak{G}_1 \equiv \{\Phi(t) | \ \Phi(t) \in L^1([0, T); W), \ \Phi_t(t) \in L^1([0, T); L^2(\Omega)) \text{ and } \Phi(t) \equiv 0 \text{ in a (variable) neighbourhood of } t = T\}.$$

We establish in § 7 that if Problem II satisfies (II.1)–(II.6), then it has at least one weak solution in the following sense.

Definition 4.1. By a weak solution of Problem II we mean a function $v(t)$ with $v(t) \in L^\infty([0, T); W)$ and $v_t(t) \in L^\infty([0, T); L^2(\Omega))$ such that

$$\int_0^T \{-(v_t(t), \Phi_t(t)) + a(t; v(t), \Phi(t)) + d(v_t(t), \Phi(t))\}\, dt = (v_1, \Phi(0)) + \int_0^T (f(t), \Phi(t))\, dt$$

for all $\Phi(t)$ in $\mathfrak{G}_1$ and $v(0) = v_0$ for almost all x in Ω.

Remarks. Many of the statements in § 3 concerning Problem I apply to Problem II as well. However, both condition (II.2) and (II.3) are considerably weaker than the corresponding conditions on Problem I. In particular, (II.3) permits less multilinear structure in the form $a(t; u, v)$. For example, let $A(t)$, $t \in [0, T)$, be given by

$$A(t)u(x) = -\Delta_2 u(x) - \left\{\left(\frac{\partial u(x)}{\partial x_1}\right) + \left(\frac{\partial u(x)}{\partial x_2}\right)\right\} + e^{-t}|u(x)|^2 u(x).$$

Then

$$r(t; u) = \frac{1}{2}\left|\left|\frac{\partial u}{\partial x_1}\right|\right|^2 + \frac{1}{2}\left|\left|\frac{\partial u}{\partial x_2}\right|\right|^2 + \tfrac{1}{4}e^{-t}||u||_{0,4}^4$$

and

$$h(t; u, v) = \left|\left|\frac{\partial u}{\partial x_1}\right|\right|^2 + \left|\left|\frac{\partial u}{\partial x_2}\right|\right|^2 + ||v||^2,$$

and condition (II.3) is satisfied.

Clearly, the existence of a weak solution of Problem II on the interval $[0, T)$ for every finite T implies the corresponding result on the half-line $[0, \infty)$.

5. Some preliminary lemmas.

LEMMA 5.1. *Let $B(t)$, $t \in E^1$, be a family of admissible lower order operators for Problem* I *and let $u^\nu(t)$ (ν an integer index) be a uniformly bounded sequence* *of functions in* $L^\infty(\tau; W)$ *such that* $D^\gamma u^\nu(t) \to D^\gamma u(t)$ *in the strong topology of* $L^p(\tau; L^p(\Omega))$ *and of* $L^p(\Omega \times S_\tau)$ *and almost everywhere on* $\Omega \times S_\tau$ *as* $\nu \to \infty$ *for all* $|\gamma| \leqq m - 1$, *where* S_τ *denotes any interval of periodicity. Then*

$$B_\beta(x, t, u^\nu(t), \ldots, D^{m-1}u^\nu(t)) \to B_\beta(x, t, u(t), \ldots, D^{m-1}u(t))$$

in the weak topology of $L^{p/(p-1)}(\tau; L^{p/(p-1)}(\Omega))$ *and of* $L^{p/(p-1)}(\Omega \times S_\tau)$ *for all* $|\beta| \leqq m - 1$ *as* $\nu \to \infty$.

Proof. Let $\beta = (\beta_1, \ldots, \beta_N)$, $|\beta| \leqq m - 1$, be fixed. By Definition 3.1, $B_\beta(x, t, z_1, \ldots, z_{M'})$ is jointly continuous in $z_1, \ldots, z_{M'}$ for almost every fixed (x, t) on $\Omega \times S_\tau$ and measurable in (x, t) on $\Omega \times S_\tau$ for fixed $z_1, \ldots, z_{M'}$ (see [**11**, p. 69]). We delete those subsets of measure zero of $\Omega \times S_\tau$ for which B_β is not jointly continuous in $z_1, \ldots, z_{M'}$ for fixed (x, t) and for which each of the $D^\gamma u^\nu(t)$, $|\gamma| \leqq m - 1$, does not converge everywhere in $\Omega \times S_\tau$. We denote by $(\Omega \times S_\tau)'$, $\Omega \times S_\tau$ with these subsets removed.

Let $N_1 \leqq N_2 \leqq \ldots \leqq N_k \leqq \ldots$ be an increasing sequence of positive numbers tending to infinity. Let E_k be the set of all (x, t) in $(\Omega \times S_\tau)'$ for which $\nu \geqq N_k$ implies that

$$|B_\beta(x, t, u^\nu(t), \ldots, D^{m-1}u^\nu(t)) - B_\beta(x, t, u(t), \ldots, D^{m-1}u(t))| < \epsilon.$$

Clearly E_k is contained in E_{k+1}. Let $F_k = (\Omega \times S_\tau)' - E_k$. Suppose that (x_0, t_0) is in $\bigcap_{k=1}^\infty F_k$. Then there exists a subsequence ν_k of ν with $\nu_k \geqq N_k$ and

$$|B_\beta(x_0, t_0, u^{\nu_k}(t_0), \ldots, D^{m-1}u^{\nu_k}(t_0)) - B_\beta(x_0, t_0, u(t_0), \ldots, D^{m-1}u(t_0))| \geqq \epsilon$$

for all $k = 1, 2 \ldots$. But this contradicts the continuity hypothesis on the functions B_β. Hence, $\bigcap_{k=1}^\infty F_k = \emptyset$ and it follows that

$$\limsup_{k\to\infty} \mu(F_k) \leqq \mu\left(\limsup_{k\to\infty} F_k\right) = \mu\left(\bigcap_{k=1}^\infty F_k\right) = 0.$$

That is,

$$B_\beta(x, t, u^\nu(t), \ldots, D^{m-1}u^\nu(t)) \to B_\beta(x, t, u(t), \ldots, D^{m-1}u(t))$$

in measure on $\Omega \times S_\tau$ as $\nu \to \infty$.

Now, since the $u^\nu(t)$ are uniformly bounded in $L^\infty(\tau; W)$ and the function $g_1(s)$ in Definition 3.1 is continuous in s, $g_1(||u^\nu(t)||_{m,p}) \leqq K_1$ and $g_1(||u(t)||_{m,p}) \leqq K_1$ for some fixed constant K_1. Let $w(t)$ be any element of $L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$. Setting

$$g^\nu(t) = K_1\left\{\sum_{|\gamma|\leqq m-1} |D^\gamma u^\nu(t)|^{p-1} + 1\right\},$$

$$g(t) = K_1\left\{\sum_{|\gamma|\leqq m-1} |D^\gamma u(t)|^{p-1} + 1\right\},$$

$$h_\beta^\nu(t) = B_\beta(x, t, u^\nu(t), \ldots, D^{m-1}u^\nu(t))/g^\nu(t),$$

and

$$h_\beta(t) = B_\beta(x, t, u(t), \ldots, D^{m-1}u(t))/g(t),$$

we have by hypothesis that $g^\nu(t) \to g(t)$ strongly in $L^{p/(p-1)}(\tau; L^{p/(p-1)}(\Omega))$

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and in $L^{p/(p-1)}(\Omega \times S_\tau)$ as $\nu \to \infty$ and by the Lebesgue Dominated Convergence Theorem that $\bar{h}_\beta{}^\nu(t)w(t) \to \bar{h}_\beta(t)w(t)$ strongly in $L^p(\tau; L^p(\Omega))$ and in $L^p(\Omega \times S_\tau)$ as $\nu \to \infty$. Hence

$$\int \langle B_\beta(x, t, u^\nu(t), \ldots, D^{m-1}u^\nu(t)), w(t)\rangle\, dt = \int \langle g^\nu(t), \bar{h}_\beta{}^\nu(t)w(t)\rangle\, dt$$
$$\to \int \langle g(t), \bar{h}_\beta(t)w(t)\rangle\, dt = \int \langle B_\beta(x, t, u(t), \ldots, D^{m-1}u(t)), w(t)\rangle\, dt$$

as $\nu \to \infty$ for all $w(t) \in L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$. This proves the lemma.

LEMMA 5.2. *Let $A(t)$, $t \in E^1$, be the family of partial differential operators given by* (1–2) *and let the functions A_α satisfy* (I.1). *Let $u(t)$ and $w(t)$ be any two functions in $L^\infty(\tau; W)$ with $D^\gamma u(t)$ and $D^\gamma w(t)$ in $L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$ for every γ with $|\gamma| \leqq m$. Let π be a bounded sequence of real numbers tending to zero. Then there exists a subsequence ρ of π such that for every fixed α with $|\alpha| \leqq m$ and every fixed β with $|\beta| \leqq m - 1$, we have:*

$$A_\alpha(x, t, u(t) + \rho w(t), \ldots, D^m(u(t) + \rho w(t))) \to A_\alpha(x, t, u(t), \ldots, D^m u(t))$$

and

$$B_\beta(x, t, u(t) + \rho w(t), \ldots, D^{m-1}(u(t) + \rho w(t)))$$
$$\to B_\beta(x, t, u(t), \ldots, D^{m-1}u(t))$$

weakly in $L^{p/(p-1)}(\tau; L^{p/(p-1)}(\Omega))$ and in $L^{p/(p-1)}(\Omega \times S_\tau)$ as $\rho \to 0$.

Proof. By hypothesis, $D^\gamma(u(t) + \pi w(t)) \to D^\gamma u(t)$ strongly in $L^p(\tau; L^p(\Omega))$ and in $L^p(\Omega \times S_\tau)$ as $\pi \to 0$ for all γ with $|\gamma| \leqq m$. Hence, there exists a subsequence ρ, obtained after M refinements of π, such that each

$$D^\gamma(u(t) + \rho w(t)), \qquad |\gamma| \leqq m,$$

converges almost everywhere on $\Omega \times S_\tau$ to $D^\gamma u(t)$ as $\rho \to 0$. The proof follows as in Lemma 5.1.

Clearly, lemmas analogous to Lemmas 5.1 and 5.2 for Problem I can be proved for Problem II.

6. The existence theory for Problem I. It shall be assumed throughout this section that conditions (I.1)–(I.5) are satisfied. Since W is separable, we can choose a countable set of distinct basis elements $\{w_j\}$, $j = 1, 2, \ldots$, which generate this space. Let P_n be the projection in $L^2(\Omega)$ onto the subspace $\{w_1, \ldots, w_n\}$ generated by the distinct basis elements $w_1, \ldots, w_n$. For each positive integer n, we denote by $u^n(t)$ the solution of the system of non-linear ordinary differential equations

$$(6\text{–}1) \quad (u_{tt}{}^n(t), w_j) + a(t; u^n(t), w_j) + d(u_t{}^n(t), w_j) = (f(t), w_j),$$
$$t \in E^1, u^n(t) \in P_n L^2(\Omega) \text{ for all } t \text{ in } E^1, j = 1, \ldots, n.$$

Since the w_j, $j = 1, \ldots, n$, are linearly independent for each fixed n, $u^n(t)$, if it exists, must be of the form

$$u^n(t) = \sum_{k=1}^{n} c_{nk}(t) w_k.$$

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We consider the following system which depends on the parameter λ in $[0, 1]$,

$$(u_{tt}^n(t, \lambda), w_j) + \lambda\{a(t; u^n(t, \lambda), w_j) - \tilde{k}(u^n(t, \lambda), w_j)\}$$
$$+ d(u_t^n(t, \lambda), w_j) + \tilde{k}(u^n(t, \lambda), w_j) = (f(t), w_j),\ t \in E^1,\ j = 1, \ldots, n,$$

where $u^n(t, \lambda) = \sum_{k=1}^n c_{nk}(t, \lambda) w_k$ and $\tilde{k}$ is a fixed constant satisfying $0 < \tilde{k} < k_0$ with k_0 as in condition (I.2) (i). For $\lambda = 0$, this system is linear and has a unique periodic solution $u^n(t, 0)$ periodic in t of period τ. For $\lambda = 1$, it becomes (6–1). Using the continuity and periodicity properties of the functions A_α together with the theory of Green's functions, the Leray-Schauder degree theory can be employed to prove for each fixed n the existence of at least one function $u^n(t, 1) = u^n(t)$ with $u_{tt}^n(t) \in L^2(\tau; W)$ which satisfies (6–1) almost everywhere on E^1 and for which each $c_{nk}(t)$ has period τ and is once continuously differentiable on E^1. The required *a priori* estimates independent of λ are obtained in the usual way with the application of (I.3) and (I.2) (i).

It remains to prove that the solutions $u^n(t)$ of (6–1) for $n = 1, 2, \ldots$, or a subsequence $u^\nu(t)$ of $u^n(t)$, satisfy

$$\int (u_t^\nu(t), \Phi_t(t))\, dt \to \int (u_t(t), \Phi_t(t))\, dt$$

and

$$\int a(t; u^\nu(t), \Phi(t))\, dt \to \int a(t; u(t), \Phi(t))\, dt$$

for all $\Phi(t)$ in $\mathfrak{F}_1$ as $\nu \to \infty$ and for some $u(t) \in L^\infty(\tau; W)$ with $u_t(t) \in L^\infty(\tau; L^2(\Omega))$.

Let G be defined by $\langle Gu, v\rangle \equiv \int a(t; u(t), v(t))\, dt$. Then it follows from (I.1) that G is a unique, well-defined mapping from $L^\infty(\tau; W)$ into $L^\infty(\tau; W^*)$ for all $v(t)$ in $L^1(\tau; W)$. Let $B(t)$, $t \in E^1$, be a family of admissible lower order operators for Problem I and let $b(t; u, v)$ be the associated family of forms. Then C defined by $\langle Cu, v\rangle \equiv \int b(t; u(t), v(t))\, dt$ is also a unique, well-defined mapping from $L^\infty(\tau; W)$ into $L^\infty(\tau; W^*)$ for all $v(t)$ in $L^1(\tau; W)$. In order to accomplish the above passage to the limit, we require the following lemma.

LEMMA 6.1 *Let $u^n(t)$ be the sequence of solutions of* (6–1) *for $n = 1, 2, \ldots$, and let S_τ denote any interval of periodicity. There exist a subsequence $u^\nu(t)$ of $u^n(t)$ and elements $u(t)$ and $G_0(t)$ with $u(t) \in L^\infty(\tau; W)$, $u_t(t) \in L^\infty(\tau; L^2(\Omega))$, $D^\gamma u(t) \in L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$ for all γ with $|\gamma| \leqq m$ and*

$$G_0(t) \in L^\infty(\tau; W^*)$$

such that, as $\nu \to \infty$,

$u^\nu(t) \to u(t)$ *in the weak* topology of* $(L^1(\tau; W^*))^*$,

$u_t^\nu(t) \to u_t(t)$ *in the weak* topology of* $(L^1(\tau; L^2(\Omega)))^*$,

$D^\gamma u^\nu(t) \to D^\gamma u(t)$ *in the weak topology of $L^p(\tau; L^p(\Omega))$ and of $L^p(\Omega \times S_\tau)$ for all $|\gamma| \leqq m$, in the strong topology of $L^p(\tau; L^p(\Omega))$ and of $L^p(\Omega \times S_\tau)$ for all $|\gamma| \leqq m - 1$, and almost everywhere on $\Omega \times S_\tau$ for all $|\gamma| \leqq m - 1$,* *and*

$Gu^\nu(t) \to G_0(t)$ *in the weak* topology of* $(L^1(\tau; W))^*$.

Proof. Multiplying (6–1) by

$$\overline{c_{nj}'(t)} \equiv \frac{d}{dt}\overline{c_{nj}(t)},$$

summing over $j = 1, \ldots, n$, taking the real part of both sides, and integrating over any interval of periodicity gives by the Schwarz inequality

$$\int \operatorname{Re} a(t; u^n(t), u_t^n(t))\, dt + d\int ||u_t^n(t)||^2\, dt \leqq \left(\int ||f(t)||^2\, dt\right)^{1/2}\left(\int ||u_t^n(t)||^2\, dt\right)^{1/2}$$

and since $\int \operatorname{Re} a(t; u^n(t), u_t^n(t))\, dt \geqq 0$ by (I.3) and $d > 0$ by (I.4),

$$\int ||u_t^n(t)||^2\, dt \leqq K_2, \tag{6–2}$$

where K_2 is a constant independent of n. Multiplying (6–1) by $\overline{c_{nj}(t)}$, summing over $j = 1, \ldots, n$, taking the real part of both sides, integrating over any interval of periodicity and integrating the first term by parts yields

$$-\int ||u_t^n(t)||^2\, dt + \int \operatorname{Re} a(t; u^n(t), u^n(t))\, dt \leqq \left(\int ||f(t)||^2\, dt\right)^{1/2}\left(\int ||u^n(t)||^2\, dt\right)^{1/2}$$

and, by (I.2) (i) and (6–2),

$$\int ||u^n(t)||^2\, dt \leqq K_3 \tag{6–3}$$

and

$$\int \operatorname{Re} a(t; u^n(t), u^n(t))\, dt \leqq K_4, \tag{6–4}$$

where K_3 and K_4 are independent of n. Finally, multiplying (6–1) by $\overline{c_{nj}'(t)}$, summing over $j = 1, \ldots, n$, taking the real part of both sides, and integrating over $[t_0, t_1]$, $t_0 < t_1$, yields by (I.3) and (6–2)

$$\tfrac{1}{2}||u_t^n(t_1)||^2 + r(t_1; u^n(t_1)) \leqq K_5 + \tfrac{1}{2}||u_t^n(t_0)||^2 + r(t_0; u^n(t_0)),$$

where K_5 is independent of n. Integrating this expression with respect to t_0 over the interval of periodicity $[t_1 - \tau, t_1]$ yields by (I.3), (6–2), and (6–4),

$$||u_t^n(t)|| + ||u^n(t)||_{m,p} \leqq K_6 \tag{6–5}$$

independent of n. It follows that

$$\int_{\Omega\times S_\tau} |D^\gamma u^n(t)|^p\, dxdt \leqq K_7 \tag{6–6}$$

for every γ with $|\gamma| \leqq m$ and

$$||Gu^n||_{L^\infty(\tau; W^*)} = \sup_{||v||_{L^1(\tau;W)}\leqq 1} |\langle Gu^n, v\rangle| \leqq K_8 \tag{6–7}$$

for constants K_7 and K_8 independent of n.

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The desired subsequence is obtained after several refinements of the original sequence $u^n(t)$. However, for brevity in notation all subsequences shall be denoted by $u^\nu(t)$. By (6–5), (6–6), and (6–7) there exist a subsequence $u^\nu(t)$ of $u^n(t)$ and elements $u(t) \in L^\infty(\tau; W)$, $\tilde{u}(t) \in L^\infty(\tau; L^2(\Omega))$,

$$u_\gamma(t) \in L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$$

with $|\gamma| \leqq m$ and $G_0(t) \in L^\infty(\tau; W^*)$ such that

$u^\nu(t) \to u(t)$ in the weak* topology of $(L^1(\tau; W^*))^*$,
$u_t{}^\nu(t) \to \tilde{u}(t)$ in the weak* topology of $(L^1(\tau; L^2(\Omega)))^*$,
$D^\gamma u^\nu(t) \to u_\gamma(t)$ in the weak topology of $L^p(\tau; L^p(\Omega))$ and of $L^p(\Omega \times S_\tau)$ for all $|\gamma| \leqq m$,

and

$Gu^\nu(t) \to G_0(t)$ in the weak* topology of $(L^1(\tau; W))^*$

as $\nu \to \infty$. It follows easily that $\tilde{u}(t) = u_t(t)$ and $u_\gamma(t) = D^\gamma u(t)$ for each $|\gamma| \leqq m$, where the derivatives are considered in the sense of distributions. In addition, since $u^\nu(t) \to u(t)$ and $u_t{}^\nu(t) \to u_t(t)$ in the indicated weak* topologies as $\nu \to \infty$, there exists (see [**16**, p. 60]) a further subsequence $u^\nu(t)$ such that

(6–8) $u^\nu(t) \to u(t)$ in the strong topology of $L^2(\tau; L^2(\Omega))$ and of $L^2(\Omega \times S_\tau)$

as $\nu \to \infty$. Since the embedding of $W \equiv W_0{}^{m,p}(\Omega)$ into $W_0{}^{m-1,p}(\Omega)$ is compact, given $\epsilon > 0$ there exists (again see [**16**, p. 59]) a constant c_ϵ such that

$$||u||_{m-1,p} \leqq \epsilon||u||_{m,p} + c_\epsilon||u||$$

for all $u(x)$ in W provided $p \geqq 2$. Thus, for every $\epsilon > 0$, there exists c_ϵ such that

$$\int ||u^\nu(t) - u(t)||_{m-1,p}{}^p\, dt \leqq K_9\epsilon^p \int ||u^\nu(t) - u(t)||_{m,p}{}^p\, dt + K_9 c_\epsilon \int ||u^\nu(t) - u(t)||^p\, dt,$$

where K_9 depends only on p. Let $\epsilon_0 > 0$ be given. Then by (6–5) and (6–6) we can choose $\epsilon > 0$ so small that

$$\int ||u^\nu(t) - u(t)||_{m-1,p}{}^p\, dt < \tfrac{1}{2}\epsilon_0 + K_{10} c_\epsilon \int ||u^\nu(t) - u(t)||^2\, dt,$$

where K_{10} depends only on p and K_6. This implies by (6–8) that $D^\gamma u^\nu(t) \to D^\gamma u(t)$ strongly in $L^p(\tau; L^p(\Omega))$ and $L^p(\Omega \times S_\tau)$ for all $|\gamma| \leqq m - 1$ as $\nu \to \infty$. The convergence almost everywhere on $\Omega \times S_\tau$ is obtained after M' refinements of this $u^\nu(t)$. This proves the lemma.

We now set $n = \nu$ in (6–1) and pass to the limit.

THEOREM 1. *There exists at least one weak solution of Problem* I *in the sense of Definition* 3.2.

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Proof. Setting $n = \nu$ in (6–1) we have:

$$(u_{tt}^{\nu}(t), w_j) + a(t; u^{\nu}(t), w_j) + d(u_t^{\nu}(t), w_j) = (f(t), w_j)$$

is valid for every $j = 1, \ldots, \nu$ and for almost all t in E^1. Thus

$$\text{(6–9)} \quad -\int (u_t^{\nu}(t), \psi_t(t))\, dt + \int a(t; u^{\nu}(t), \psi(t))\, dt + d\int (u_t^{\nu}(t), \psi(t))\, dt = \int (f(t), \psi(t))\, dt$$

for all

$$\psi(t) = \sum_{k=1}^{l} \eta_k(t) w_k,$$

where each $\eta_k(t)$, $k = 1, \ldots, l$, has period τ and is once continuously differentiable on E^1 and $l \leqq \nu$. Since these $\psi(t)$ are dense in $\mathfrak{F}_1$ for l arbitrary, taking the limit in (6–9) yields by Lemma 6.1,

$$-\int (u_t(t), \Phi_t(t))\, dt + \langle G_0, \Phi\rangle + d\int (u_t(t), \Phi(t))\, dt = \int (f(t), \Phi(t))\, dt$$

for all $\Phi(t)$ in $\mathfrak{F}_1$. Hence, to show that $u(t)$ is a weak solution of Problem I, it is sufficient to show that $G_0(t) = Gu(t)$.

Substituting $u(t)$ for $\Phi(t)$ in the above equation and taking the real part of both sides yields

$$\text{(6–10)} \qquad -\int ||u_t(t)||^2\, dt + \operatorname{Re}\langle G_0, u\rangle = \operatorname{Re}\int (f(t), u(t))\, dt,$$

where $\operatorname{Re}\int (u_t(t), u(t))\, dt = 0$ since $u(t)$ is equal almost everywhere to a continuous function from E^1 to $L^2(\Omega)$ (see [**25**]) so that, in particular, the integration by parts formula

$$\int_{t_0}^{t_1} \{(u_t(t), u(t)) + (u(t), u_t(t))\}\, dt = ||u(t_1)||^2 - ||u(t_0)||^2$$

is valid. Now if we multiply (6–1) by $\overline{c_{nj}(t)}$, sum over $j = 1, \ldots, n$, take the real part of both sides, integrate over any interval of periodicity, integrate the first term by parts, set $n = \nu$, and take the limit inferior of both sides as $\nu \to \infty$, we have

$$\text{(6–11)} \qquad -\int ||u_t(t)||^2\, dt + \liminf_{\nu\to\infty} \operatorname{Re}\langle Gu^{\nu}, u^{\nu}\rangle \leqq \operatorname{Re}\int (f(t), u(t))\, dt.$$

Comparing (6–10) and (6–11), we can conclude that

$$\text{(6–12)} \qquad \operatorname{Re}\langle G_0, u\rangle \geqq \liminf_{\nu\to\infty} \operatorname{Re}\langle Gu^{\nu}, u^{\nu}\rangle.$$

Hence, if $\theta(t)$ is an arbitrary element in $L^{\infty}(\tau; W)$ with

$$D^{\alpha}\theta(t) \in L^p(\tau; L^p(\Omega)) \cap L^p(\Omega \times S_\tau)$$

for every fixed α with $|\alpha| \leqq m$, then by (6–12), Lemma 5.1, and (I.2) (ii),

$$\operatorname{Re}\{\langle G_0 - G\theta, u - \theta\rangle + \langle Cu - C\theta, u - \theta\rangle\} \geqq \liminf_{\nu\to\infty} \operatorname{Re}\{\langle Gu^{\nu} - G\theta, u^{\nu} - \theta\rangle + \langle Cu^{\nu} - C\theta, u^{\nu} - \theta\rangle\} \geqq 0,$$

where C is the mapping associated with that $B(t)$ in (I.2) (ii). Finally, we set $\theta(t) = u(t) - \pi\theta_1(t)$ where $\pi \in (0, 1]$. Then

$$\operatorname{Re}\{\langle G_0 - G(u - \pi\theta_1), \theta_1\rangle + \langle Cu - C(u - \pi\theta_1), \theta_1\rangle\} \geqq 0.$$

Let π tend to zero and let ρ be the subsequence of π in Lemma 5.2. Then

$$\operatorname{Re}\langle G_0 - Gu, \theta_1\rangle \geqq 0$$

and since $\theta(t)$ was arbitrary, so is $\theta_1(t)$ and $G_0(t) = Gu(t)$ almost everywhere on E^1. This proves the theorem.

7. The existence theory for Problem II. Let conditions (II.1)–(II.6) be satisfied. Let $\{w_j\}$, $j = 1, 2, \ldots$, be a basis of W and let P_n be the projection in $L^2(\Omega)$ onto the subspace generated by the distinct basis elements $w_1, \ldots, w_n$. Denoting by $v^n(t)$ the solution of the system

$$(7\text{–}1)\quad (v_{tt}^n(t), w_j) + a(t; v^n(t), w_j) + d(v_t^n(t), w_j) = (f(t), w_j),\qquad t \in [0, T),\ j = 1, \ldots, n,$$
$$v^n(t) \in P_nL^2(\Omega) \text{ for all } t \text{ in } [0, T) \text{ and } v^n(0) = P_nv_0,\ v_t^n(0) = P_nv_1,$$

we have $v^n(t) = \sum_{k=1}^n c_{nk}(t)w_k$, $\sum_{k=1}^n c_{nk}(0)w_k = P_nv_0$, and $\sum_{k=1}^n c_{nk}'(0)w_k = P_nv_1$.

It is well known from the theory of ordinary differential equations that for each positive integer n there exist, provided (II.1) is satisfied, functions $c_{nk}(t)$, $k = 1, \ldots, n$, with $c_{nk}(0)$ and $c_{nk}'(0)$ equal to arbitrary preassigned constants such that $v^n(t) = \sum_{k=1}^n c_{nk}(t)w_k$ satisfies (7–1) almost everywhere on $[0, T_n)$ for some T_n with $0 < T_n \leqq T$. As the following *a priori* estimates show, each interval of existence $[0, T_n)$ can be taken to be $[0, T)$.

Multiplying (7–1) by $\overline{c_{nj}'(t)}$, summing over $j = 1, \ldots, n$, taking the real part of both sides, and integrating from 0 to t yields

$$(1/2)\{||v_t^n(t)||^2 - ||v_t^n(0)||^2\} + \int_0^t \operatorname{Re} a(s; v^n(s), v_s^n(s))\, ds + d\int_0^t ||v_s^n(s)||^2\, ds = \operatorname{Re}\int_0^t (f(s), v_s^n(s))\, ds.$$

This becomes by (II.3) and the Schwarz inequality

$$(1/2)||v_t^n(t)||^2 + c_2(||v^n(t)||_{m,p}) \leqq (1/2)||v_t^n(0)||^2 + g_2(||v^n(0)||_{m,p}) - d\int_0^t ||v_s^n(s)||^2\, ds + k_1\int_0^t \{c_2(||v^n(s)||_{m,p}) + ||v_s^n(s)||^2\}\, ds + tk_2 + \int_0^t ||f(s)||^2\, ds + \int_0^t ||v_s^n(s)||^2\, ds.$$

By construction, $||v^n(0)||_{m,p} \leqq ||v_0||_{m,p}$ and $||v_t^n(0)|| \leqq ||v_1||$ for every n. In addition, since g_2 is continuous and $f(t)$ is a fixed element in $L^2([0, T); L^2(\Omega))$,

$$||v_t^n(t)||^2 + 2c_2(||v^n(t)||_{m,p}) \leqq K_{11} + K_{12}\int_0^t \{||v_s^n(s)||^2 + 2c_2(||v^n(s)||_{m,p})\}\, ds,$$

where K_{11} and K_{12} are constants independent of n and t. Thus, since c_2 is continuous with $\lim_{s\to\infty} c_2(s) = \infty$, there exists a constant K_{13} independent of n and of $t \in [0, T)$ such that

$$||v_t^n(t)|| + ||v^n(t)||_{m,p} \leqq K_{13} \tag{7–2}$$

and it follows that each T_n can be taken to be T.

In order to establish the existence of a weak solution of Problem II in the sense of Definition 4.1, it is sufficient to show that a subsequence $v^\nu(t)$ of $v^n(t)$ satisfies

$$\int_0^T (v_t^\nu(t), \Phi_t(t))\, dt \to \int_0^T (v_t(t), \Phi_t(t))\, dt$$

and

$$\int_0^T a(t; v^\nu(t), \Phi(t))\, dt \to \int_0^T a(t; v(t), \Phi(t))\, dt$$

for all $\Phi(t)$ in $\mathfrak{G}_1$ as $\nu \to \infty$ and that $v(0) = v_0$ almost everywhere on Ω.

G and C defined by

$$\langle Gv, w\rangle \equiv \int_0^T a(t; v(t), w(t))\, dt \text{ and } \langle Cv, w\rangle \equiv \int_0^T b(t; v(t), w(t))\, dt$$

are unique, well-defined mappings from $L^\infty([0, T); W)$ into $L^\infty([0, T); W^*)$ for all $w(t) \in L^1([0, T), W)$ where $b(t; u, v)$, $t \in [0, T)$, is the family of forms associated with a family $B(t)$, $t \in [0, T)$, of admissible lower order operators for Problem II. Just as in Lemma 6.1, we have by (7–2) that there exists a subsequence $v^\nu(t)$ such that as $\nu \to \infty$,

$v^\nu(t) \to v(t)$ in the weak* topology of $(L^1([0, T); W^*))^*$,
$v_t^\nu(t) \to v_t(t)$ in the weak* topology of $(L^1([0, T); L^2(\Omega)))^*$,

$D^\gamma v^\nu(t) \to D^\gamma v(t)$ weakly in $L^p(\Omega \times [0, T))$ for all $|\gamma| \leqq m$, strongly in $L^p(\Omega \times [0, T))$ for all $|\gamma| \leqq m - 1$, and almost everywhere on $\Omega \times [0, T)$ for all $|\gamma| \leqq m - 1$,

and

$Gv^\nu(t) \to G_0(t)$ in the weak* topology of $(L^1([0, T); W))^*$.

We are now in a position to pass to the limit in (7–1). Setting $n = \nu$ we have:

$$(v_{tt}^\nu(t), w_j) + a(t; v^\nu(t), w_j) + d(v_t^\nu(t), w_j) = (f(t), w_j)$$

is valid for every $j = 1, \ldots, \nu$ and for almost all $t \in [0, T)$. Since $v_t^\nu(0) \to v_t(0)$ weakly in $L^2(\Omega)$ while $v_t^\nu(0) \to v_1$ strongly in $L^2(\Omega)$ as $\nu \to \infty$, we have:

$$-\int_0^T (v_t(t), \Phi_t(t))\, dt + \langle G_0, \Phi\rangle + d\int_0^T (v_t(t), \Phi(t))\, dt$$
$$= (v_1, \Phi(0)) + \int_0^T (f(t), \Phi(t))\, dt$$

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for all $\Phi(t) \in \mathfrak{G}_1$ and setting $\Phi(t) = v(t)$ and taking the real part of both sides yields

$$(7\text{–}3)\quad -\int_0^T ||v_t(t)||^2\,dt + \operatorname{Re}\langle G_0, v\rangle + \tfrac{1}{2}d||v(T)||^2 + ||v_t(T)||^2 = \tfrac{1}{2}d||v_0||^2 + ||v_1||^2 + \operatorname{Re}\int_0^T (f(t), v(t))\,dt,$$

where the various integration by parts are justified as in Theorem 1. Now if we multiply (7–1) by $\overline{c_{nj}(t)}$, sum over $j = 1, \ldots, n$, take the real part of both sides, integrate over $[0, T)$, integrate the first term by parts, set $n = \nu$, and take the limit inferior of both sides as $\nu \to \infty$, we have, recalling that $v_t^\nu(0) \to v_1$ and $v^\nu(0) \to v_0$ strongly in $L^2(\Omega)$,

$$(7\text{–}4)\quad -\int_0^T ||v_t(t)||^2\,dt + \liminf_{\nu\to\infty} \operatorname{Re}\langle Gv^\nu, v^\nu\rangle + \tfrac{1}{2}d||v(T)||^2 + ||v_t(T)||^2 \leqq \tfrac{1}{2}d||v_0||^2 + ||v_1||^2 + \operatorname{Re}\int_0^T (f(t), v(t))\,dt.$$

Comparing (7–3) and (7–4) we can conclude that

$$\operatorname{Re}\langle G_0, v\rangle \geqq \liminf_{\nu\to\infty} \operatorname{Re}\langle Gv^\nu, v^\nu\rangle$$

and just as in § 6 we can show using lemmas analogous to Lemmas 5.1 and 5.2 that $G_0(t) = Gv(t)$. Thus we have proved the following theorem.

THEOREM 2. *There exists at least one weak solution of Problem* II *in the sense of Definition* 4.1.

8. Some examples. *Problem* I. Several specific examples of families $A(t)$, $t \in E^1$, in Problem I have already been mentioned in § 3. These are

$$A(t)u(x) = -\Delta_2 u(x) + |u(x)|^\gamma u(x) \qquad (\gamma \geqq 0)$$

and

$$A(t)u(x) = -\frac{\partial}{\partial x_1}\left(\frac{\partial u(x)}{\partial x_1}\right)^3 - \frac{\partial}{\partial x_2}\left(\frac{\partial u(x)}{\partial x_2}\right) + k \sin u(x) \quad (u(x) \text{ real-valued})$$

on Ω, where Ω is a bounded domain in E^2, (for simplicity, most of the examples mentioned are restricted to one and two space dimensions). A very simple example is the family

$$A(t)u(x) = -\frac{\partial^2 u(x)}{\partial x^2} + ku(x) + \epsilon u^3(x)$$

for the non-linear, one-dimensional wave equation

$$u_{tt}(t) - \frac{\partial^2 u(t)}{\partial x^2} + ku(t) + \epsilon u^3(t) + du_t(t) = f(t)$$

considered by Ficken and Fleishman [**9**]. Here ϵ is a small parameter and k is a positive constant.

The verification that (I.1)–(I.3) (or these conditions with W replaced by $W^{m,p}(\Omega)$ in the case of more general boundary conditions) are satisfied for the above examples is relatively easy with the aid of the following lemma.

LEMMA. *The function* $\theta(z) = |z|^{\rho-1}z$, $\rho \geqq 1$, *satisfies*

(i) $\operatorname{Re}\{\theta(z)\bar{z}\} \geqq c_1|z|^{\rho+1}$,

(ii) $|\theta(z)| \leqq c_2|z|^{\rho}$,

(iii) $\operatorname{Re}\{\theta_z'(\zeta)\bar{\zeta}\} \geqq c_3|z|^{\rho-1}|\zeta|^2$,

(iv) $|\theta_z'(\zeta)| \leqq c_4|z|^{\rho-1}|\zeta|$,

where c_1, c_2, c_3, *and* c_4 *are positive constants and* $\theta_z'(\zeta)$ *denotes the derivative of* θ *at the point* z *in the direction* ζ.

Proof. The proof of the above lemma can be found in [**17**, p. 54].

Problem II. Since (I.1)–(I.3) restricted to $t \in [0, T)$ for all finite T are far stronger than the corresponding conditions on Problem II, the examples of $A(t)$ for Problem I restricted to t in $[0, T)$ are also valid examples of $A(t)$ in Problem II. Some additional examples of $A(t)$ in Problem II are

$$A(t)u(x) = \Delta_2^{\,2}u(x) - k\left\{\left(\frac{\partial u(x)}{\partial x_1}\right)^2 + \left(\frac{\partial u(x)}{\partial x_2}\right)^2\right\}^{1/2}$$

and

$$A(t)u(x) = -\frac{\partial}{\partial x_1}\left(\left|\frac{\partial u(x)}{\partial x_1}\right|\frac{\partial u(x)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\left|\frac{\partial u(x)}{\partial x_2}\right|\frac{\partial u(x)}{\partial x_2}\right) + e^{-t}|u(x)|^3u(x)$$

on Ω, where again Ω is a bounded domain in E^2 and k is some constant.

The equation

$$v_{tt}(t) - \Delta_2 v(t) + e^{v(t)} = 0, \qquad t \geqq 0,$$

describes the vibrations of a uniformly charged plasma, such as an ionized gas or an electron gas in a vacuum tube [**13**], and equations of the form

$$v_{tt}(t) - \left(1 + \left(\frac{\partial v(t)}{\partial x}\right)^2\right)\frac{\partial^2 v(t)}{\partial x^2} = 0, \qquad t \geqq 0,$$

arise in the study of the propagation of high intensity sound waves in a fluid (macrosonics) [**15**]. Corresponding to these equations are the families

$$A(t)u(x) = -\Delta_2 u(x) + e^{u(x)}$$

and

$$A(t)u(x) = -\left(1 + \left(\frac{\partial u(x)}{\partial x}\right)^2\right)\frac{\partial^2 u(x)}{\partial x^2},$$

respectively for Problem II.

The verification that (II.1)–(II.3) are satisfied for these examples of $A(t)$ is left to the reader.

An initial-boundary value problem associated with Problem II involves the general non-linear Euler-Poisson-Darboux equation

$$v_{tt}(t) + A(t)v(t) + (d/t)v_t(t) = f(t),$$

where $A(t)$ is, as usual, given by (1–2). With only slight modification, the existence results for Problem II in § 7 are valid for initial-boundary value problems for the above mildly singular equation. For work done on the linear and semi-linear Euler-Poisson-Darboux equation, the reader is referred to [**8**; **14**].

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