The maximal injective crossed product

ALCIDES BUSS†, SIEGFRIED ECHTERHOFF‡ and RUFUS WILLETT§

† *Departamento de Matemática, Universidade Federal de Santa Catarina,*
88.040-900 Florianópolis-SC, Brazil
(e-mail: alcides@mtm.ufsc.br)
‡ *Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62,*
48149 Münster, Germany
(e-mail: echters@uni-muenster.de)
§ *Mathematics Department, University of Hawaiʻi at Mānoa, Keller 401A,*
2565 McCarthy Mall, Honolulu, HI 96822, USA
(e-mail: rufus@math.hawaii.edu)

Abstract. A crossed product functor is said to be *injective* if it takes injective morphisms to injective morphisms. In this paper we show that every locally compact group G admits a maximal injective crossed product $A \mapsto A \rtimes_{\text{inj}} G$. Moreover, we give an explicit construction of this functor that depends only on the maximal crossed product and the existence of G-injective C^*-algebras; this is a sort of 'dual' result to the construction of the minimal exact crossed product functor, the latter having been studied for its relationship to the Baum–Connes conjecture. It turns out that $\rtimes_{\text{inj}}$ has interesting connections to exactness, the local lifting property, amenable traces, and the weak expectation property.

1. *Introduction*

Let G be a a locally compact group. A C^*-algebra equipped with a continuous action of G will be called a G-*algebra*. An (exotic) crossed product functor $\rtimes_\mu$ for G is a functor $A \mapsto A \rtimes_\mu G$ from the category of G-algebras and equivariant $*$-homomorphisms into the category of C^*-algebras and $*$-homomorphisms in which $A \rtimes_\mu G$ is a C^*-completion of the algebraic crossed product $A \rtimes_{\text{alg}} G = C_c(G, A)$ with respect to a C^*-norm $\| \cdot \|_\mu$ which satisfies

$$\| \cdot \|_r \leq \| \cdot \|_\mu \leq \| \cdot \|_{\max},$$

where $\| \cdot \|_r$ and $\| \cdot \|_{\max}$ denote the norms of the reduced and maximal crossed products, respectively. Thus, we see that for any exotic crossed product, the identity map on $C_c(G, A)$ induces surjections

$$A \rtimes_{\max} G \twoheadrightarrow A \rtimes_\mu G \twoheadrightarrow A \rtimes_r G.$$

Recently, the study of exotic crossed products has become a focus of research not only because of interesting connections to the Baum–Connes conjecture, as revealed in [3, 11, 12], but also because of the fact that exotic crossed products and group algebras provide interesting new examples of C^*-algebras attached to locally compact groups and their actions (see, e.g., [8, 18, 25, 27, 28]).

A crossed product functor $A \mapsto A \rtimes_\mu G$ is said to be *injective* if it takes injective morphisms to injective morphisms. Our goal in this paper is to show that there is always a maximal injective crossed product $\rtimes_{\inj}$; this is a sort of 'dual' result to the existence of a minimal exact crossed product functor that has been studied for its relationship to the Baum–Connes conjecture (we refer to [13] for the most recent results on this functor). There are no applications given here to Baum–Connes, but it turns out that $\rtimes_{\inj}$ has some interesting connections to exactness, the local lifting property (LLP), amenable traces, and the weak expectation property (WEP), as well as G-injective algebras generally; our goal is to elucidate these. We hope these show that $\rtimes_{\inj}$ is a natural object.

After this introduction, we start with a preliminary section giving a self-contained introduction to G-injective C^*-algebras, which were first studied by Hamana in [17]. The main fact we need in this paper is the observation that every G-algebra embeds equivariantly into a G-injective one (see Corollary 2.5 below). The construction of the maximal injective crossed product $A \rtimes_{\inj} G$ is given in §3. Our results show that it can be described as the completion of the algebraic crossed product $A \rtimes_{\alg} G$ inside the maximal crossed product $B \rtimes_{\max} G$ if $A \hookrightarrow B$ is any G-equivariant embedding of A into a G-injective C^*-algebra B. We show that this construction gives an injective crossed product functor which is maximal among all injective crossed product functors for G. It follows from this that $A \rtimes_{\inj} G = A \rtimes_{\max} G$ for every G-injective algebra A. In fact, in §3 we show that a similar statement holds for all G-algebras which satisfy a G-equivariant version of Lance's WEP (see Proposition 3.12).

In §4, we study some connections between properties of the injective crossed product $\rtimes_{\inj}$, exactness of G, and the LLP of $C^*_{\max}(G)$. Using a recent characterization of exact locally compact groups as those groups which admit amenable actions on compact spaces, due to Brodzki, Cave and Li in [6], it is fairly easy to show that the injective functor coincides with the reduced crossed product functor for all exact G. This implies the interesting observation that for exact G, the reduced crossed product functor is the *only injective crossed product functor* for G, although (if G is not amenable) there are often uncountably many crossed product functors with other good properties (see, e.g., [11] for a detailed discussion). To make sure that we do not talk about reduced crossed products only, we show that for a certain class of non-exact groups as constructed by Osajda [20], we do have $\rtimes_{\inj} \neq \rtimes_r$. Another class of groups for which $\rtimes_{\inj} = \rtimes_r$ is given by those discrete groups G for which the maximal group algebra $C^*_{\max}(G)$ has the LLP (see Definition 4.1). Together with the above result on Osajda's groups, this shows that for all these groups the

group algebra $C_{\max}^*(G)$ does not have the LLP. Indeed, our results suggest that the LLP for $C_{\max}^*(G)$ would possibly imply exactness, while we observe that the converse direction fails, since the only other known examples of groups with $C_{\max}^*(G)$ not having the LLP, due to Thom in [**26**], turn out to be exact.

In §5, we study the group algebra $C_{\mathrm{inj}}^*(G) = \mathbb{C} \rtimes_{\mathrm{inj}} G$ associated to $\rtimes_{\mathrm{inj}}$. Extending a well-known result for $C_r^*(G)$, we show that $C_{\mathrm{inj}}^*(G)$ admits an amenable trace if and only if G is amenable, which hints at some similarity between $C_{\mathrm{inj}}^*(G)$ and $C_r^*(G)$. Since the trivial representation $1_G : C_{\max}^*(G) \to \mathbb{C}$ is always an amenable trace, this directly implies that $C_{\mathrm{inj}}^*(G) = C_{\max}^*(G)$ if and only if G is amenable. There is a certain similarity between the defining properties of an amenable trace and Lance's WEP for a C^*-algebra B, and for discrete groups we can show that if $A \rtimes_{\mathrm{inj}} G$ has the WEP, then $A \rtimes_{\mathrm{inj}} G = A \rtimes_{\max} G$. If $A = \mathbb{C}$, this implies that $C_{\mathrm{inj}}^*(G)$ has the WEP if and only if G is amenable, which gives another variant of a well-known result for the reduced group algebra $C_r^*(G)$ (see [**9**, Proposition 3.6.9]). In particular, if G is discrete and exact, then our result shows that the WEP for $A \rtimes_r G$ implies that $A \rtimes_r G \cong A \rtimes_{\max} G$, which indicates that, in general, the WEP for $A \rtimes_r G$ should be related to some kind of amenability for the action of G on A.

In §6, we show that the injective crossed product behaves quite naturally with respect to closed subgroups. It turns out that the injective crossed product functor for a locally compact group G always 'restricts' to the injective functor for M, for every closed subgroup M of G. Moreover, if M is an open supgroup of G and A is a G-algebra, then $A \rtimes_{\mathrm{inj}} M$ always embeds faithfully into $A \rtimes_{\mathrm{inj}} G$, a fact well known for the maximal and the reduced crossed products. We close this paper with a short discussion in §7 of various open questions regarding the maximal injective crossed product functor.

2. *Preliminaries on G-injectivity*

In this section, we give some background on G-injectivity for a locally compact group G. This is presumably well known to at least some experts, but we provide details for the reader's convenience and because we could not find exactly what we wanted in the literature.

Definition 2.1. An equivariant ccp map $\phi : A \to B$ between G-algebras is *G-injective* if for any equivariant injective $*$-homomorphism $\iota : A \to C$ the dashed arrow in the diagram

$$
\begin{array}{ccc}
C & & \\
{\scriptstyle \iota} \uparrow & \searrow^{\tilde{\phi}} & \\
A & \xrightarrow{\ \phi\ } & B
\end{array}
$$
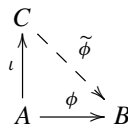
can be filled in with an equivariant contractive completely positive (ccp) map.

A G-algebra B is *G-injective* if any equivariant ccp map $A \to B$ is G-injective.

For the next result, recall that a C^*-algebra B is *injective* if it is G-injective in the above sense for G the trivial group. The most important example is $\mathcal{B}(H)$ for any Hilbert space H; this is a consequence of Arveson's extension theorem (see [**9**, Theorem 1.6.1]). The following result is essentially the same as [**17**, Lemma 2.2], but as we work in a slightly different context, we give a proof for the reader's convenience.

Recall that if B is a C^*-algebra then a function $f : G \to B$ is *right uniformly continuous* if for any $\epsilon > 0$ there exists a neighbourhood U of the identity in G such that $\|f(g) - f(h)\| < \epsilon$ whenever gh^{-1} is in U. We let $C_{ub}(G, B)$ denote the C^*-algebra of all bounded right uniformly continuous functions from G to B, and write $C_{ub}(G)$ if $B = \mathbb{C}$. We equip $C_{ub}(G, B)$ with the action γ induced by the *left* translation action of G on itself:

$$\gamma_g(f)(h) := f(g^{-1}h).$$

This action is continuous, and thus $C_{ub}(G, B)$ is a G-algebra.

PROPOSITION 2.2. *Let B be an injective C^*-algebra. Then $C_{ub}(G, B)$ is G-injective.*

Proof. Let

$$\phi : A \to C_{ub}(G, B)$$

be an equivariant ccp map, and let $\iota : A \to C$ be an equivariant embedding. Let $\delta_e : C_{ub}(G, B) \to B$ be defined by $f \mapsto f(e)$, and let $\psi : A \to B$ be defined by $\psi = \delta_e \circ \phi$, which is ccp. Injectivity of B gives a ccp extension $\widetilde{\psi} : C \to B$ of ψ to C. Let α denote the action of G on C, and define

$$\widetilde{\phi} : C \to C_{ub}(G, B), \quad \widetilde{\phi}(c)(g) := \widetilde{\psi}(\alpha_{g^{-1}}(c)).$$

This function is equivariant and has the property that $\widetilde{\phi}(c)(g)$ is positive or contractive for each $g \in G$ whenever a has these properties. Using the identification $M_n(C_{ub}(G, B)) = C_{ub}(G, M_n(B))$, this gives that $\widetilde{\phi}$ is ccp; we leave the straightforward check that it extends ϕ to the reader. $\quad\square$

Remark 2.3. If B is equipped with a non-trivial G-action β, we may also consider $C_{ub}(G, B)$ equipped with the 'diagonal' action

$$\gamma_g(f)(h) := \beta_g f(g^{-1}h).$$

The resulting G-algebra is equivariantly isomorphic to the one from Proposition 2.2, so it is also G-injective whenever B is (non-equivariantly) injective.

COROLLARY 2.4. *For any G-algebra A, there exists an equivariant embedding $A \to B$ of A into a G-injective G-algebra B.*

Proof. Choose a faithful representation $\pi : A \to \mathcal{B}(H)$. Since $\mathcal{B}(H)$ is injective, $B = C_{ub}(G, \mathcal{B}(H))$ is G-injective by the result of Proposition 2.2. Then, if α denotes the action of G on A, the map

$$\widetilde{\pi} : A \to B, \quad \widetilde{\pi}(a)(g) := \pi(\alpha_{g^{-1}}(a))$$

is an equivariant embedding. $\quad\square$

Lemma 2.2 also allows us to give another example of G-injective maps; this will be useful later.

COROLLARY 2.5. *Let (A, α) be a G-algebra, and fix a (not necessarily equivariant, or faithful) non-degenerate $*$-representation $\pi : A \to \mathcal{B}(H)$. Let $\widetilde{\pi} : A \to \mathcal{B}(L^2(G, H))$ denote the representation used in the definition of the left regular representation: explicitly,*

$$(\widetilde{\pi}(a)\xi)(g) := \pi(\alpha_{g^{-1}}(a))\xi(g);$$

note that $\widetilde{\pi}$ is equivariant for the inner action on $\mathcal{B}(L^2(G, H))$ induced by the amplification of the regular representation. We let $\mathcal{B}(L^2(G, H))_c$ denote the G-continuous part of this algebra.

Then, the $*$-homomorphism $\widetilde{\pi} : A \to \mathcal{B}(L^2(G, H))_c$ is G-injective. In particular, if π is injective, $\widetilde{\pi}$ will be a G-injective embedding.

Proof. The map $\widetilde{\pi} : A \to \mathcal{B}(L^2(G, H))_c$ factors through $C_{ub}(G, \mathcal{B}(H))$ as in Proposition 2.2, when the latter is included by multiplication operators in $\mathcal{B}(L^2(G, H))_c$ in the natural way. It is straightforward to see that a map that factors through a G-injective algebra is G-injective. □

3. *The maximal injective crossed product*

In this section, G denotes a general locally compact group.

Definition 3.1. Let A be a G-algebra. For each equivariant injective $*$-homomorphism $\iota : A \to B$, define the ι-norm on $C_c(G, A)$ by

$$\|a\|_\iota := \|(\iota \rtimes G)(a)\|_{B \rtimes_{\max} G},$$

and let $A \rtimes_\iota G$ denote the corresponding completion. Define the *injective norm* on $C_c(G, A)$ by

$$\|a\|_{\text{inj}} := \inf\{\|a\|_\iota \mid \iota : A \to B \text{ an equivariant injection}\}$$

and the *injective crossed product* $A \rtimes_{\text{inj}} G$ to be the corresponding completion.

Note that $\|\cdot\|_{\text{inj}}$ dominates the reduced norm (by injectivity of the latter), so it is a norm on $C_c(G, A)$, not just a seminorm.

In order to study $\rtimes_{\text{inj}}$, we will make heavy use of the material on G-injective maps in the previous section.

LEMMA 3.2. *Let $\pi : A \to B$ be a G-injective embedding. Then $A \rtimes_{\text{inj}} G$ identifies with the closure of $C_c(G, A)$ under its natural embedding in $B \rtimes_{\max} G$.*

Proof. Let $\phi : A \to C$ be any G-equivariant embedding. We need to show that $\|a\|_\pi \leq \|a\|_\phi$ for any $a \in C_c(G, A)$. As π is G-injective, there exists a ccp equivariant map $\psi : C \to B$ making the diagram

$$
\begin{array}{ccc}
 & C & \\
{\scriptstyle \phi}\nearrow & & \searrow{\scriptstyle \psi} \\
A & \xrightarrow{\ \pi\ } & B
\end{array}
$$
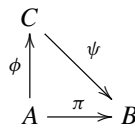
commute. The maximal crossed product is functorial for ccp maps, as follows from, for example, [**12**, Theorem 4.9]. Taking maximal crossed products, we get a commutative diagram

$$
\begin{array}{ccc}
C \rtimes_{\max} G & & \\
\phi \rtimes G \uparrow & \overset{\psi \rtimes G}{\searrow} & \\
A \rtimes_{\max} G & \overset{\pi \rtimes G}{\longrightarrow} & B \rtimes_{\max} G
\end{array}
$$

where $\phi \rtimes G$ and $\pi \rtimes G$ are $*$-homomorphisms and $\psi \rtimes G$ is ccp. It follows that for any $a \in C_c(G, A)$,

$$\|a\|_\pi = \|(\pi \rtimes G)(a)\| = \|(\psi \rtimes G) \circ (\phi \rtimes G)(a)\| \leq \|(\phi \rtimes G)(a)\| = \|a\|_\phi$$

as required.   □

From the above lemma, we get the following immediate corollary.

COROLLARY 3.3. *Suppose that A is G-injective. Then $A \rtimes_{\mathrm{inj}} G = A \rtimes_{\max} G$.*   □

PROPOSITION 3.4. *The injective crossed product defines a crossed product functor.*

Proof. Let $\phi : A \to C$ be an arbitrary $*$-homomorphism. Let

$$\pi_A : A \to B_A \quad \text{and} \quad \pi_C : C \to B_C$$

be G-injective $*$-homomorphisms as in Corollary 2.4. Now, in the diagram

$$
\begin{array}{ccc}
B_A & - - \to & B_C \\
\pi_A \uparrow & & \uparrow \pi_C \\
A & \overset{\phi}{\longrightarrow} & C
\end{array}
$$

the definition of injectivity for B_C applied to the inclusion $A \to B_A$ and to the composition $\pi_C \circ \phi : A \to B_C$ allows us to fill in the dashed arrow with an equivariant ccp map, say $\widetilde{\phi}$. Applying Lemma 3.2 and using functoriality of the ccp maps, we get a diagram

$$
\begin{array}{ccc}
B_A \rtimes_{\max} G & \overset{\widetilde{\phi} \rtimes G}{\longrightarrow} & B_C \rtimes_{\max} G \\
\pi_A \rtimes G \uparrow & & \uparrow \pi_C \rtimes G \\
A \rtimes_{\mathrm{inj}} G & & C \rtimes_{\mathrm{inj}} G
\end{array}
$$

where the vertical maps are injections. Identifying $A \rtimes_{\mathrm{inj}} G$ and $C \rtimes_{\mathrm{inj}} G$ with their images under the vertical maps, the restriction of $\widetilde{\phi} \rtimes G$ to $A \rtimes_{\mathrm{inj}} G$ is the map required by functoriality.   □

PROPOSITION 3.5. *The functor $\rtimes_{\mathrm{inj}}$ is the maximal injective crossed product functor.*

Proof. We must first show that $\rtimes_{\mathrm{inj}}$ is injective. Let $\phi : A \to C$ be an injective equivariant $*$-homomorphism. Let $\pi_A : A \to B_A$ and $\pi_C : C \to B_C$ be G-injective embeddings as in

Corollary 2.4, and consider the following diagram.

$$
\begin{array}{ccc}
B_A & \longleftarrow\!-\!- & B_C \\
\uparrow{\scriptstyle \pi_A} & & \uparrow{\scriptstyle \pi_C} \\
A & \xrightarrow{\ \phi\ } & C
\end{array}
$$

The composition $\pi_C \circ \phi : A \to B_C$ is injective, so as the inclusion $A \to B_A$ is G-injective, the dashed arrow can be filled in with an equivariant ccp map, say ψ. Taking crossed products gives a diagram

$$
\begin{array}{ccc}
B_A \rtimes_{\max} G & \xleftarrow{\ \psi \rtimes G\ } & B_C \rtimes_{\max} G \\
\uparrow{\scriptstyle \pi_A \rtimes G} & & \uparrow{\scriptstyle \pi_C \rtimes G} \\
A \rtimes_{\mathrm{inj}} G & \xrightarrow{\ \phi \rtimes G\ } & C \rtimes_{\mathrm{inj}} G
\end{array}
$$

where the vertical maps are injective by Lemma 3.2. The composition

$$(\psi \rtimes G) \circ (\pi_C \rtimes G) \circ (\phi \rtimes G)$$

is thus injective as it agrees with the left-hand vertical map; hence, $\phi \rtimes G$ is injective as required.

To see that $\rtimes_{\mathrm{inj}}$ is the maximal injective crossed product, let $\rtimes_{\mu}$ be any other injective crossed product and let $A \to B_A$ be a G-injective embedding. Then we look at the following diagram.

$$
\begin{array}{ccc}
A \rtimes_{\mathrm{inj}} G & \longrightarrow & B_A \rtimes_{\max} G \\
 & & \downarrow \\
A \rtimes_{\mu} G & \longrightarrow & B_A \rtimes_{\mu} G
\end{array}
$$

The composition $A \rtimes_{\mathrm{inj}} G \to B_A \rtimes_{\mu} G$ has image isomorphic to $A \rtimes_{\mu} G$, as $\rtimes_{\mu}$ is injective, and is the identity on $C_c(G, A)$; this yields a homomorphism $A \rtimes_{\mathrm{inj}} G \twoheadrightarrow A \rtimes_{\mu} G$ extending the identity on $C_c(G, A)$. This completes the proof. $\qquad\square$

Remark 3.6. Recall from [**12**] that a crossed product functor $\rtimes_{\mu}$ for G is called a *correspondence functor* if it is functorial for G-equivariant correspondences in the sense that if E is a G-equivariant correspondence from A to B, then there is a canonical construction of a crossed product correspondence $E \rtimes_{\mu} G$ from $A \rtimes_{\mu} G$ to $B \rtimes_{\mu} G$. It was shown in [**11**, **12**] that correspondence functors enjoy many nice properties. For example, they admit dual coactions and a descent in Kasparov's G-equivariant bivariant K-theory (see [**12**, §§5 and 6]). Moreover, [**12**, Theorem 4.9] shows (among other things) that for a given crossed product functor $\rtimes_{\mu}$, the following are equivalent:
(1) $\rtimes_{\mu}$ is a correspondence functor;
(2) $\rtimes_{\mu}$ is injective on G-invariant hereditary subalgebras in the sense that if $B \subseteq A$ is G-invariant hereditary subalgebra of A, then $B \rtimes_{\mu} G$ injects into $A \rtimes_{\mu} G$;
(3) $\rtimes_{\mu}$ is functorial for ccp maps.
Hence, the following corollary is immediate from the fact that $\rtimes_{\mathrm{inj}}$ is injective.

COROLLARY 3.7. *The injective functor $\rtimes_{\mathrm{inj}}$ is a correspondence functor.* □

Among the many nice implications of being a correspondence functor, we mention the following, which we shall use in §5 below. It follows directly from Corollary 3.7 and [**12**, Theorem 5.6].

PROPOSITION 3.8. *Let G be a locally compact group and let (A, α) be a G-algebra. Then, the crossed product functor $\rtimes_{\mathrm{inj}}$ is a* duality functor, *meaning that the canonical representation*

$$(A, G) \to \mathcal{M}(A \rtimes_{\mathrm{inj}} G \otimes C^*_{\max}(G))$$

sending $a \mapsto a \otimes 1$ and $g \mapsto \delta_g \otimes \delta_g$ extends to an (injective) homomorphism (called the dual coaction*)*

$$\widehat{\alpha}_{\mathrm{inj}} \colon A \rtimes_{\mathrm{inj}} G \hookrightarrow \mathcal{M}(A \rtimes_{\mathrm{inj}} G \otimes C^*_{\max}(G)).$$

The G-WEP. We saw in Corollary 3.3 that for G-injective algebras A, the injective crossed product by G coincides with the maximal crossed product by G. We shall now introduce a larger class of G-algebras which enjoy the same property. Recall that a C^*-algebra A has Lance's WEP if every embedding $A \hookrightarrow B$ into another C^*-algebra B admits a weak conditional expectation, that is, a ccp map $p \colon B \to A^{**}$ which restricts to the identity on A (see [**9**, Definition 3.6.7]). We now introduce a G-equivariant version of this property.

Definition 3.9. Let G be a locally compact group. A G-algebra A has the *G-equivariant weak expectation property* (G-WEP) if for every G-equivariant embedding $\iota \colon A \hookrightarrow B$ into some other G-algebra B, there is an equivariant ccp map $p \colon B \to A^{**}$ whose composition with ι coincides with the canonical inclusion $A \hookrightarrow A^{**}$.

Here, we consider A^{**} endowed with the double dual action $\alpha^{**} \colon G \to \mathrm{Aut}(A^{**})$ of the given action $\alpha \colon G \to \mathrm{Aut}(A)$. Let A^{**}_c denote the subalgebra of A^{**} consisting of all G-continuous elements of A^{**}. Then, for any G-algebra B, the image of any norm decreasing G-equivariant map $B \to A^{**}$ lies in A^{**}_c. In particular, this applies to the ccp map $p \colon B \to A^{**}$ in the above definition.

PROPOSITION 3.10. *Let G be a locally compact group. Then the following hold.*
(1) *Every G-injective G-algebra has the G-WEP.*
(2) *If A is a G-algebra such that there exists a G-invariant C^*-subalgebra $C \subseteq A^{**}_c$ which is G-injective and contains A, then A has the G-WEP.*

Proof. It suffices to show (2). Assume that $A \subseteq C \subseteq A^{**}_c$ are as in (2). Let $\iota \colon A \hookrightarrow B$ be a G-equivariant embedding into a G-algebra B. Then, the G-injectivity of C applied to the inclusion $i \colon A \hookrightarrow C$ (which is the co-restriction of the canonical embedding $A \hookrightarrow A^{**}$) implies the existence of a G-equivariant ccp map $p \colon B \to C \subseteq A^{**}$ with $p \circ \iota = i$. □

Example 3.11. Let $B = \mathcal{B}(H)$ be the C^*-algebra of bounded operators on some Hilbert space H endowed with the trivial G-action. We know that the G-algebra $C_{ub}(G, B)$ is G-injective (with respect to the translation G-action). Since this is canonically embedded into the double dual of the G-algebra $C_0(G, \mathcal{K})$, where $\mathcal{K} := \mathcal{K}(H)$, it follows from (2) in

the above proposition that any G-algebra A lying between $C_0(G, \mathcal{K})$ and $C_{ub}(G, B)$ has the G-WEP.

PROPOSITION 3.12. *If A is a G-algebra with the G-WEP, then $A \rtimes_{\text{inj}} G = A \rtimes_{\text{max}} G$.*

Proof. Given a G-equivariant embedding $\iota \colon A \hookrightarrow B$, there is a G-equivariant ccp map $p \colon B \to A_c^{**}$ with $p \circ \iota(a) = a$ for all $a \in A$. Notice that $A \rtimes_{\text{max}} G$ embeds into $A_c^{**} \rtimes_{\text{max}} G$. Indeed, by the universal property of $\rtimes_{\text{max}}$, we have a canonical homomorphism $A_c^{**} \rtimes_{\text{max}} G \to (A \rtimes_{\text{max}} G)^{**}$ whose composition

$$A \rtimes_{\text{max}} G \to A_c^{**} \rtimes_{\text{max}} G \to (A \rtimes_{\text{max}} G)^{**}$$

is the canonical bidual embbeding. We can therefore identify $A \rtimes_{\text{max}} G \subseteq A_c^{**} \rtimes_{\text{max}} G$. Now, by functoriality of the maximal crossed product for ccp maps, p induces a ccp map $p \rtimes_{\text{max}} G \colon B \rtimes_{\text{max}} G \to A_c^{**} \rtimes_{\text{max}} G$ satisfying $(p \rtimes_{\text{max}} G) \circ (\iota \rtimes_{\text{max}} G)(x) = x$ for all $x \in A \rtimes_{\text{max}} G$ so that $\iota \rtimes_{\text{max}} G \colon A \rtimes_{\text{max}} G \to B \rtimes_{\text{max}} G$ is injective. Since B was arbitrary, the result follows from the definition of the injective crossed product. $\qquad\square$

Remark 3.13. Although above we identified a class of G-algebras such that $A \rtimes_{\text{inj}} G = A \rtimes_{\text{max}} G$, we shall see later that for a locally compact group G, the maximal injective functor $\rtimes_{\text{inj}}$ coincides with $\rtimes_{\text{max}}$ if and only if G is amenable. Indeed, we show in Proposition 5.3 below that the corresponding group algebras coincide if and only if G is amenable.

Remark 3.14. The only property of the maximal crossed product functor used in our constructions for the maximal injective crossed product $\rtimes_{\text{inj}}$ is its functoriality for G-equivariant ccp maps. Therefore, the constructions of this section could be carried out without change starting with an arbitrary correspondence functor $\rtimes_\mu$ in place of the maximal crossed product functor $\rtimes_{\text{max}}$. Everything goes through as before, and the resulting crossed product functor, say $\rtimes_{\text{inj}(\mu)}$, is the largest injective crossed product functor that is dominated by $\rtimes_\mu$. Moreover, for any G-injective algebra A we then have $A \rtimes_{\text{inj}(\mu)} G \cong A \rtimes_\mu G$. An analogous statement for algebras A with the G-WEP is not clear, since the proof of Proposition 3.12 uses the universality of the maximal crossed product. However, the proof goes through if we start with an *exact* correspondence functor $\rtimes_\mu$ by making use of [**13**, Theorem 3.5].

4. *Connections with exactness and the LLP*

There are two interesting cases where we can show that the injective crossed product agrees with the reduced crossed product. Our goal in this section is to discuss these cases and deduce some consequences: perhaps most notable of these is that we give examples where $\rtimes_r \neq \rtimes_{\text{inj}}$, and use this to give new examples of groups G for which $C_{\text{max}}^*(G)$ does not have the LLP.

The first such case occurs when G is exact. We give an ad hoc definition of exactness that is convenient for our purposes. See [**6**, Theorem A] for a proof that this is equivalent to more standard definitions (the result of [**6**, Theorem A] is only stated for second countable G, but the proof works in general with minor modifications). First, recall from [**1**] that a continuous action of the locally compact group G on a locally compact space X is said

to be *amenable* if there exists a net $(m_i)_{i \in I}$ of continuous maps $x \mapsto m_i^x$ from X into the space $\mathrm{Prob}(G)$ of probability measures on G such that

$$\lim_i \| g \cdot m_i^x - m_i^{gx} \| = 0$$

uniformly on compact subsets of $X \times G$.

Definition 4.1. A locally compact group G is *exact* if it admits an amenable continuous action on a compact space X.

PROPOSITION 4.2. *Let G be an exact locally compact group. Then, for any G-algebra A, $A \rtimes_{\mathrm{inj}} G = A \rtimes_r G$.*

Proof. As G is exact, G acts continuously and amenably on some compact space X. Thus, by [**1**, Theorem 5.3], for any G-algebra A, integrating the covariant representation of (A, G) in $(A \otimes C(X)) \rtimes_{\max} G$ given by

$$a \mapsto a \otimes 1, \quad g \mapsto \delta_g$$

gives a ∗-homomorphism

$$A \rtimes_{\max} G \to (A \otimes C(X)) \rtimes_{\max} G = (A \otimes C(X)) \rtimes_r G.$$

As $\rtimes_r$ is injective, this factors through $A \rtimes_r G$. Thus, the reduced norm on $C_c(G, A)$ is one of the norms that $\| \cdot \|_{\mathrm{inj}}$ is the infimum over, and the result follows. □

For the second example where $\rtimes_{\mathrm{inj}} = \rtimes_r$, we need to restrict to the case of discrete groups. We recall an ad hoc definition of the LLP that is convenient for our purposes. See [**9**, Corollary 13.2.5] for a proof that this is equivalent to the usual definition.

Definition 4.3. A C^*-algebra A has the *local lifting property* (LLP) if for any Hilbert space H, $A \otimes \mathcal{B}(H) = A \otimes_{\max} \mathcal{B}(H)$, that is, if there is a unique C^*-norm on the algebraic tensor product $A \odot \mathcal{B}(H)$ for every H.

PROPOSITION 4.4. *Let G be a discrete group such that $C^*_{\max}(G)$ has the LLP. Then, for any G-algebra A, $A \rtimes_{\mathrm{inj}} G = A \rtimes_r G$.*

Proof. Let $\pi : A \to \mathcal{B}(H)$ be any faithful (non-equivariant) ∗-representation, where $\mathcal{B}(H)$ is equipped with the trivial G-action. Let $\tilde{\pi} : A \to \mathcal{B}(\ell^2(G, H))$ be the amplified form of this representation as in Corollary 2.5, where we equip $\mathcal{B}(\ell^2(G, H))$ with the conjugation action associated to the amplification of the left regular representation λ. Then, Lemma 3.2 implies that the integrated form

$$\tilde{\pi} \rtimes G : C_c(G, A) \to \mathcal{B}(\ell^2(G, H)) \rtimes_{\max} G$$

extends to an inclusion

$$A \rtimes_{\mathrm{inj}} G \to \mathcal{B}(\ell^2(G, H)) \rtimes_{\max} G.$$

Identify $\ell^2(G, H)$ with $H \otimes \ell^2(G)$ in the usual way. As the action of G on $\mathcal{B}(H \otimes \ell^2(G))$ is inner, there is a canonical 'untwisting isomorphism'

$$\Phi : \mathcal{B}(H \otimes \ell^2(G)) \rtimes_{\max} G \to \mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G)$$
$$T \delta_g \mapsto T(1 \otimes \lambda_g) \otimes \delta_g.$$

On the other hand, using the LLP for $C^*_{\max}(G)$ gives a canonical identification

$$\mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G) = \mathcal{B}(H \otimes \ell^2(G)) \otimes C^*_{\max}(G),$$

so we may identify the image of Φ with the algebra on the right-hand side above.

Consider, finally, the commutative diagram

$$
\begin{array}{ccc}
A \rtimes_{\mathrm{inj}} G & \longrightarrow & \mathcal{B}(H \otimes \ell^2(G)) \rtimes_{\max} G \\
& \searrow{\scriptstyle \psi} & \downarrow{\scriptstyle \Phi} \\
& & \mathcal{B}(H \otimes \ell^2(G)) \otimes C^*_{\max}(G)
\end{array}
$$

where the diagonal arrow ψ is by definition the composition of the other two maps and so is, in particular, injective. Computing, the diagonal arrow is the integrated form of the covariant pair given on $a \in A$ and $g \in G$ by

$$a \mapsto \tilde{\pi}(a) \otimes 1, \quad g \mapsto 1 \otimes \lambda_g \otimes \delta_g.$$

The image of this map therefore agrees precisely with the image of $A \rtimes_r G$ under the (injective) composition of the coaction

$$\delta : A \rtimes_r G \to A \rtimes_r G \otimes C^*_{\max}(G)$$

as in [**15**, Definition A.27] and of the tensor product $*$-homomorphism

$$(\tilde{\pi} \rtimes (1 \otimes \lambda)) \otimes \mathrm{id} : A \rtimes_r G \otimes C^*_{\max}(G) \to \mathcal{B}(H \otimes \ell^2(G)) \otimes C^*_{\max}(G).$$

As we already remarked that the diagonal arrow ψ is injective, we thus have that the identity map on $C_c(G, A)$ extends to an injection $A \rtimes_{\mathrm{inj}} G \to A \rtimes_r G$, and we are done. $\square$

COROLLARY 4.5. *If G is an exact locally compact group, or if G is discrete and $C^*_{\max}(G)$ has the LLP, then the reduced crossed product is the* only *injective crossed product functor.*

Proof. If $\rtimes_\mu$ is injective, then Proposition 3.5 gives that $\rtimes_r \leq \rtimes_\mu \leq \rtimes_{\mathrm{inj}}$. Hence, by Propositions 4.2 and 4.4, all three are equal. $\square$

This is in stark contrast to the case of exact crossed products: indeed, if G is any non-amenable group, then there is a large class of exotic exact crossed products arising, for example from the Brown–Guentner construction, as discussed in [**11**, Definition 3.6].

At this point, it is reasonable to ask whether $\rtimes_{\mathrm{inj}}$ *ever* differs from the reduced crossed product! We can show that indeed it does using the relatively explicit construction of non-exact groups, due to Osajda [**20**]. For the proof we need the following fact, which is immediate from Lemmas 2.2 and 3.2.

COROLLARY 4.6. *For any discrete group G, $\ell^\infty(G) \rtimes_{\max} G = \ell^\infty(G) \rtimes_{\mathrm{inj}} G$.* $\square$

We can now show that $\rtimes_{\mathrm{inj}}$ is at least sometimes not equal to $\rtimes_r$. Osajda shows that groups as in the statement exist [**20**].

LEMMA 4.7. *Let G be a non-exact group equipped with an isometric embedding $X \to G$, where X is an expander that is a coarse union of a sequence of finite connected graphs with a uniform bound on vertex degrees, and with girth tending to infinity. Then $\ell^\infty(G) \rtimes_{\mathrm{inj}} G \neq \ell^\infty(G) \rtimes_r G$.*

Proof. Using Corollary 4.6, it suffices to prove that $\ell^\infty(G) \rtimes_{\max} G \neq \ell^\infty(G) \rtimes_r G$. Let $\chi_X \in \ell^\infty(G)$ be the characteristic function of X. Then, using that $\chi_X(\ell^\infty(G) \rtimes_{\mathrm{alg}} G)\chi_X$ identifies with the algebraic uniform Roe algebra $\mathbb{C}_u[X]$, it is not too difficult to see that the corners

$$\chi_X(\ell^\infty(G) \rtimes_{\max} G)\chi_X \quad \text{and} \quad \chi_X(\ell^\infty(G) \rtimes_r G)\chi_X$$

identify respectively with the maximal and reduced uniform Roe algebras of X, denoted by $C^*_{u,\max}(X)$ and $C^*_u(X)$. Hence, it suffices to show that $C^*_{u,\max}(X)$ and $C^*_u(X)$ are not equal. This can be done K-theoretically using the main ideas of [**29**, **30**]: the basic point is that the maximal coarse Baum–Connes conjecture for X is true, but the usual version is false. We give a somewhat more direct proof, however, based on [**31**, §8].

Let $\Delta \in \mathbb{C}_u[X]$ denote the graph Laplacian on X; thus, if $X = \bigsqcup X_n$ is the decomposition of X into finite connected graphs, we have that Δ has matrix coefficients given by

$$\Delta_{xy} = \begin{cases} \mathrm{degree}(x) & x = y, \\ -1 & x, y \text{ connected by an edge in some } X_n, \\ 0 & \text{otherwise.} \end{cases}$$

According to the definition of X being an expander, there is some $c > 0$ such that the spectrum $\mathrm{spec}_{C^*_u(X)}(\Delta)$ of Δ considered as an element of $C^*_u(X)$ is contained in $\{0\} \cup [c, \infty)$. On the other hand, [**31**, Lemma 8.9], combined with the assumption that the girth of the sequence (X_n) tends to infinity, implies that the spectrum $\mathrm{spec}_{C^*_{u,\max}(X)}(\Delta)$ of Δ considered as an element of $C^*_{u,\max}(X)$ contains points in $(0, c]$ for any $c > 0$. Hence, $C^*_{u,\max}(X) \neq C^*_u(X)$, as required. $\qquad \square$

The following corollary is immediate from Lemmas 4.4 and 4.7.

COROLLARY 4.8. *Let G be as in the hypotheses of Lemma 4.7. Then $C^*_{\max}(G)$ does not have the LLP.* $\qquad \square$

There seem to be very few examples where $C^*_{\max}(G)$ is known to not have the LLP. We discuss this, and the connection between this property and exactness, in the next few remarks.

Remark 4.9. The class of discrete groups G for which $C^*_{\max}(G)$ has the LLP contains all amenable groups and is closed under taking subgroups, and free products with finite amalgam; see [**22**, Proposition 3.21] and following discussion. However, it is not clear to us that it contains, for example, any non-exact group, or even a group without the Haagerup approximation property. On the other hand, it appears that the only known examples where $C^*_{\max}(G)$ does not have the LLP, other than those of Corollary 4.8, are those constructed by Thom in [**26**] (other examples where $C^*_{\max}(G)$ does not have the LP were constructed by Ozawa [**23**]).

Remark 4.10. It is natural to ask whether the LLP for $C^*_{\max}(G)$ implies that G is exact. Some evidence for this goes as follows. If $C^*_{\max}(G)$ has the LLP, then Lemma 4.4 and Corollary 4.6 imply that $\ell^\infty(G) \rtimes_{\max} G = \ell^\infty(G) \rtimes_r G$. It would be reasonable (well, arguably...) to expect that this implies that the action of G on the maximal ideal space βG of $\ell^\infty(G)$ is amenable, and thus that G is exact. Note that if $\partial G := \beta G \setminus G$ is the

associated corona of G, then the equality $C(\partial G) \rtimes_{\max} G = C(\partial G) \rtimes_r G$ does imply—indeed characterizes—that G is exact, using the results of [**24**, §5.1].

On the other hand, if one could produce a non-exact group with $C^*_{\max}(G)$ having the LLP, this would give an example of a non-amenable action on a compact space such that the associated maximal and reduced crossed products are the same. This would answer a long-standing open question.

Remark 4.11. The converse question, whether exactness of G implies that $C^*_{\max}(G)$ has the LLP, has a negative answer. Indeed, Thom's example of a group without the LLP from [**26**, §2] is exact. To construct his example G, Thom starts with a specfic (countable) subgroup G_0 of $\mathrm{GL}_5(R)$, where $R = \mathbb{F}_p[t, t^{-1}]$ is the ring of Laurent polynomials over the finite field with p elements for some prime p. He then defines G to be the quotient of G_0 by some specific subgroup C of its center. Now, G_0 is a countable subgroup of $\mathrm{GL}_n(R)$, where R is a commutative ring with unit, and therefore has Yu's property A by [**16**, Theorems 4.6 and 5.2.1]. Hence, $C^*_r(G_0)$ is exact by the main result of [**21**]. On the other hand, as C is a central subgroup of G_0, it is abelian, so in particular amenable, and so the quotient map $G_0 \to G$ induces a surjective $*$-homomorphism $C^*_r(G_0) \to C^*_r(G)$. In particular, $C^*_r(G)$ is a quotient of an exact C^*-algebra, so exact by [**9**, Corollary 9.4.3]. Hence, G is exact. Similar reasoning shows that the other example of a group not satisfying the LLP given in §3 of Thom's paper is exact as well.

5. *The injective group algebra, amenability, and the WEP*
We now study the group algebra $C^*_{\mathrm{inj}}(G) := \mathbb{C} \rtimes_{\mathrm{inj}} G$.

The first result we are aiming for is a direct analogue of a well-known property for the reduced group C^*-algebra of a discrete group [**7**, Corollary 4.1.2], and provides some evidence that we might have $C^*_{\mathrm{inj}}(G) = C^*_r(G)$ in general; it does at least show that $C^*_{\mathrm{inj}}(G) \neq C^*_{\max}(G)$ for a general discrete non-amenable group (and hence that $\rtimes_{\mathrm{inj}} \neq \rtimes_{\max}$ if G is not amenable).

To state the result, we recall one of the definitions of an amenable trace [**7**, Theorem 3.1.6].

Definition 5.1. Let $\tau : A \to \mathbb{C}$ be a tracial state on a unital C^*-algebra, let $\pi_\tau : A \to \mathcal{B}(L^2(A, \tau))$ be the associated GNS representation, and let $\pi_\tau(A)''$ be the von Neumann algebra generated by the image of A in this representation. Then τ is *amenable* if for any faithful representation $A \subseteq \mathcal{B}(H)$ there is a unital completely positive (ucp) map $\phi : \mathcal{B}(H) \to \pi_\tau(A)''$ such that $\phi(a) = \pi_\tau(a)$ for all $a \in A$.

We say that a tracial state on a non-unital C^*-algebra is *amenable* if its canonical extension to a tracial state on the unitization is amenable†.

In other words, the trace τ is amenable if its GNS representation π_τ is an injective ccp map (in the sense of our Definition 2.1 for the trivial group) when viewed as a map $A \to \pi_\tau(A)''$. In particular, τ is amenable if $\pi_\tau(A)''$ is an injective von Neumann algebra (e.g. if $\pi_\tau(A)$ is a nuclear C^*-algebra).

---

† We are not sure if there is a standard definition of amenability of a trace on a non-unital C^*-algebra; this ad hoc one is convenient for our purposes.

Example 5.2. Let A be a C^*-algebra, let $\pi : A \to M_n(\mathbb{C})$ be a finite-dimensional representation, and let tr : $M_n(\mathbb{C}) \to \mathbb{C}$ be the canonical tracial state. Then the pull-back of tr to (the unitization of) A is amenable. Indeed, in this case, $L^2(A, \tau)$ is finite-dimensional by uniqueness of GNS representations, and hence $\pi_\tau(A)''$ is finite-dimensional, so in particular injective. The existence of an appropriate ϕ thus follows, as $\pi_\tau(A)''$ is injective.

PROPOSITION 5.3. *The group algebra $C^*_{\mathrm{inj}}(G)$ has an amenable trace if and only if G is amenable†. In particular, if G is non-amenable then $C^*_{\mathrm{inj}}(G)$ has no finite-dimensional representations, and is therefore not equal to $C^*_{\max}(G)$.*

Proof. If G is amenable, then $C^*_{\max}(G) = C^*_r(G)$, which forces $C^*_{\max}(G) = C^*_{\mathrm{inj}}(G)$. In particular, the trivial representation extends to $C^*_{\mathrm{inj}}(G)$, and this gives an amenable trace by (a very simple case of) Example 5.2. Hence, in this case, $C^*_{\mathrm{inj}}(G)$ has an amenable trace.

Conversely, let $\tau : C^*_{\mathrm{inj}}(G) \to \mathbb{C}$ be an amenable trace. Let $A = \widetilde{C^*_{\mathrm{inj}}(G)}$ be the unitization of $C^*_{\mathrm{inj}}(G)$ in the non-unital case, or just $A = C^*_{\mathrm{inj}}(G)$ if this is already unital. Abuse notation by also writing $\tau : A \to \mathbb{C}$ for the canonical extension. Fix a non-degenerate embedding $C_{ub}(G) \rtimes_{\max} G \subseteq \mathcal{B}(H)$ and note that Lemmas 2.2 and 3.2 give us an embedding

$$C^*_{\mathrm{inj}}(G) \subseteq C_{ub}(G) \rtimes_{\max} G \subseteq \mathcal{B}(H),$$

and thus also a unital embedding of A into $\mathcal{B}(H)$. Let $\phi : \mathcal{B}(H) \to \pi_\tau(A)''$ be the ucp map given by the definition of an amenable trace, and let $\tau : \pi_\tau(A)'' \to \mathbb{C}$ be the tracial state induced by τ. We thus get a state

$$\widetilde{m} : \mathcal{B}(H) \to \mathbb{C}, \quad \widetilde{m} := \tau \circ \phi.$$

We claim that the restriction $m : C_{ub}(G) \to \mathbb{C}$ of $\widetilde{m}$ to $C_{ub}(G)$ is an invariant mean. Indeed, let $a \in C_{ub}(G)$, write α for the translation action of G on $C_{ub}(G)$, let $g \in G$, and let $(f_i)_{i \in I}$ be an approximate unit in $C_c(G) \subseteq C^*_{\mathrm{inj}}(G)$. For each i, let $\delta_g * f_i \in C_c(G)$ denote the convolution of the Dirac mass at g with f_i. Then we have that the net

$$((\delta_g * f_i)a(\delta_g * f_i)^*)_{i \in I}$$

converges in the norm of $C_{ub}(G) \rtimes_{\max} G$ to $\alpha_g(a)$. On the other hand, each $\delta_g * f_i$ is in the multiplicative domain of ϕ, and hence

$$m(\alpha_g(a)) = \lim_i \tau(\phi((\delta_g * f_i)a(\delta_g * f_i)^*)) = \lim_i \tau(\pi_\tau(\delta_g * f_i)\phi(a)\pi_\tau(\delta_g * f_i)^*).$$

Using that τ is a trace, this equals $\lim_i \tau(\pi_\tau(f_i^* f_i)\phi(a))$. As $\pi_\tau : A \to \mathcal{B}(L^2(A, \tau))$ restricts to a non-degenerate representation of $C^*_{\mathrm{inj}}(G)$, and as (f_i) is an approximate unit for $C^*_{\mathrm{inj}}(G)$, we have that $\tau(f_i^* f_i)$ converges strongly to the identity operator on $L^2(A, \tau)$; moreover, the canonical extension $\tau : \pi_\tau(A)'' \to \mathbb{C}$ is normal, and hence in particular strongly continuous on bounded sets. Thus, the net $\lim_i \tau(\pi_\tau(f_i^* f_i)\phi(a))$ converges to $\tau(\phi(a)) = m(a)$, completing the proof of invariance of m, and thus m is indeed an invariant mean and G is amenable.

---

† The same property holds for $C^*_r(G)$ in place of $C^*_{\mathrm{inj}}(G)$, with essentially the same proof; this is well known, at least when G is discrete [**7**, Corollary 4.1.2].

The remaining comments about non-amenable G follow from Example 5.2 and the fact that $C^*_{\max}(G)$ always has at least one finite-dimensional representation (the trivial representation). $\square$

Notice that the amenability condition on a trace $\tau : A \to \mathbb{C}$ has some similarity with the WEP, which we briefly discussed at the end of §3. Recall that a C^*-algebra A has the WEP if every embedding $A \hookrightarrow B$ admits a ccp map $B \to A^{**}$ which restricts to the identity on A. By [**9**, Proposition 3.6.8], this is equivalent to the property that every embedding $A \hookrightarrow B$ induces an embedding $A \otimes_{\max} D \hookrightarrow B \otimes_{\max} D$ for every C^*-algebra D. The archetypal example of a C^*-algebra with the WEP is the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. On the other hand, the reduced group C^*-algebra $C^*_r(G)$ of a discrete group G has the WEP if and only if G is amenable, see [**9**, Proposition 3.6.9]. We want to arrive at a similar result for $C^*_{\mathrm{inj}}(G)$ which gives another hint that $C^*_{\mathrm{inj}}(G)$ might be equal to $C^*_r(G)$. Indeed, we can prove the following general result.

PROPOSITION 5.4. *Let G be a discrete group. If A is a G-algebra for which $A \rtimes_{\mathrm{inj}} G$ has the WEP, then $A \rtimes_{\mathrm{inj}} G = A \rtimes_{\max} G$.*

Proof. As in the proof of Proposition 4.4, we choose a faithful non-degenerate representation $\pi \colon A \hookrightarrow \mathcal{B}(H)$ and embed $A \rtimes_{\mathrm{inj}} G$ into $\mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G)$ via the diagonal homomorphism $a\delta_g \mapsto \tilde{\pi}(a)(1 \otimes \lambda_g) \otimes \delta_g$. Since $A \rtimes_{\mathrm{inj}} G$ is assumed to have the WEP, we get an embedding

$$A \rtimes_{\mathrm{inj}} G \otimes_{\max} C^*_{\max}(G) \hookrightarrow \mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G) \otimes_{\max} C^*_{\max}(G).$$

Now we consider the embedding (the comultiplication)

$$\Delta \colon C^*_{\max}(G) \hookrightarrow C^*_{\max}(G) \otimes_{\max} C^*_{\max}(G)$$

sending $\delta_g \mapsto \delta_g \otimes \delta_g$. We then get a map

$$\mathrm{id} \otimes_{\max} \Delta : \mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G)$$
$$\to \mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G) \otimes_{\max} C^*_{\max}(G).$$

This map sends the image of $A \rtimes_{\mathrm{inj}} G$ in $\mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G)$ into the image of $A \rtimes_{\mathrm{inj}} G \otimes_{\max} C^*_{\max}(G)$ in $\mathcal{B}(H \otimes \ell^2(G)) \otimes_{\max} C^*_{\max}(G) \otimes_{\max} C^*_{\max}(G)$. We therefore get a map

$$A \rtimes_{\mathrm{inj}} G \hookrightarrow A \rtimes_{\mathrm{inj}} G \otimes_{\max} C^*_{\max}(G)$$

sending $a\delta_g \mapsto a\delta_g \otimes \delta_g$. Since the composition of this map with $\mathrm{id}_{A \rtimes_{\mathrm{inj}} G} \otimes 1_G$ gives the identity on $A \rtimes_{\mathrm{inj}} G$, this map is injective. It follows from [**10**, Theorem 5.1] that the dual coaction $\widehat{\alpha}$ from Proposition 3.8 is maximal, which means $A \rtimes_{\mathrm{inj}} G = A \rtimes_{\max} G$, as desired. $\square$

COROLLARY 5.5. *For a discrete group G, its injective group algebra $C^*_{\mathrm{inj}}(G) := \mathbb{C} \rtimes_{\mathrm{inj}} G$ has the WEP if and only if G is amenable.*

Proof. This follows directly from Propositions 5.3 and 5.4. $\square$

Remark 5.6. Asking a crossed product to have the WEP is probably a strong restriction. In the above situation, it seems to be related to the amenability of the underlying action.

For example, if G is exact, we know that $A \rtimes_{\mathrm{inj}} G = A \rtimes_r G$, so the assumption that $A \rtimes_{\mathrm{inj}} G$ has the WEP implies that $A \rtimes_{\max} G = A \rtimes_r G$. If this holds and the crossed product C^*-algebra has the WEP, then so does the algebra A, as remarked in [**4**, §4]. Moreover, the main result of [**4**] asserts that, assuming the G-action on a unital A to be amenable (as defined in [**9**]), the crossed product $A \rtimes_{\max} G = A \rtimes_r G$ has the WEP if and only if A has the WEP.

6. *Passing to subgroups*

Since the construction of $\rtimes_{\mathrm{inj}}$ gives a canonical way to associate a crossed product functor to any group G, it is interesting to see how it behaves with respect to passing to subgroups. Recall from [**11**, §6] that given a crossed product functor $\rtimes_\mu$, its *restriction* $\rtimes_{\mu|M}$ to a closed subgroup M is linked to $\rtimes_\mu$ via Green's imprimitivity theorem. To be more precise, let (A, α) be an M-algebra. Then, the induced G-algebra $(\mathrm{Ind}_M^G A, \mathrm{Ind}\alpha)$ is defined as

$$\mathrm{Ind}_M^G A := \left\{ F \in C_b(G, A) : \begin{array}{l} \alpha_h(F(sh)) = F(s) \ \forall s \in G, h \in M, \\ \text{and } (sM \mapsto \|F(s)\|) \in C_0(G/M) \end{array} \right\}$$

and $(\mathrm{Ind}\alpha_s(F))(t) = F(s^{-1}t)$ for $F \in \mathrm{Ind}_H^G A$ and $s, t \in G$. Green's imprimitivity theorem then provides a canonical $\mathrm{Ind}_M^G A \rtimes_{\max} G - A \rtimes_{\max} M$ Morita equivalence $X_M^G(A)$ which is functorial in A (see, e.g., [**14**, ch. 2] for a detailed discussion of this theory). Now, given any crossed product functor $\rtimes_\mu$ for G, the crossed product $\mathrm{Ind}_M^G A \rtimes_\mu G$ is a quotient of $\mathrm{Ind}_M^G A \rtimes_{\max} G$ by some ideal $I_\mu \subseteq \mathrm{Ind}_M^G A \rtimes_{\max} G$ which corresponds to a unique ideal $J_\mu \subseteq A \rtimes_{\max} M$ via the Rieffel correspondence such that the quotient $X_M^G(A)_\mu := X_M^G(A)/(X_M^G(A) \cdot I_\mu)$ becomes an

$$\mathrm{Ind}_M^G A \rtimes_\mu G - A \rtimes_{\mu|M} M := (A \rtimes_{\max} M)/J_\mu$$

equivalence bimodule. We show in [**11**, §6] that $(A, \alpha) \mapsto A \rtimes_{\mu|M} G$ is indeed a crossed product functor for M which inherits many important properties from the given functor $\rtimes_\mu$ for G. We want to show the following proposition.

PROPOSITION 6.1. *Let M be a closed subgroup of the locally compact group G. Then the restriction $\rtimes_{\mathrm{inj}(G)|M}$ to M of the maximal injective crossed product functor $\rtimes_{\mathrm{inj}(G)}$ for G coincides with the maximal injective crossed product $\rtimes_{\mathrm{inj}(M)}$ of M.*

For the proof we need the following lemma, which is a variant of Proposition 2.2.

LEMMA 6.2. *Suppose that M is a closed subgroup of G, and let (B, β) be an M-injective M-algebra. Let*

$$I_M^G(B) := \{F \in C_{ub}(G, A) : \beta_h(F(sh)) = F(s) \ \forall s \in G, h \in M\},$$

equipped with G-action $\left(I(\beta)_s(F)\right)(t) = F(s^{-1}t)$. Then, $(I_M^G(B), I(\beta))$ is G-injective.

Proof. The proof is almost identical to the proof of Proposition 2.2 and is left to the reader.   □

Proof of Proposition 6.1. Let A be any M-algebra, and let $\varphi : A \hookrightarrow B$ be an M-equivariant embedding of A into the M-injective algebra B. By functoriality of Green's

imprimitivity bimodule, we obtain a morphism of imprimitivity bimodules $\Psi : X_M^G(A) \to X_M^G(B)$ which is compatible with the *-homomorphisms $A \rtimes_{\max} M \to B \rtimes_{\max} M$ and $\operatorname{Ind}_M^G A \rtimes_{\max} G \to \operatorname{Ind}_M^G B \rtimes_{\max} G$ induced from the equivariant morphism $\varphi : A \hookrightarrow B$ and its induced form $\operatorname{Ind}\varphi : \operatorname{Ind}_M^G A \hookrightarrow \operatorname{Ind}_M^G B$. It follows that the kernel

$$J_\alpha := \ker(\varphi \rtimes_{\max} M)$$

is matched to the kernel $I_\alpha := \ker(\operatorname{Ind}\varphi \rtimes_{\max} G)$ via the Rieffel correspondence with respect to $X_M^G(A)$. Since B is M-injective, the quotient $(A \rtimes_{\max} M)/J_\alpha$ coincides with $A \rtimes_{\operatorname{inj}(M)} M$. On the other side, we observe that $\operatorname{Ind}_M^G B$ naturally embeds as a G-invariant ideal into $I_M^G(B)$, and therefore $\operatorname{Ind}_M^G A$ embeds into the G-injective algebra $I_M^G(B)$ via the composition

$$\operatorname{Ind}_M^G A \overset{\operatorname{Ind}\varphi}{\hookrightarrow} \operatorname{Ind}_M^G B \hookrightarrow I_M^G(B).$$

Therefore, the injective crossed product $\operatorname{Ind}_M^G A \rtimes_{\operatorname{inj}(G)} G$ is equal to the quotient $(\operatorname{Ind}_M^G A \rtimes_{\max} G)/I_{\operatorname{inj}(G)}$, where $I_{\operatorname{inj}(G)}$ is the kernel of the composition

$$\operatorname{Ind}_M^G A \rtimes_{\max} G \to \operatorname{Ind}_M^G B \rtimes_{\max} G \to I_M^G(B) \rtimes_{\max} G.$$

But, since $\rtimes_{\max}$ enjoys the ideal property, we see that the second map in this composition is faithful. Therefore, I coincides with the kernel of the first map, which is I_α. It follows that $A \rtimes_{\operatorname{inj}(M)} M$ is linked to $\operatorname{Ind}_M^G A \rtimes_{\operatorname{inj}(G)} G$ via the Rieffel correspondence for $X_M^G(A)$, which proves that $\rtimes_{\operatorname{inj}(M)} = \rtimes_{\operatorname{inj}(G)|M}$. $\qquad\square$

Remark 6.3. Let M be an *open* subgroup of the locally compact group G, and let H be a Hilbert space. Since $\mathcal{B}(H)$ is an injective C^*-algebra, it follows from Proposition 2.2 that $C_{ub}(G, \mathcal{B}(H))$ is an injective G-algebra. We claim that $C_{ub}(G, \mathcal{B}(H))$ is also M-injective with respect to the restriction of the translation action to M. To see this, we choose a section $\mathfrak{s} : M\backslash G \to G$ for the space of left M-cosets in G. Since M is open in G, the quotient $M\backslash G$ is a discrete space, and we obtain an M-equivariant isomorphism

$$\Psi : C_{ub}(G, \mathcal{B}(H)) \to C_{ub}(M, \ell^\infty(M\backslash G, \mathcal{B}(H))); \ \Psi(f)(m, \dot{g}) = f(m \cdot \mathfrak{s}(\dot{g})).$$

Since $\ell^\infty(M\backslash G, \mathcal{B}(H))$ is an injective von Neumann algebra (because it is type I), it follows from Proposition 2.2 that $C_{ub}(M, \ell^\infty(M\backslash G, \mathcal{B}(H)))$, and hence $C_{ub}(G, \mathcal{B}(H))$ is an injective M-algebra.

The above result does not hold without the assumption that M is open in G. For instance, it is not true if M is the trivial group and G is not discrete, because the C^*-algebra $C_{ub}(G, \mathcal{B}(H))$ is not injective if G is not discrete. Indeed, if $C_{ub}(G, \mathcal{B}(H))$ were injective, then it would be an AW^*-algebra [**5**, IV.2.1.7], and so would be its center $ZC_{ub}(G, \mathcal{B}(H)) \cong C_{ub}(G)$ [**19**, Theorem 2.4]. However, the spectrum of a commutative AW^*-algebra is an extremally disconnected space. Since G embeds as a (dense) open subset of the spectrum of $C_{ub}(G)$, this would imply that G itself is extremally disconnected. This is impossible, however, if G is not discrete, as it is locally compact, and compact subsets of extremally disconnected topological groups must be finite, by [**2**, Theorem 2.10].

For the maximal and reduced crossed products, it is well known that for any open subgroup M of a locally compact group G and any G-algebra A, we get an injective

embedding of the crossed product by M into the crossed product by G extending the canonical inclusion $\iota : C_c(M, A) \to C_c(G, A)$. From Remark 6.3, we immediately obtain the same property for the injective crossed product.

PROPOSITION 6.4. *Suppose that M is an open subgroup of the locally compact group G. Then, if (A, α) is a G-algebra, the inclusion $\iota : C_c(M, A) \hookrightarrow C_c(G, A)$ extends to a faithful inclusion $A \rtimes_{\mathrm{inj}} M \hookrightarrow A \rtimes_{\mathrm{inj}} G$.*

Proof. Let $\pi : A \to \mathcal{B}(H)$ be a faithful representation of A on a Hilbert space. Let $\tilde{\pi} : A \to B := C_{ub}(G, \mathcal{B}(H))$ be the map sending a to the function $[g \mapsto \pi(\alpha_{g^{-1}}(a))] \in B$. Then it follows from the above remark together with Lemma 3.2 that we get the following commutative diagram of maps:

$$
\begin{array}{ccc}
A \rtimes_{\mathrm{inj}} G & \longhookrightarrow & B \rtimes_{\mathrm{max}} G \\
\big\uparrow & & \big\uparrow \\
A \rtimes_{\mathrm{inj}} M & \longhookrightarrow & B \rtimes_{\mathrm{max}} M
\end{array}
$$

where the broken arrow exists and extends the inclusion $\iota : C_c(M, A) \to C_c(G, A)$ because of injectivity of all other maps in the diagram and commutativity on the level of $C_c(M, A)$. $\qquad\square$

7. *Questions*

(1) What is $\mathbb{C} \rtimes_{\mathrm{inj}} G$? The only information we currently have comes from Proposition 5.3 in general, plus Propositions 4.2 and 4.4 in some special cases. All of these results provide some evidence that $C^*_{\mathrm{inj}}(G)$ might be equal to $C^*_r(G)$ in general, but we have no strong feeling about this.

For a discrete group G, using the representation from the proof of Proposition 4.4, notice that $C^*_{\mathrm{inj}}(G)$ identifies with the C^*-algebra generated by the 'diagonal' representation

$$G \to \mathcal{B}(\ell^2(G)) \otimes_{\mathrm{max}} C^*_{\mathrm{max}}(G), \quad g \mapsto \lambda_g \otimes \delta_g.$$

It follows that $C^*_{\mathrm{inj}}(G) = C^*_r(G)$ if and only if this representation factors through $C^*_r(G)$. Is this always true? We know that it is true if G is exact or $C^*_{\mathrm{max}}(G)$ has the LLP. Similarly, we have that for any locally compact G, $C^*_{\mathrm{inj}}(G)$ agrees with the image of the natural map

$$C^*_{\mathrm{max}}(G) \to C_{ub}(G) \rtimes_{\mathrm{max}} G$$

induced by the unit inclusion $\mathbb{C} \to C_{ub}(G)$, and one can ask if this map always factors through the reduced group C^*-algebra.

(2) Does the LLP for $C^*_{\mathrm{max}}(G)$ imply exactness of G? Evidence for a positive answer is provided by Remark 4.10, and the fact that Corollary 4.8 shows that the 'best understood' examples of non-exact groups are such that $C^*_{\mathrm{max}}(G)$ does not have the LLP. Note that the converse is false by Remark 4.11.

(3) More generally, is $\rtimes_{\mathrm{inj}}$ always different from $\rtimes_r$ for non-exact groups? This would be implied by $C_{ub}(G) \rtimes_{\mathrm{max}} G \neq C_{ub}(G) \rtimes_r G$ for all non-exact groups, which matches the (scant) available evidence.

(4) Is $\rtimes_{\mathrm{inj}}$ exact? More generally, can a non-exact group admit a crossed product functor that is both exact and injective? It would also be interesting to compare the injective crossed product functor $\rtimes_{\mathrm{inj}}$ with the minimal exact crossed product functor $\rtimes_{\mathcal{E}}$ of [**13**]. Both functors agree for exact groups with the reduced crossed product functor, and so far we do not know of any example of a group G for which $\rtimes_{\mathrm{inj}} \neq \rtimes_{\mathcal{E}}$.

(5) Is $\rtimes_{\mathrm{inj}}$ a KLQ-functor? This is related to Proposition 3.8. More precisely, it is equivalent to the existence of a faithful homomorphism

$$A \rtimes_{\mathrm{inj}} G \hookrightarrow \mathcal{M}(A \rtimes_{\max} G \otimes C^*_{\mathrm{inj}}(G))$$

extending the representation $a\delta_g \mapsto a\delta_g \otimes \delta_g$. Notice that if $C^*_{\mathrm{inj}}(G) = C^*_r(G)$, then $\rtimes_{\mathrm{inj}}$ can only be a KLQ-functor if it equals $\rtimes_r$.

References

[1] C. Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.* **354**(10) (2002), 4153–4178.

[2] A. Arhangel'skii. A study of extremally disconnected topological spaces. *Bull. Math. Sci.* **1**(1) (2011), 3–12.

[3] P. Baum, E. Guentner and R. Willett. Expanders, exact crossed products, and the Baum–Connes conjecture. *Ann. K-theory* **1**(2) (2015), 155–208.

[4] A. Bhattacharya and D. Farenick. Crossed products of C^*-algebras with the weak expectation property. *New York J. Math.* **19** (2013), 423–429.

[5] B. Blackadar. *Operator Algebras: Theory of C^*-Algebras and Von Neumann Algebras*. Springer, Berlin, 2006.

[6] J. Brodzki, C. Cave and K. Li. Exactness of locally compact second countable groups. *Adv. Math.* **312** (2017), 209–233.

[7] N. Brown. Invariant means and finite representation theory of C^*-algebras. *Mem. Amer. Math. Soc.* **184**(865) (2006), vi+105pp.

[8] N. Brown and E. Guentner. New C^*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.* **45**(6) (2013), 1181–1193.

[9] N. Brown and N. Ozawa. *C^*-Algebras and Finite-Dimensional Approximations (Graduate Studies in Mathematics, 88)*. American Mathematical Society, Providence, RI, 2008.

[10] A. Buss and S. Echterhoff. Maximality of dual coactions on sectional C^*-algebras of Fell bundles and applications. *Studia Math.* **229**(3) (2015), 233–262.

[11] A. Buss, S. Echterhoff and R. Willett. Exotic crossed products. *Operator Algebras and Applications (The Abel Symposium)*. Eds. T. M. Carlsen, N. S. Larsen, S. Neshveyev and C. Skau. Springer, Cham, 2015, pp. 61–108.

[12] A. Buss, S. Echterhoff and R. Willett. Exotic crossed products and the Baum–Connes conjecture. *J. Reine Angew. Math.* **740** (2018), 111–159.

[13] A. Buss, S. Echterhoff and R. Willett. The minimal exact crossed product. *Documenta Math.* **23** (2018), 2043–2077.

[14] J. Cuntz, S. Echterhoff, X. Li and G. Yu. *K-Theory for Group C^*-Algebras and Semigroup C^*-Algebras (Oberwolfach Seminars)*. Birkhäuser, Cham, 2017.

[15] S. Echterhoff, S. Kaliszewski, J. Quigg and I. Raeburn. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.* **180** (2006), viii+169 pp.

[16] E. Guentner, R. Tessera and G. Yu. Discrete groups with finite decomposition complexity. *Groups Geom. Dyn.* **7**(2) (2013), 377–402.

[17] M. Hamana. Injective envelopes of C^*-dynamical systems. *Tohoku Math. J.* **37** (1985), 463–487.

[18] S. Kaliszewski, M. Landstad and J. Quigg. Exotic group C^*-algebras in noncommutative duality. *New York J. Math.* **19** (2013), 689–711.

[19] I. Kaplansky. Projections in Banach algebras. *Ann. of Math. (2)* **53** (1951), 235–249.

[20] D. Osajda. Small cancellation labellings of some infinite graphs and application. *Preprint*, 2014, arXiv:1406.5015.

[21] N. Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), 691–695.

[22] N. Ozawa. About the QWEP conjecture. *Internat. J. Math.* **15** (2004), 501–530.

[23] N. Ozawa. There is no separable universal II_1-factor. *Proc. Amer. Math. Soc.* **132**(2) (2004), 487–490.

[24] J. Roe and R. Willett. Ghostbusting and property A. *J. Funct. Anal.* **266**(3) (2014), 1674–1684.

[25] Z.-J. Ruan and M. Wiersma. On exotic group C^*-algebras. *J. Funct. Anal.* **271**(2) (2016), 437–453.

[26] A. Thom. Examples of hyperlinear groups without the factorization property. *Groups Geom. Dyn.* **4** (2010), 195–208.

[27] M. Wiersma. L^p-Fourier and Fourier–Stieltjes algebras for locally compact groups. *J. Funct. Anal.* **269**(12) (2015), 3928–3951.

[28] M. Wiersma. Constructions of exotic group C^*-algebras. *Illinois J. Math.* **60**(3–4) (2016), 655–667.

[29] R. Willett and G. Yu. Higher index theory for certain expanders and Gromov monster groups I. *Adv. Math.* **229**(3) (2012), 1380–1416.

[30] R. Willett and G. Yu. Higher index theory for certain expanders and Gromov monster groups II. *Adv. Math.* **229**(3) (2012), 1762–1803.

[31] R. Willett and G. Yu. Geometric property (T). *Chin. Ann. Math. Ser. B* **35**(5) (2014), 761–800.