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ON RINGS ALL OF WHOSE FACTOR RINGS ARE INTEGRAL DOMAINS

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Abstract

A ring R is called a (proper) quotient no-zero-divisor ring if every (proper) nonzero factor ring of R has no zero-divisors. A characterization of a quotient no-zero-divisor ring is given. Using it, the additive groups of quotient no-zero-divisor rings are determined. In addition, for an arbitrary positive integer n, a quotient no-zero-divisor ring with exactly n proper ideals is constructed. Finally, proper quotient no-zero-divisor rings and their additive groups are classified.

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1. Introduction

As in Feigelstock [1], a ring R is called a quotient no-zero-divisor ring if every homomorphic image of R has no zero-divisors. Similarly a ring R is called a proper quotient no-zero-divisor ring if every proper homomorphic image of R has no zero-divisors. Feigelstock [1, Question 4.1.13] asked: What can be said about the quotient no-zero-divisor rings and the proper quotient no-zero-divisor rings? In this paper, we consider this question. We first give a characterization of a quotient no-zero-divisor ring. As a corollary, we obtain that the additive group of a quotient no-zero-divisor ring is either a direct sum of copies of the additive group of the field of rational numbers or a direct sum of cyclic groups of order p for some fixed prime p. In addition, we show that a quotient no-zero-divisor ring satisfying a polynomial identity is a division ring. We next construct a quotient no-zero-divisor ring with exactly n proper ideals for an arbitrary positive integer n. We do not know whether there exists a quotient no-zero-divisor ring with infinitely many ideals or not. However we give an example of a Hausdorff topological ring R such that R has infinitely many closed proper ideals and all of them are completely prime. In the second half of this paper, we study the proper quotient no-zero-divisor rings. We classify the proper quotient no-zero-divisor rings into four types and determine their structures. As a result, we describe the structure of the additive groups of proper quotient no-zero-divisor rings.

2. Quotient no-zero-divisor rings

A proper ideal P of a ring R is said to be completely prime if R/P has no zero-divisors. Thus R is a quotient no-zero-divisor ring if and only if every proper ideal of R is completely prime. For each element a of a ring R, (a) denotes the principal ideal of R generated by a, that is, $(a) = Ra + aR + RaR + \mathbb{Z}a$. A ring R is called a chain ring if the lattice of ideals of R is totally ordered. We shall now give a characterization of a quotient no-zero-divisor ring.

THEOREM 1. *The following statements are equivalent:*

(1) *R is a quotient no-zero-divisor ring;*

(2) *R is a chain ring satisfying $(a) = (a^2)$ for all elements a in R.*

PROOF. (1) $\Rightarrow$ (2). Let a be an element of R. If $(a^2) = R$, then clearly we have $(a) = (a^2)$. So suppose that $(a^2) \neq R$. Then (a^2) is completely prime by hypothesis, and so we get $a \in (a^2)$. Therefore we have $(a) = (a^2)$. To prove that R is a chain ring, let I, J be two ideals of R. Assume, to the contrary, that $I \not\subset J$ and $J \not\subset I$ and take $a \in I \backslash J$ and $b \in J \backslash I$. Then we see that $ab \in I \cap J$, and so $I \cap J$ is not completely prime. This is contrary to our hypothesis. Thus R is a chain ring.

(2) $\Rightarrow$ (1). Let P be a proper ideal of R. To prove that P is completely prime, let a, b be two elements of R such that $ab \in P$. Then, for any $x \in R \cup \mathbb{Z}$, we have $bxa \in (bxa) = ((bxa)^2) \subset (ab) \subset P$. Hence $(b)(a) \subset P$. Since R is a chain ring, without loss of generality, we may assume that $(b) \subset (a)$. Then $(b) = (b^2) \subset (b)(b) \subset (b)(a) \subset P$, that is, $b \in P$. This proves that P is a completely prime ideal.

Let $\mathbb{Q}^+$ denote the additive group of the field of rational numbers, and for any positive integer n, let $\mathbb{Z}(n)$ denote the cyclic group of order n. We shall determine the additive groups of quotient no-zero-divisor rings.

COROLLARY 1. *Let G be an abelian group. Then the following statements are equivalent:*

(1) *G is the additive group of a quotient no-zero-divisor ring;*

(2) *G is isomorphic to either $\bigoplus_\alpha \mathbb{Q}^+$ or $\bigoplus_\alpha \mathbb{Z}(p)$ for some prime p, where α is an arbitrary cardinal.*

PROOF. (1) $\Rightarrow$ (2). Suppose that G is the additive group of a quotient no-zero-divisor ring R. Since R is an integral domain, the characteristic of R is either 0 or a prime p. In the latter case, R is a vector space over a field of order p, and hence $G \cong \bigoplus_\alpha \mathbb{Z}(p)$ for some cardinal α. Suppose now that R is of characteristic 0. Take an element $a \in R$ and a positive integer n. Then, by Theorem 1, $na \in ((na)^2) = n^2(a^2)$. Since R is of characteristic 0, we conclude that $a \in nR$. This implies that the additive group of R is a torsion-free divisible group. By Fuchs [2, Theorem 23.1], $G \cong \bigoplus_\alpha \mathbb{Q}^+$ for some cardinal α.

(2) $\Rightarrow$ (1). Needless to say, all fields are quotient no-zero-divisor rings. If $G \cong \bigoplus_\alpha \mathbb{Q}^+$, then G is the additive group of a field extension of degree α of the field of rational numbers. On the other hand, if $G \cong \bigoplus_\alpha \mathbb{Z}(p)$, then G is the additive group of a field extension of degree α of a field of order p. This completes the proof.

The first Weyl algebra $A_1(\mathbb{Q})$ over the field $\mathbb{Q}$ of rational numbers is the algebra on x, y over $\mathbb{Q}$ with the defining relation $xy - yx = 1$. It is well known that $A_1(\mathbb{Q})$ is a simple domain with unity 1. It is easy to check that $R = A_1(\mathbb{Q})x$ is a simple domain without unity. Hence R is an example of a quotient no-zero-divisor ring without unity. As another corollary of Theorem 1, we have the following.

COROLLARY 2. *Let R be a quotient no-zero-divisor ring and let C denote the center of R. Then R has a unity if and only if $C \neq 0$.*

PROOF. Suppose that $C \neq 0$, and take a nonzero element a of C. By Theorem 1, we have $(a) = (a^2)$. Then we can write $a = ae$ for some $e \in (a)$. Then, for any $x \in R$, we have $a(x - xe) = xa - xae = 0$. Similarly, we have $a(x - ex) = 0$. Since R has no zero-divisors, these imply that $x = xe = ex$ for all $x \in R$. Hence e is a unity of R.

It is easy to see that a commutative quotient no-zero-divisor ring is a field. More generally, we have the following.

PROPOSITION 1. *Let R be a quotient no-zero-divisor ring. If R satisfies a polynomial identity, then R is a division ring.*

PROOF. Let C denote the center of R. Since R is prime, $C \neq 0$ by Herstein [3, Theorem 1.4.2]. Hence, by Corollary 1, R has a unity 1. Let a be a nonzero element of C. By Theorem 1, $(a) = (a^2)$. Hence there exists $b \in R$ such that $a = a^2 b$. Since R is an integral domain, b is the inverse of a. Clearly b is in C. Hence C is a field. By Rowen [6, Corollary 1.6.28], R is a simple Artinian ring. Since R has no zero-divisors, R must be a division ring.

Let R be a chain ring. If the number of proper ideals of R is n, then R is called an n-chain ring. We shall show that there exist n-chain quotient no-zero-divisor rings for any positive integer n. Note that a 1-chain quotient no-zero-divisor ring is nothing but a simple domain. Let K be a field and let A be an algebra over K. Then, for each positive integer m, we let $T_K^m(A) = A \otimes_K \cdots \otimes_K A$ (tensor product taken m times). To construct n-chain quotient no-zero-divisor rings, we use a simple domain$D$ with center K such that $T_K^m(D)$ is an integral domain for any positive integer m. We shall now give some examples of such simple domains.

EXAMPLE 1. (1) Let K be a field of characteristic 0. Then, for each positive integer n, the n-th Weyl algebra $A_n(K)$ is a simple domain with 1 which is not a division ring. The center of $A_n(K)$ is K, and $T_k^m(A_n(k)) \cong A_{mn}(K)$. Hence $T_K^m(A_n(K))$ is an integral domain for any positive integer m.

(2) Let F be a field (of any characteristic) and let $F(\lambda)$ be the field of rational functions in λ over F. Let $B = B(\lambda, F(\lambda))$ be a ring defined as the $F(\lambda)$-algebra generated by x, x^{-1}, y and y^{-1} subject to the relation $xy = \lambda yx$. By Jategaonkar [4, Theorem 2.1], B is a simple domain with 1 which is not a division ring. Clearly the center of B is $F(\lambda)$. Since $T_{F(\lambda)}^m(B) = B(\lambda, T_{F(\lambda)}^{m-1}(B))$, using induction on m and Jategaonkar [4, Proposition 1.1(a)], we can show that $T_{F(\lambda)}^m(B)$ is an integral domain for any positive integer m.

THEOREM 2. *Let D be a simple domain with 1 which is not a division ring, and let K denote the center of D. Suppose that $T_K^m(D)$ is an integral domain for any positive integer m. Take a non-unit $0 \neq x \in D$ and let $R_1 = K + xD$.*

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Define a subalgebra R_n of $T_K^n(D)$ inductively: $R_n = K + axD \otimes_K R_{n-1}$. Then R_n is an $(n+1)$-chain quotient no-zero-divisor ring.

PROOF. Let I be a nonzero proper ideal of R_1. Since D is a simple domain, we see that $I \supset xDIxD = xD$. Since R_1/xD is isomorphic to the field K, xD is a maximal ideal of R_1, and hence we conclude that $I = xD$. Thus 0, xD and R_1 are all the ideals of R_1. Obviously, 0 and xD are completely prime. Therefore R_1 is a 2-chain quotient no-zero-divisor ring. By induction on n, we shall prove that the proper ideals of R_n are 0, $T_K^n(xD)$, $T_K^{n-1}(xD) \otimes_K R_1$, $T_K^{n-2}(xD) \otimes_K R_2, \dots, xD \otimes_K R_{n-1}$ and these are completely prime ideals. Let us set $R_0 = K$. Assume now that $n > 1$ and that the proper ideals of R_{n-1} are $I_0 = 0$, $I_1 = T_K^{n-1}(xD), \dots, I_{n-1} = xD \otimes_K R_{n-2}$ and these are completely prime. Since D is a simple ring with center K, by Renault [5, Theorem 5.1.3] the proper ideals of $D \otimes_K R_{n-1}$ are $D \otimes_K I_0 = 0$, $D \otimes_K I_1, \dots, D \otimes_K I_{n-1}$. Let I be a proper ideal of R_n. Then $(D \otimes -KR_{n-1})I(xD \otimes_K R_{n-1}) = D \otimes_K I_m$ for some $m < n$. Then we obtain that $(x \otimes 1)I(x \otimes 1) \subset xD \otimes_K I_m$. Since $R_n/(xD \otimes_K I_m) \cong K + xD \otimes_K (R_{n-1}/I_m) \cong K + xD \otimes_K R_{n-m-1} = R_{n-m}$, $xD \otimes_K I_m$ is completely prime by the induction hypothesis. Since $(x \otimes 1)I(x \otimes 1) \subset xD \otimes_K I_m \subset I$ and $x \otimes 1 \notin xD \otimes_K I_m$, we conclude that $I = xD \otimes_K I_m$, and hence I is a completely prime ideal of R_n. If $m \neq 0$, then $I = xD \otimes_K T_K^{n-m}(xD) \otimes_K R_{m-1} = T_K^{n-m+1}(xD) \otimes_K R_{m-1}$, and if $m = 0$, then $I = 0$. This completes the induction.

We do not know whether there exists a quotient no-zero-divisor ring with infinitely many ideals or not. However we shall construct a Hausdorff topological ring R such that R has infinitely many closed proper ideals and all of them are completely prime.

Let D be a simple domain with center K satisfying the hypothesis of Theorem 2, $R_n (n \geq 1)$ be the rings defined in Theorem 2 using D and let $R_0 = K$. Let n be a positive integer, and I_n the unique minimal ideal of R_n. As shown in the proof of Theorem 2, we have a canonical isomorphism $R_n/I \cong R_{n-1}$. Let $f_n : R_n \to R_{n-1}$ be the epimorphism induced by this isomorphism. Then $\{R_n, f_n\}$ is an inverse system of rings.

THEOREM 3. *The inverse limit $S = \varprojlim R_n$ becomes a Hausdorff topological ring all of whose closed proper ideals are completely prime.*

PROOF. Let $p_n : S \to R_n$ denote the canonical projection, $M_{-1} = S$, and $M_n = \operatorname{Ker} p_n$ for each $n \geq 0$. By taking $\{M_n\}$ to be a base of neighborhoods of 0, S becomes a topological ring. Since $\bigcap_{n \geq -1} M_n = 0$, S is Hausdorff. Since

$M_n = S - \bigcup_{a \in S \setminus M_n}(a + M_n)$, M_n is open and closed. Since $S/M_n \cong R_n$ and each R_n is an integral domain, M_n is a completely prime ideal for any $n \geq 0$. Now let M be a nonzero closed ideal of S. We shall prove that $M = M_m$ for some $m \geq -1$. Since $\bigcap_{n \geq -1} M_n = 0$ and $M_{-1} = S \supset M_0 \supset M_1 \supset \dots$, there exists an integer $m \geq -1$ such that $M_m \supset M$ and $M_{m+1} \not\supset M$. Since $S/M_k \cong R_k$ for each k, by taking the structure of the lattice of ideals of R_k into consideration, we conclude that $M_k + M = M_m$ for all $k \geq m$. Therefore $M_m \subset M + M_k$ for all $k \geq 0$. To prove that $M_m \subset M$, let $a \in M_m$. Then $a \in M + M_k$ for all $k \geq 0$. Hence $(a + M_k) \cap M \neq 0$ for all $k \geq 0$. Since M is closed and $\{a + M_k \mid k \geq 0\}$ is a base of neighborhoods of a, we conclude that $a \in M$. This proves that $M = M_m$.

3. Proper quotient no-zero-divisor rings

We shall classify the proper quotient no-zero-divisor rings.

THEOREM 4. *Let R be a ring. Then R is a proper quotient no-zero-divisor ring if and only if one of the following holds:*

(a) *R is a quotient no-zero-divisor ring;*

(b) *R is a simple ring with zero-divisors;*

(c) *R is not an integral domain and R has a unique minimal ideal P such that R/P is a quotient no-zero-divisor ring;*

(d) *R is the direct sum of two simple domains;*

(e) *There exist two quotient no-zero-divisor rings A and B with unique proper minimal ideals P and Q respectively, such that there is an isomorphism*

$$\sigma : A/P \to B/Q, \text{ and } R = \{(a, b) \in A \times B | \sigma(a + P) = b + Q\}.$$

PROOF. Let R be a proper quotient no-zero-divisor ring such that 0 is not a completely prime ideal. Suppose first that R is a chain ring. If R has no nonzero proper ideals, then R satisfies (b). So we may assume that R has a nonzero proper ideal. Let P denote the intersection of all nonzero proper ideals of R. Since R is a chain ring and since every nonzero proper ideal is completely prime, P is completely prime. Hence P is nonzero by hypothesis. In this case, R satisfies (c). Suppose next that R has two ideals P and Q such that $P \not\subset Q$ and $Q \not\subset P$. Then $R/P \cap Q$ is not an integral domain, and so $P \cap Q$ must be zero. Let I be a nonzero ideal of R different from P and Q. Since R/I is an integral domain, either $P \subset I$ or $Q \subset I$. Suppose that $P \not\subset I$. Then $Q \subset I$ and $I \cap P \subset P$. If $I \cap P \neq 0$, then $Q \subset I \cap P$, because $P \not\subset I \cap P$. Then $Q = (I \cap P) \cap Q \subset P \cap Q = 0$, which is a contradiction. Hence $I \cap P = 0 \subset Q$, and so $I = Q$. This contradicts the choice of I. Therefore I contains P. Similarly we can prove that I contains Q. Hence $P + Q$ is minimum among the nonzero ideals of R different from P and Q. Now if $P + Q = R$, then R is the direct sum of the simple domains P and Q, and so R satisfies (d). So assume that $P + Q \neq R$. Let us set $A = R/Q$ and $B = R/P$. By hypothesis A and B are quotient no-zero-divisor rings. Since $P \cap Q = 0$, P and Q can be viewed as ideals of A and B, respectively. By the above observation, P and Q are minimal ideals of A and B respectively. Then we see $A/P \cong R/(P + Q) \cong B/Q$. Let $\sigma : A/P \to B/Q$ denote this isomorphism. Consider the natural embedding $f : R \to A \times B$; $f(a) = (a + Q, a + P)$ for all $a \in R$. Let S denote the set $\{(x, y) \in A \times B \mid \sigma(x + P) = y + Q\}$. We shall show that $Im(f) = S$. Clearly $Im(f)$ is contained in S. Let $(x, y) = (a + Q, b + P) \in S$, where a, b are elements of R. Then $a + (P + Q) = b + (P + Q)$ in $R/(P + Q)$, that is, $a - b \in P + Q$. Hence we can write $a - b = p + q$ for some $p \in P$ and $q \in Q$. Now it is easy to see that $f(a - q) = (x, y)$. This proves $Im(f) = S$, and hence R satisfies (e).

Conversely, if R satisfies one of (a)–(d), then clearly R is a proper quotient no-zero-divisor ring. Assume now that R satisfies the condition (e). Then we can easily see that the proper nonzero ideals of R are $(P, 0) = \{(p, 0) \in A \times B \mid p \in P\}$, $(0, Q)$, and

$$H(I) = \{(a, b) \in A \times B \mid a \in I,\ \sigma(a + P) = b + Q\},$$

where I runs over all proper ideals of A containing P. Since $R/(P, 0) \cong B$, $R/(0, Q) \cong A$ and $R/H(I) \cong A/I$, all of these nonzero ideals are completely prime. This completes the proof.

We shall give examples of rings satisfying the conditions (c) or (e) in Theorem 4.

EXAMPLE 2. (1) Let $n > 1$ be an integer and let R be an n-chain quotient no-zero-divisor ring with unique minimal ideal P. Consider the set

$$S = \left\{ \begin{pmatrix} a & p \\ 0 & a \end{pmatrix} \,\middle|\, a \in R,\ p \in P \right\}.$$

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By defining addition and multiplication in S as in ordinary matrices, S becomes a ring. We can easily see that S satisfies the condition (c) in Theorem 4 and S is not a prime ring. On the other hand, let F be a field and let A be the ring of countable matrices over F of the form

$$\begin{pmatrix} C_n & 0 & & & \\ 0 & a & & & \\ & & \ddots & & \\ & & & a & \\ & & & & \ddots \end{pmatrix}$$

where $a \in F$ and C_n is an arbitrary $n \times n$ matrix over F and n is allowed to be any integer. Then A is a prime ring satisfying (c) in Theorem 4.

(2) Let $n > 1$ be an integer and let R be an n-chain quotient no-zero-divisor ring with unique minimal ideal P. Then $S = \{(a, b) \in R \times R \mid a - b \in P\}$ is a ring satisfying the condition (e) in Theorem 4.

As a result of Theorem 4, we can describe the structure of the additive groups of proper quotient no-zero-divisor rings.

COROLLARY 3. *Let R be a proper quotient no-zero-divisor ring and let R^+ denote the additive group of R. Then one of the following holds:*

(1) $R^+ = \bigoplus_\alpha \mathbb{Q}^+$ *where α is a cardinal;*

(2) $R^+ = \bigoplus_\alpha \mathbb{Z}(p)$ *where p is a prime and α is a cardinal;*

(3) $R^+ = \bigoplus_\alpha \mathbb{Q}^+ \oplus \bigoplus_\beta \mathbb{Z}(p)$ *is a prime and α, β are cardinals;*

(4) $R^+ = \bigoplus_\alpha \mathbb{Z}(p) \oplus \bigoplus_\beta \mathbb{Z}(q)$ *where p and q are distinct primes and α, β are cardinals;*

(5) $R^+ = \bigoplus_\alpha \mathbb{Z}(p^2)$ *where p is a prime and α is a cardinal;*

(6) $R^+ = \bigoplus_\alpha \mathbb{Z}(p) \oplus \bigoplus_\beta \mathbb{Z}(p^2)$ *where p is a prime and α, β are cardinals.*

PROOF. If R satisfies the condition (a) in Theorem 4, then either (1) or (2) holds by Corollary 1. Now suppose that R satisfies (b) in Theorem 4. Then the characteristic of R is either a prime p or 0. If $\mathrm{char}(R) = p$, then (2) holds. On the other hand, if $\mathrm{char}(R) = 0$, then $nR = R$ for any positive integer n, because nR is a nonzero ideal of R. This implies that R^+ is divisible. Hence (1) holds by Fuchs [2, Theorem 23.1]. Suppose next that R satisfies the condition (c) in Theorem 4. Then R is a chain ring and P is the unique minimal ideal of R. If $P^2 \neq 0$, then $P^2 = P$. In this case, R is a prime ring and $I^2 = I$ for any ideals I of R, because every ideal of R/P is idempotent by Theorem 1. If the characteristic of R is a prime p, then (2) holds. So assume that $\mathrm{char}(R) = 0$ and let n be an arbitrary positive integer. Then, since $nR = (nR)^2 = n^2R^2 = n^2R$, we obtain $nR = R$. Then R^+ is a torsion-free divisible group, and hence (1) holds again by Fuchs [2, Theorem 23.1]. In case $P^2 = 0$, P is a right R/P-module. If $\mathrm{char}(R/P) = 0$, then R/P is a vector space over $\mathbb{Q}$ by Corollary 1, and hence the right R/P-module P is also a vector space over $\mathbb{Q}$. Then R^+ is isomorphic to $P^+ \oplus (R/P)^+$, and hence (1) holds. On the other hand, if $\mathrm{char}(R/P) = p$, then $\mathrm{char}(R)$ equals either p or p^2. Then, by Fuchs [2, Theorem 17.2], R^+ satisfies either (2), (5) or (6). If R satisfies (d) in Theorem 4, then R^+ satisfies one of (1)–(4). Finally, suppose that R satisfies the condition (e) in Theorem 4. Then, by Corollary 1, A is a vector space over either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Similarly B is a vector space over either $\mathbb{Q}$ or $\mathbb{Z}/q\mathbb{Z}$ for some prime q. Since A/P is isomorphic to B/Q, it holds that $\mathrm{char}(A) = \mathrm{char}(B)$. If $\mathrm{char}(A) = \mathrm{char}(B) = p > 0$, then R^+ satisfies (2), because R is a subring of $A \times B$. Suppose now that A and B are vector spaces over $\mathbb{Q}$. Then we can easily see that the isomorphism $\sigma : A/P \to B/Q$ is a $\mathbb{Q}$-space isomorphism. Therefore R is also a vector space over $\mathbb{Q}$, and hence (1) holds in this case.

REMARK. Let p be a prime. Then $\mathbb{Z}/p^2\mathbb{Z}$ is a proper quotient no-zero-divisor ring satisfying (5) of Corollary 3. However we do not know the existence of a proper quotient no-zero-divisor ring satisfying (6) of Corollary 3.

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