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THREE REMARKABLE GRAPHS

NORMAN BIGGS

1. Introduction. In the development of any mathematical theory it is often advisable to test the known theoretical results in particular cases, for in this way we are able to judge the state of the subject, its strengths and its weaknesses. It is the purpose of this paper to put forward three remarkable graphs which can be used to indicate the progress being made in research in graph theory. We shall undertake three tasks – the construction of the graphs themselves, the investigation of their properties (insofar as the theory allows), and the indication of possible lines for further enquiry.

We begin by introducing the cast. First, we have the graph discussed by Petersen [**12**] in 1891, as an example of a trivalent graph not having an edge 3-colouring. This graph has already achieved eminence among graph-theorists and is a standard example or counter-example. Next, there is the trivalent graph discovered by Coxeter and investigated by Tutte [**16**] in 1959; this graph has 28 vertices, and its group of automorphisms is transitive on arcs of three consecutive edges. Our third graph was found by D. H. Smith and the present author during their investigation of trivalent distance-transitive graphs in 1969. In the course of that work a computer was programmed to find possible intersection matrices for such graphs, and among the few possibilities which appeared, a matrix which would correspond to a graph with 102 vertices was prominent. The existence of a graph realizing this matrix was then inferred from certain group-theoretical investigations, as explained in Section 4.

When the list of trivalent distance-transitive graphs was completed [**5**], it appeared that the three graphs which we have just introduced are the only ones whose automorphism group acts primitively on the vertices (with trivial addition of the complete graph K_4). For this reason it was desirable to have a more uniform description of them, and this was found and published in [**5**]. Later, we discovered that J. H. Conway had independently constructed our third graph in this way.

It is necessary to say a few words about K_4. This graph could be considered as the first in a series of four, since its properties coincide very closely with those of the three remarkable graphs, as the information in the Table of Results shows. However, it is hardly admissible to speak of it as a remarkable graph, and so we shall not discuss it explicitly, but merely include it in our tabulation of results for the sake of interest.

The three graphs to be discussed have 10, 28, and 102 vertices. The first is small enough for us to be able to answer any question about it by direct and

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straightforward calculation, whereas the second is in a range where subtle and ingenious use of symmetry properties is usually required. For our third graph, even if we use its remarkably large group of automorphisms, there is usually little prospect at the moment of a neat and conclusive result. This graph remains an enigma – which is a fair justification for the claim that it should be used to test the state of our knowledge.

2. Description of the graphs. Petersen's graph is often **[10**, p. 89] drawn as five edges with one set of five ends joined by a pentagon and the remaining five ends joined by a pentagram (that is, the star pentagon $5|2$). Coxeter's graph has a similar description, as seven copies of a "Y-configuration" with free ends joined by the heptagons $7|1$, $7|2$ and $7|4$. Finally the graph with 102 vertices is constructed by taking seventeen copies of an "H-configuration" and joining the free ends by the 17-gons $17|1$, $17|2$, $17|4$, $17|8$, in the manner shown in Figure 1.

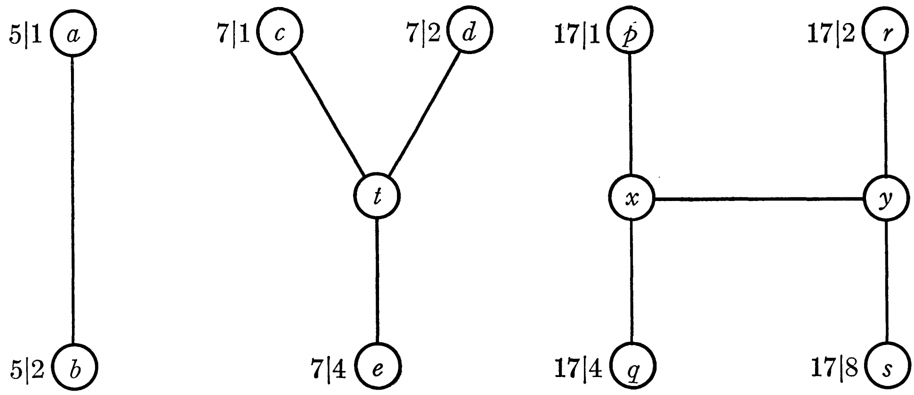

FIGURE 1

We shall use the mnemonic letters **I**, **Y**, **H** for the three graphs and in Figure 1 we display the notation to be employed for their vertices. For example **Y** has vertices c_i, d_i, e_i, t_i $(i = 1, 2, \ldots 7)$, where for each suffix i the vertex t_i is joined to c_i, d_i, e_i, the vertex c_i is joined also to c_{i-1} and c_{i+1}, the vertex d_i is joined also to d_{i-2} and d_{i+2}, and the vertex e_i is joined also to c_{i-4} and e_{i+4}, all suffixes being taken modulo 7.

We shall employ the notation u/v for the edge joining vertices u and v; thus $a1/b1$ is an edge of **I**.

Since each graph is distance-transitive there is an alternative, important, and useful, method of drawing it. We take any vertex v, (the choice is immaterial since all vertices are equivalent under the action of the automorphism group), and arrange the remaining vertices in circles centred on v, the jth circle $\Delta_j(v)$ containing all vertices u for which the distance $d(u, v)$ is j. Then an edge incident with a vertex in $\Delta_j(v)$ joins that vertex to a vertex in one of

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the circles $\Delta_{j-1}(v)$, $\Delta_j(v)$, $\Delta_{j+1}(v)$, and at each vertex in $\Delta_j(v)$ there is the same number of edges going to each of these circles. These numbers are the entries of the intersection matrix [**2**] of the graph, and they are given for the graphs **I**, **Y**, and **H** in the Table of Results. The three rows displayed there are the three main diagonals of the intersection matrix, in accordance with notation employed on p. 18 of [**2**].

Now the drawings of **I** and **Y** as described in the previous paragraph are practical possibilities, and the results are shown in Figure 2. For the graph **H**, the full drawing is too complicated, but the idea is valuable and the partial drawing given in Figure 3 will prove useful. In this drawing the circles are replaced by levels, and we omit the edges which join Δ_3 to Δ_4, Δ_4 to Δ_5, Δ_5 to Δ_6 and Δ_6 to Δ_7. However, in order that the graph may be reconstructed from our diagram, we label the vertices according to the scheme indicated in Figure 1.

From the intersection matrices and our diagrams we see that the girths of **I**, **Y**, **H** are $g = 5, 7, 9$ respectively, and their diameters are $d = 2, 4, 7$. Of course, Petersen's graph is the trivalent 5-cage; that is, it has the least number of vertices possible for a graph of valency 3 and girth 5. However **Y** is not the trivalent 7-cage (that is the McGee graph [**17**, p. 77] with 24 vertices), and **H** is not the trivalent 9-cage, for there is a graph of valency 3 and girth 9 with 60 vertices, and there may be even smaller ones.

3. Spectra and complexities. The *spectrum* of graph is the set of eigenvalues of its adjacency matrix A. In the case of a distance-transitive graph of diameter d there are just $d + 1$ distinct eigenvalues, which are also the eigenvalues of the intersection matrix B; the multiplicities of these numbers as eigenvalues of A can be calculated from a formula involving only the eigenvectors of B [**3**, p. 94]. Thus complete information about the spectra of **I**, **Y** and **H** can be calculated simply from their intersection numbers, and this information is shown in our Table of Results.

The *complexity* $\kappa(\Gamma)$ of a graph Γ is the number of spanning trees which it possesses. A result of Kirchhoff [**10**, p. 152] tells us that the complexity is the value of each cofactor in a certain matrix, which in the case of a regular graph of valency k is the matrix $kI - A$. That is

$$\operatorname{adj}(kI - A) = \kappa J,$$

where A is the adjacency matrix and J is the matrix each of whose entries is 1. Now, if $\phi(z) = \det(zI - A)$ is the characteristic polynomial of A, the preceding equation tells us that $\phi'(k) = n\kappa$, where n the size of A (the number of vertices of Γ). Also, the eigenvalues of A are k, with multiplicity 1, and $\lambda_1, \ldots, \lambda_r$ with multiplicities $m_1, m_2 \ldots m_r$, say. Thus

$$(k - \lambda_1)^{m_1}(k - \lambda_2)^{m_2} \ldots (k - \lambda_r)^{m_r} = n\kappa.$$

Applying this to the three graphs under consideration we find

$$\kappa(\mathbf{I}) = 2^4 \cdot 5^3 = 2000,\ \kappa(\mathbf{Y}) = 2^{18} \cdot 7^5, \text{ and } \kappa(\mathbf{H}) = 2^{18} \cdot 3^{32} \cdot 17^{15}.$$

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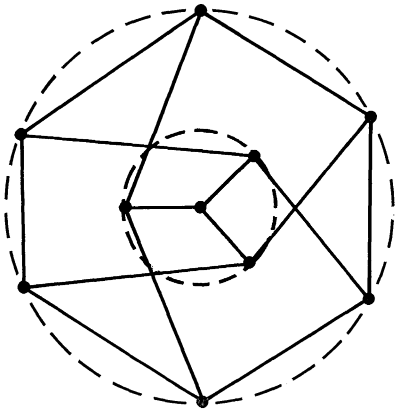

FIGURE 2(i). Petersen's graph

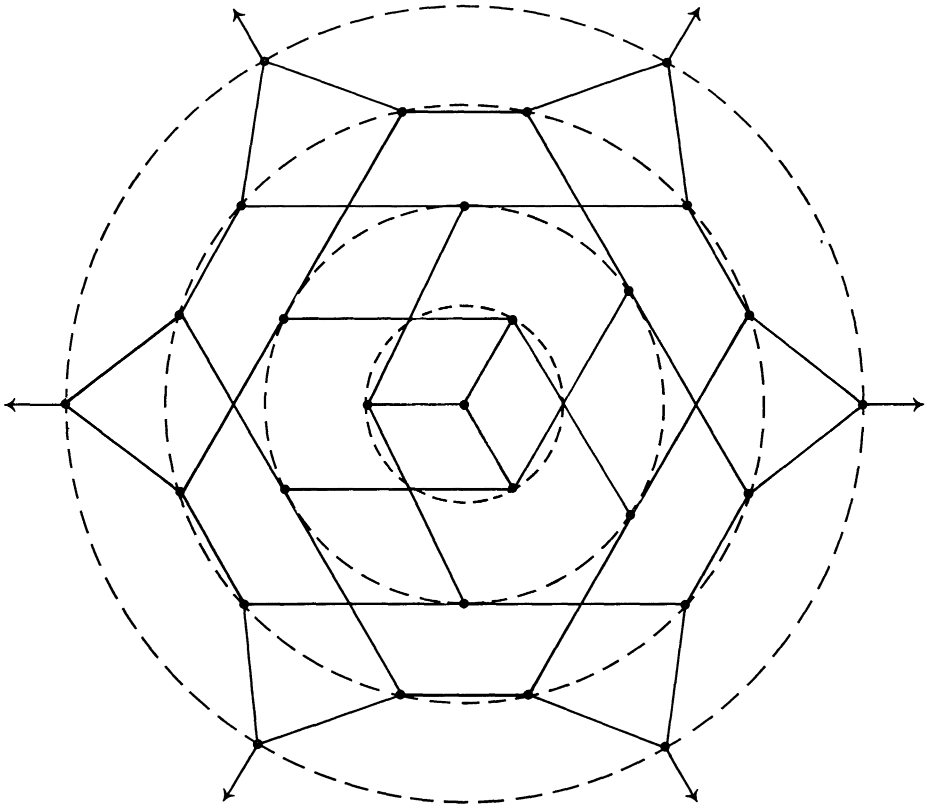

FIGURE 2(ii). Coxeter's graph

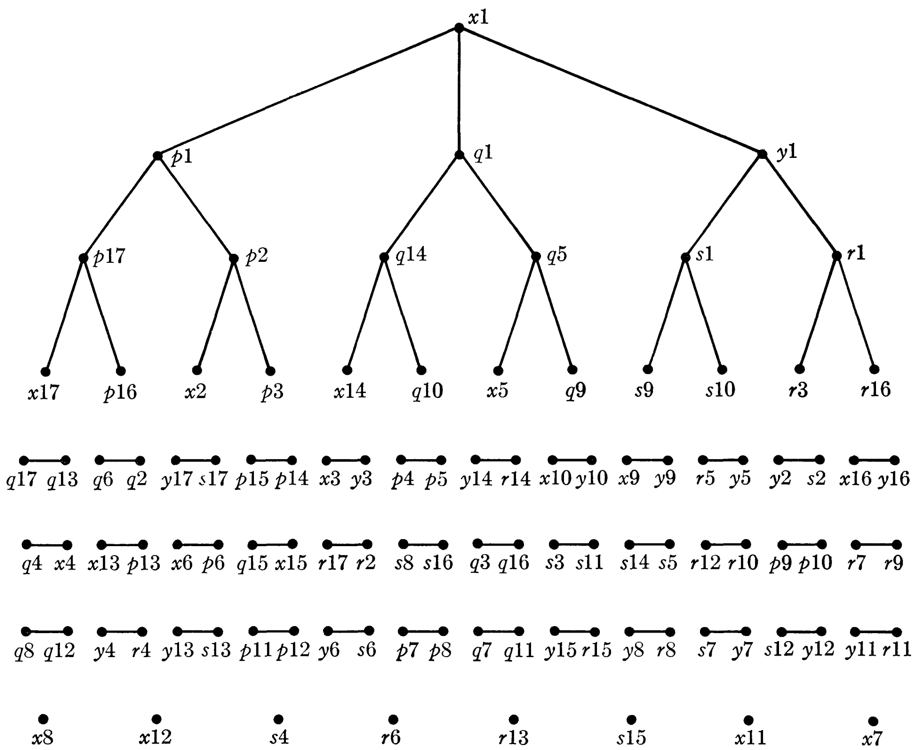

FIGURE 3. The graph **H**

4. Automorphism groups. An unexpected and remarkable analogy among the graphs K_4, **I**, **Y**, **H** is found in their groups of automorphisms. The full group of automorphisms contains PSL(2, 3), PSL(2, 5), PSL(2, 7) and PSL(2, 17) respectively – in the first three cases as a subgroup of index two, and in the last case as the whole group. It might be thought that there should be a uniform method of deriving this fact, but no such method is known to the author.

The group of K_4 is clearly the symmetric group S_4 which contains the alternating group A_4, isomorphic to PSL(2, 3). The group of **I** is S_5, as was proved by Frucht [**9**] in 1937. This is most easily seen by means of an alternative construction of **I** in which the vertices are the ten unordered pairs of five objects and the edges join disjoint pairs. The symmetric group S_5 contains as a subgroup of index two the alternating group A_5, isomorphic to PSL(2, 5).

For the other two graphs it is most convenient to rely on the results of W. J. Wong [**21**]. He lists the primitive groups in which the stabilizer has an orbit of length 3, and in his list we find the group PGL(2, 7) with stabilizer the dihedral group D_{12}, and the group PSL(2, 17) with stabilizer S_4. The degrees of these permutation groups are 28 and 102 respectively, and from these

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transitive permutation representations we can construct graphs, as explained, for example in [**3**, p. 83]. If the orbits on which we base our construction are the orbits of length 3, we obtain two trivalent graphs with automorphism groups PGL(2, 7) and PSL(2,17) respectively. These graphs can actually be constructed by a variant of the well-known coset enumeration technique and they turn out to be the graphs **Y** and **H**. The existence of the graph **H** was verified in this way (by D. H. Smith) before its presentation in the format of Figure 1 was known to the author.

Trivalent distance-transitive graphs have automorphism groups which are certainly transitive on the set of edges, and so the results of Tutte [**17**, p. 63] on *s-regularity* apply. That is, such a graph has a group of automorphisms which is regular on the set of arcs of s successive edges, for $1 \leqq s \leqq 5$. Consequently the order of a vertex-stabilizer is $3 \cdot 2^{s-1}$ and the order of a stabilizer of a directed edge is 2^{s-1}; the abstract structure of these groups is also determined [**21**, p. 238]. In our case, the graphs **I**, **Y**, **H**, have $s = 3, 3, 4$, respectively, so the vertex-stabilizers are D_{12}, D_{12}, S_4, and the directed-edge-stabilizers are $(\mathbf{Z}_2)^2$, $(\mathbf{Z}_2)^2$, D_8.

5. Edge-colourings and Hamiltonian circuits. It is clear that at least three colours are necessary for an edge-colouring of a trivalent graph, and in fact four colours are always sufficient, following the general result of Vizing [**20**]. The problem of characterizing trivalent graphs which do not have an edge-3-colouring is one of the hardest questions in graph theory – the four colour conjecture is equivalent to the statement that every bridgeless planar trivalent graph has an edge-3-colouring. Petersen's graph is the classic example of a bridgeless trivalent graph needing four edge colours and one might hope that one of the graphs **Y** and **H** would share this property. Unfortunately this hope is unfounded.

Our studies in this area will be based on an analysis of the *1-factors* (or complete matchings) of the graphs; Petersen's theorem [**12**] shows that every bridgeless trivalent graph has a 1-factor, that is, a set of edges covering each vertex exactly once. The complement of a 1-factor in a trivalent graph is a 2-factor, or set of disjoint circuits. If this 2-factor consists entirely of even circuits then we may construct an edge-3-colouring by colouring the edges of the 1-factor with one colour and the edges of the circuits alternately with two other colours. Also we notice that the graph is Hamiltonian if and only if there is a 1-factor whose complement consists of a single circuit. Thus a complete description of the 1-factors answers all our questions on this topic.

If $\mathscr{M}$ is a 1-factor in one of the graphs **I**, **Y**, **H**, then we shall adopt the following notation. We consider a presentation of the graph in distance-transitive format (Figures 2 and 3) with respect to some chosen vertex, and let μ_{ij} denote the number of edges in $\mathscr{M}$ which join a vertex in Δ_i to one in Δ_j. We refer to the edges of $\mathscr{M}$ as *blue* edges, and to those of the complementary 2-factor as *red* edges.

In the case of Petersen's graph the description of all 1-factors is very simple. We must have $\mu_{01} = 1$, $\mu_{12} = 2$, and consequently $\mu_{22} = 2$. The hexagon formed by the edges joining two vertices in Δ_2 must therefore have two blue edges and four red edges. The two blue edges cannot be adjacent, and it is easy to check that they cannot be opposite in the hexagon, thus the edges of the hexagon must be blue-red-blue-red-red-red, and there are just six possibilities. Each possibility forces a unique 1-factor in the graph and these 1-factors are all equivalent under the group of automorphisms. In other words, Aut **I** acts transitively on the set of 1-factors of **I**, and we see that stabilizer of a particular 1-factor is a group of order 20. The complement of any 1-factor is easily found; it consists of two circuits of length 5. Thus the complements of all 1-factors have this form, and Petersen's graph is not Hamiltonian, nor does it have an edge-3-colouring.

A similar analysis applies to Coxeter's graph **Y**. We must have $\mu_{01} = 1$, $\mu_{12} = 2$, $\mu_{23} = 4$, and for the three remaining quantities μ_{33}, μ_{34}, μ_{44} there are the equations

$$\begin{aligned} 4 + 2\mu_{33} + \mu_{34} &= 12, \\ \mu_{34} + 2\mu_{44} &= 6. \end{aligned}$$

Putting $\mu_{44} = \gamma$ $(0 \leqq \gamma \leqq 3)$, we have $\mu_{34} = 6 - 2\gamma$, $\mu_{33} = \gamma + 1$.

This information reduces the enumeration of the 1-factors of **Y** to a straightforward, if tedious, piece of work. There are about seven cases to be considered (the exact number depends upon the precise system adopted) and the result is that **Y** possesses just 84 1-factors. Examination of the cases will show that in every possibility the complement of the 1-factor consists of two 14-gons, and so we can verify in this way the known result [**16**] that **Y** is non-Hamiltonian. However, there is another way of proving this fact, as a consequence of a very much stronger and more surprising property of the 1-factors of **Y**: we show that the automorphism group of **Y** (like that of **I**) acts transitively on the set of 1-factors.

Let $\mathscr{M}_0$ denote the particular 1-factor of **Y** consisting of those edges marked by bold lines in Figure 4. Then $\mathscr{M}_0$ contains all three edges ($c4/c5$, $d2/d7$, $e3/e6$) which are "extreme" with respect to $t1$, and so if an automorphism ϕ of **Y** fixes $\mathscr{M}_0$ (setwise) then $\mathscr{M}_0$ must also contain the three edges extreme with respect to $\phi(t1)$. These edges are easily found for all vertices of **Y** and are as follows:

Vertex:	t, i	c, i
Extreme edges:	$c, i-3/c, i+3$	$t, i-3/d, i-3$
	$d, i-1/d, i+1$	$t, i+3/d, i+3$
	$e, i-2/e, i+2$	$e, i-2/e, i+2$
Vertex:	d, i	e, i
Extreme edges:	$t, i-1/e, i-1$	$t, i-2/c, i-2$
	$t, i+1/e, i+1$	$t, i+2/c, i+2$
	$c, i-3/c, i+3$	$d, i-1/d, i+1$.

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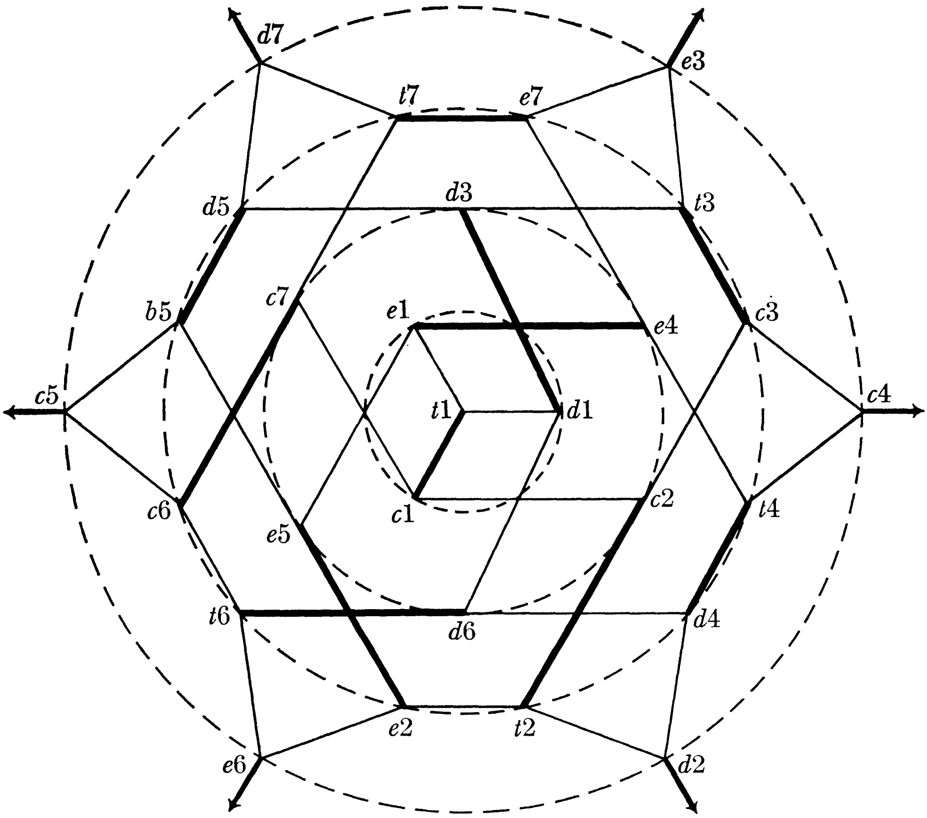

FIGURE 4

The 1-factor $\mathscr{M}_0$ in **Y**

$c1/t1$	$e1/e4$	$d1/d3$	$e2/e5$	$d6/t6$	$c6/c7$	$c2/t2$
$t7/e7$	$t3/c3$	$t4/d4$	$t5/d5$	$d2/d7$	$e3/e6$	$c4/c5$

Checking this list against the edges of $\mathscr{M}_0$ it appears that ϕ must take $t1$ to either $t1$ or $c1$, that is, ϕ must fix the edge $t1/c1$. Now we know that such automorphisms form a group of order 8, and one such automorphism is that which is induced by the permutation (1)(27)(36)(45) of the numerical parts of the vertex-labels. This automorphism clearly does not fix $\mathscr{M}_0$ and so the stabilizer of $\mathscr{M}_0$ has order at most 4. But the following automorphism of Y fixes $\mathscr{M}_0$ and has order 4

$$\theta = (t1\ c1)\ (t2\ d3\ c6\ e4)\ (d1\ c7\ e1\ c2)\ (d4\ d7\ t5\ c4)$$
$$(e3\ e6)\ (d6\ t7\ e5\ c3)\ (t3\ t6\ e7\ e2)\ (t4\ d2\ d5\ c5);$$

and so $\mathscr{M}_0$ has exactly $336/4 = 84$ distinct images under the action of Aut **Y**.

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From our enumerative result we conclude that these are all 1-factors of **Y** and so Aut **Y** is transitive on the 1-factors.

The author surmises that graphs which share this property with **I** and **Y** are rare. In a sense, these are graphs for which Petersen's theorem on the existence of 1-factors is "barely true".

The fact that each 1-factor of **I** has complement consisting of 2 5-gons, whereas each 1-factor of **Y** has complement consisting of 2 14-gons, suggests that each 1-factor of H might have a complementary 2-factor consisting of 2 51-gons, in which case **H** would have no edge-3-colouring. Although the complete description of the 1-factors of **H** is not known, we can do enough to squash the preceding suggestion; in fact **H** is Hamiltonian!

A Hamiltonian circuit in **H** can be found by considering a 1-factor $\mathscr{M}$ in **H**. We begin with the same calculations as for **I** and **Y**, giving $\mu_{01} = 1$, $\mu_{12} = 2$, $\mu_{23} = 4$, $\mu_{34} = 8$, and for μ_{44}, μ_{45}, μ_{55}, μ_{66} and μ_{67} the equations

$$\begin{aligned} 8 + 2\mu_{44} + \mu_{45} &= 24, \\ \mu_{45} + 2\mu_{55} + \mu_{56} &= 24, \\ \mu_{56} + 2\mu_{66} + \mu_{67} &= 24, \\ \mu_{67} &= 8. \end{aligned}$$

Take $\mu_{44} = \beta$, $\mu_{66} = \alpha$ so that $\mu_{45} = 16 - 2\beta$, $\mu_{55} = \alpha + \beta - 4$, $\mu_{56} = 16 - 2\alpha$.

Now if we suppose $\beta = 0$, the $1 + 2 + 4 + 8$ blue edges joining Δ_0 to Δ_1 to Δ_2 to Δ_3 to Δ_4 can be chosen in only a few essentially different ways. As luck would have it one of the first choices investigated was that of the following blue edges:

$$\begin{array}{llllllll} x1/q1 \\ p1/p2 & y1/s1 \\ p17/p16 & q14/x14 & q5/q9 & r1/r3 \\ x17/y17 & x2/q2 & p3/x3 & q10/x10 & x5/y5 & s9/y9 & s10/y10 & r16/y16. \end{array}$$

This choice, together with $\beta = 0$, forces a unique 1-factor in **H**, whose complement is a Hamiltonian circuit.

It seems unlikely that Aut **H** is transitive on the 1-factors of **H**.

6. Vertex colourings and chromials. In the case of vertex colourings our fundamental existence theorem is the result of Brooks [**6**] which implies that every trivalent graph except K_4 has a vertex-3-colouring. Since each of the graphs **I**, **Y**, **H** has odd girth, none of them has a vertex-2-colouring.

We begin our study of vertex-colourings of Petersen's graph by remarking that the maximum number of independent vertices (sometimes called the *point independence number* β_0, or *coefficient of internal stability*) is 4. For if an independent set contains a vertex v the remainder of its vertices must lie in $\Delta_2(v)$ and there can be only three such independent vertices, since $\Delta_2(v)$ spans a hexagon. Also, it is clear that no two sets of four independent vertices are disjoint and so in a vertex-3-colouring of **I** the colour-classes must have 4, 3, and 3 members. There are $10 \cdot 2/4 = 5$ independent sets of four vertices, and for each of these the remaining six vertices can be partitioned into two threes in four different ways, so that the number of vertex-3-colourings is 20, or 120 if we take account of permutations of the colours.

To answer all questions on the enumeration of vertex-colourings of **I** we may resort to calculation of the chromatic polynomial (*chromial*) $C(\mathbf{I}; z)$, whose value when z is a positive integer n is the number of vertex-colourings of **I** with at most n colours, taking account of permutations of the colours. This polynomial can be computed by hand using the contraction and reduction process described by Read [**13**]. The result is

$$C(\mathbf{I}; z) = z(z-1)(z-2)(z^7 - 12z^6 + 67z^5 - 230z^4 + 529z^3 - 814z^2 + 775z - 352),$$

whose value when $z = 3$ is 120, in agreement with our direct calculation.

Another approach to the chromial is via the two-variable polynomial defined by Tutte [**18**] in terms of the spanning trees of the graph. For any graph Γ we set

$$\chi(\Gamma; x, y) = \sum t_{ij} x^i y^j$$

where t_{ij} is the number of spanning trees of Γ which have internal activity i and external activity j; although the definitions initially depend on an ordering of the edges of Γ, the numbers t_{ij} are independent of this ordering. For Petersen's graph this polynomial can be found either by hand in a manner similar to that used for the chromial, or by using a computer algorithm developed by D. A. Sands [**14**] which relies on matrix operations involving the incidence matrix.

The matrix (t_{ij}) for **I** is found to be

$$\begin{bmatrix} 0 & 36 & 84 & 75 & 35 & 9 & 1 \\ 36 & 168 & 171 & 65 & 10 & & \\ 120 & 240 & 105 & 15 & & & \\ 180 & 170 & 30 & & & & \\ 170 & 70 & & & & & \\ 114 & 12 & & & & & \\ 56 & & & & & & \\ 21 & & & & & & \\ 6 & & & & & & \\ 1 & & & & & & \end{bmatrix}.$$

From this we check immediately that $\kappa(\mathbf{I}) = \chi(\mathbf{I}; 1, 1) = 2000$, as we found from our knowledge of the spectrum of **I**. We can also check our calculation of the chromial, since it is known [**19**] that

$$C(\mathbf{I}; z) = -z\chi(\mathbf{I}; 1 - z, 0).$$

But χ also gives a great deal of new information, because of its link with the rank polynomial:

$$\xi^{n-1}\chi(\Gamma;\, \xi^{-1}+1, \eta+1) = \sum r_{ij}\xi^i\eta^j$$

where r_{ij} is the number of edge-generated subgraphs of Γ with *cycle-rank* (edges minus vertices plus components) equal to i and *coboundary-rank* (vertices minus components) equal to j. From the matrix (t_{ij}) for **I** we deduce that the matrix (r_{ij}) for **I** is

$$\begin{bmatrix} 1 & 15 & 105 & 455 & 1565 & 2991 & 4875 & 5805 & 4780 & 2000 \\ & & & & 12 & 130 & 630 & 1725 & 2665 & 2172 \\ & & & & & & 30 & 240 & 816 & 1230 \\ & & & & & & & 14 & 135 & 455 \\ & & & & & & & & 10 & 105 \\ & & & & & & & & & 15 \\ & & & & & & & & & 1 \end{bmatrix}.$$

Thus we have a complete enumeration of subgraphs, giving very full information about the structure of **I** and checking with several other computations in preceding sections.

When we turn to the graphs **Y** and **H** we discover the limitations of our present knowledge in this field. For not only are the chromials of these graphs unknown, but even the number of vertex-3-colourings is incredibly difficult to calculate. For the graph **Y** it is not hard to see that $\beta_0 = 12$; however this information is not sufficiently restrictive to make the explicit listing of vertex-3-colourings a practical possibility.

7. Imbeddings. An imbedding of a simple graph (that is, one which has no loops or multiple edges) in an *orientable* surface may be represented combinatorially by giving a cyclic ordering of the vertices adjacent to each vertex (see, for example, [**4**]). Such a set of cyclic permutations $\{\rho_v\}$, one for each vertex v of the graph, is called a *rotation* on the graph. In the case of a trivalent graph there are just two possibilities for each ρ_v, and so the number of possible rotations is 2^n, and this the number of ways of drawing the graph on an orientable surface.

Some reduction in this large number of possibilities can be made when the graph Γ under consideration has a high degree of symmetry. For if $\rho = \{\rho_v\}$ is a rotation on Γ and θ is an automorphism of Γ, then we may define a new rotation $\rho^{(\theta)}$ by the rule

$$\rho^{(\theta)}_{\theta v}(\theta u) = \theta\rho_v(u), \qquad \text{or } \rho^{(\theta)}_{\theta v} = \theta\rho_v\theta^{-1}.$$

In other words we have an action of Aut Γ on the set of rotations on Γ, given by $(\theta, \rho) \to \rho^{(\theta)}$.

If $\rho^{(\theta)} = \rho$ we shall say that θ is an *automorphism* of the *map* (Γ, ρ); thus the group of automorphisms of (Γ, ρ) is the stabilizer of ρ in the group of automorphisms of Γ. Further, the number of rotations equivalent to ρ under the action of Aut Γ is the index of the group of (Γ, ρ) in the group of Γ.

For Petersen's graph **I** there are $2^{10} = 1024$ possible rotations, which fall into a mere 12 classes under the action of Aut **I**. The minimum genus g_0 of **I** is at least 1 (from Euler's formula, using the fact that the girth is 5), and there is a single class of 40 rotations with genus 1. These maps have automorphism group $\mathbf{Z}_3$ and faces of size 5, 5, 5, 6, 9. An example is shown in Figure 5.

The 2^{28} possible rotations on **Y** can likewise be arranged in classes, but the number of classes (of the order of 2^{19}) is too large to be amenable. However the orientable imbeddings of this graph do have at least one noteworthy feature, which we now discuss. Using the fact that the girth of **Y** is 7, Euler's formula tells us that the minimum genus g_0 is at least 2, and that $g_0 = 2$ if and only if there is an imbedding whose 12 faces are each 7-gons. Now **Y** has 24 7-gons and extrapolation from known results about the complete graphs, bipartite complete graphs, cubes, and so forth, might lead us to suppose that, when a graph has the required number of "small" circuits, the lower bound for its genus given by Euler's formula (and its girth) is attained. However **Y** is a counter-example for this supposition, since it is quite easy to see that we cannot even choose the three faces at any one vertex so that each is a 7-gon. In fact the best we can do is an imbedding of genus 3, whose 10 faces are of size 7, 7, 7, 7, 7, 7, 8, 8, 8, 18. This is illustrated in Figure 6. The author conjectures that there is only one class of genus 3 imbeddings of **Y**.

We end this section with a remark about *non-orientable* imbeddings. It is well-known that **I** can be imbedded in the projective plane as a map of six 5-gons, for example by identifying antipodal vertices of a dodecahedron. But our comments show **Y** has no similar imbeddings with 12 7-gons, and we conjecture that **H** has no such imbeddings with 36 9-gons.

8. Other remarkable graphs? A natural question which arises from this paper is: Are there any other graphs which can properly be called remarkable? The graphs we have considered are distance-transitive and one might begin by insisting that any remarkable graph should have this property. Now the results of D. H. Smith show that some distance-transitive graphs are more remarkable than others, for such graphs in which the group of automorphisms

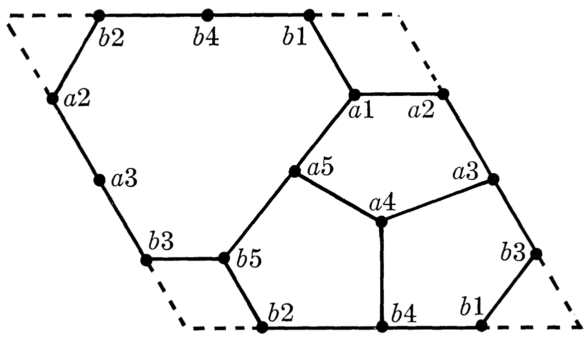

FIGURE 5

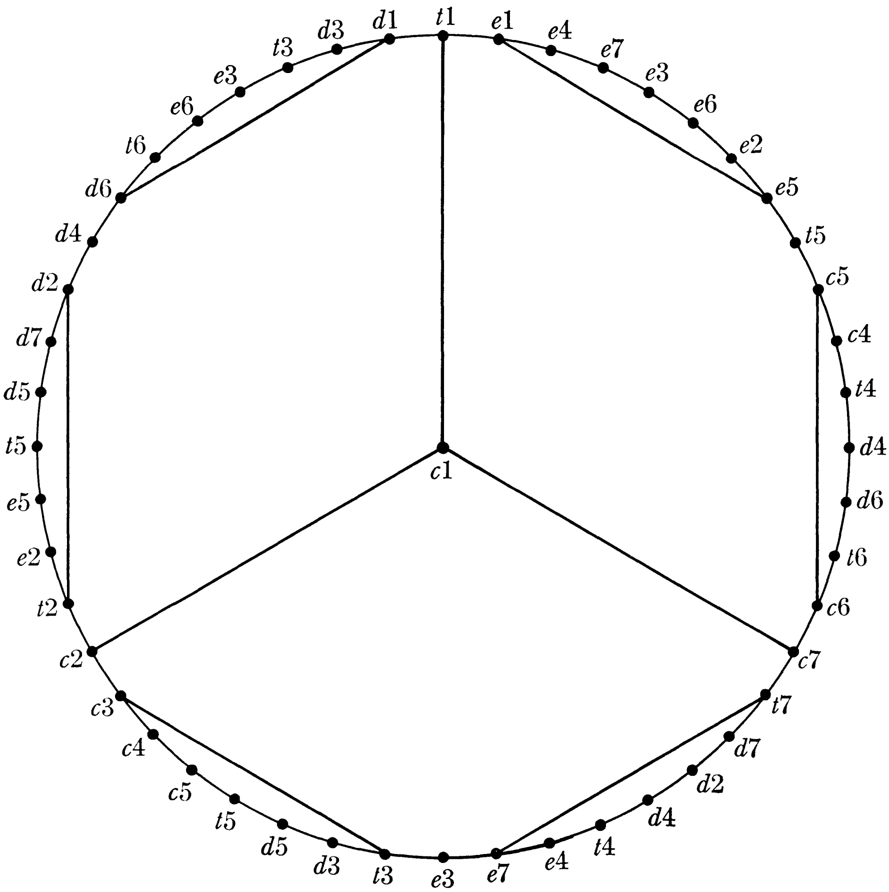

FIGURE 6

acts imprimitively on the vertices must be either *bipartite* or *antipodal* [**15**, p. 554], which means that the imprimitive distance-transitive graphs are somewhat easier to analyse than the primitive ones. To be sure, many interesting and famous graphs, such as Tutte's 8-cage and Heawood's graph are to be found among the imprimitive distance-transitive graphs, but we can give two other reasons why these are not truly remarkable. The first is that these particular graphs are members of infinite families, and the second is that a search for the intersection matrices of distance-transitive graphs shows that the bipartite and antipodal types are noticeably more common than the primitive type.

Thus we can justify the claim that the primitive distance-transitive graphs are worthy of special study. If we exclude the complete graphs we are left with the class of graphs called *automorphic* in [**5**], and the subjects of this paper are then the only trivalent automorphic graphs. It is thought that the 4-valent

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automorphic graphs are almost certainly few in number, and there is a strong suspicion that the number of automorphic graphs with any given valency is finite.

Few automorphic graphs are known, but among those that have been studied there are two relatively small ones which seem especially notable. These are the graph of Hoffman and Singleton [**11**; **1**] with 50 vertices, and the graph with 266 vertices described in [**7**, p. 223]. Hoffman and Singleton's graph is the *Moore graph* of diameter 2 and valency 7, whereas Petersen's graph is the Moore graph of diameter 2 and valency 3. A result of Damerell [**8**] shows that these are the only Moore graphs for any diameter greater than 1, and any valency greater than 2, with the possible exception of diameter 2 and valency 57. If this is not enough to make Hoffman and Singleton's graph remarkable, we may add that the group of this graph contains, as a subgroup of index two, the simple group PSU$(3, 5^2)$ of order 126,000.

The second graph mentioned above has valency 11, diameter 4, and intersection array

$$\begin{array}{ccccc} \cdot & 1 & 1 & 5 & 11 \\ 0 & 0 & 4 & 5 & 0 \\ 11 & 10 & 6 & 1 & \cdot\, \cdot \end{array}$$

Its group is the "little" group of Janko, of order 175,560, the first of the recently discovered sporadic simple groups; and this fact alone should make the graph remarkable.

Table of Results

Graph, Γ	K_4	I	Y	H
Number of vertices, n	4	10	28	102
Diameter, d	1	2	4	7
Girth, g	3	5	7	9
Automorphism group, Aut Γ	S_4	S_5	PGL(2, 7)	PSL(2, 17)
Vertex-stabilizer	S_3	D_{12}	D_{12}	S_4
Directed-edge-stabilizer	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$	D_8
Arc-regularity, s	2	3	3	4
Intersection matrix, B	$\cdot$ 1	$\cdot$ 1 1	$\cdot$ 1 1 1 2	$\cdot$ 1 1 1 1 1 1 3
	0 2	0 0 2	0 0 0 1 1	0 0 0 0 1 1 1 0
	3 $\cdot$	3 2 $\cdot$	3 2 2 1 $\cdot$	3 2 2 2 1 1 1 $\cdot$
Eigenvalues, λ	$3, -1$	$3, 1, -2$	$3, 2, -1,$	$3, 2, 0, \frac{1}{2}(1\pm\sqrt{17})$
			$-1\pm\sqrt{2}$	$\alpha_1, \alpha_2, \alpha_3$*
Multiplicities, m	1, 3	1, 5, 4	1, 8, 7, 6, 6	1, 18, 17, 9, 9,
				16, 16, 16
Complexity, κ	2^4	$2^4\cdot 5^3$	$2^{18}\cdot 7^5$	$2^{18}\cdot 3^{32}\cdot 17^{15}$
Number of 1-factors	3	6	84	?
Number of edge-3-colourings	1	0	56	?
Hamiltonian	Yes	No	No	Yes
Vertex-independence number, β_0	1	4	12	?
Number of vertex-3-colourings	0	20	?	?
Minimum genus, g_0	0	1	3	?

*$\alpha_1, \alpha_2, \alpha_3$ are the roots of the cubic equation $x^3 + 3x^2 - 3 = 0$.

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