Now let $x = a$, and we have :

$$f(a) = (a - a)Q(a) + R = R.$$

The question is often asked : Is it not possible to avoid the difficulty, by adopting the alternative proof, depending on the fact that $x - a$ is a factor of $x^n - a^n$? It is not. There is and can be no method of proving a *remainder* theorem which does not require a clear knowledge of the meaning of the term.

Further, the proof given above has an advantage over the alternative proof, in addition to that of brevity. To complete our definition of the process of division, we require to prove its validity. That is, we must prove, *inter alia*, the existence of a remainder with the properties stated; and the enunciation of the Theorem presupposes that this has already been done. But the definition does not guarantee the existence of a unique remainder, and we have, apart from proof, no reason to expect that the remainder is unique. The alternative proof shows that division can be carried out in such a way that $f(a)$ is the remainder, but it leaves us in doubt as to whether there might not be another mode of division leading to a different result. The method above leaves no such doubt. Suppose the division carried out in any way, then the remainder is $f(a)$.

JAMES HYSLOP.

Linear Transformations and Geometry.

The following note suggests certain connections between the theory of linear transformations and quadratic forms on the one hand, and the geometry of second degree surfaces on the other. It is hoped that the note may prove useful to those who may have to teach either theory to students who already possess an elementary knowledge of the other. The general ideas may be such as may well have occurred to anyone familiar with both theories, but the examples given may be new to readers.

In the geometry Cartesian co-ordinates are used throughout, and the axes of reference are rectangular, unless the contrary is stated ; it may however be noted that many of the results, including those of § 1, are valid also when the axes are oblique. For simplicity the discussion is restricted to three dimensions, but the results hold in a space of any number of dimensions.

§ 1. *The Transformation.*—A linear transformation means a set of equations of the type :

$$y_r = a_{r1} x_1 + a_{r2} x_2 + a_{r3} x_3 \quad [r = 1, 2, 3] \qquad \ldots.(1)$$

If the determinant of the coefficients is not zero, equations (1) can be solved for x_1, x_2, x_3, giving a solution of the form :

$$x_r = b_{r1} y_1 + b_{r2} y_2 + b_{r3} y_3 \quad [r = 1, 2, 3] \qquad \ldots.(2)$$

We shall assume in what follows that our transformation is of type (1), with real coefficients, and having a reciprocal given by (2).

By the transformation, to each point P, (x_1, x_2, x_3) is made to correspond one and only one point Q, (y_1, y_2, y_3), and conversely. Further, if two points P_1, P_2 are transformed into Q_1, Q_2, then the point P_3, which lies on the straight line P_1P_2 and divides P_1P_2 in the ratio $m:n$, is transformed into Q_3, which divides Q_1Q_2 in the ratio $m:n$. This is readily verified by substituting the appropriate co-ordinates in (1). Thus the transformation transforms points of a straight line into collinear points, segments of a line into segments of the corresponding line, and it conserves the ratio of two segments of the same line.

Now the diagonals of a parallelogram bisect one another, and conversely if two line segments bisect one another, the joins of their extremities form a parallelogram ; further, a pair of lines which bisect one another is transformed into another such pair ; thus a parallelogram is transformed into a parallelogram, and equal parallel lines into equal parallel lines. It follows that parallelism and ratios of parallel segments are conserved by the transformation.

Again, if the point P moves on a surface of n^{th} degree, its co-ordinates continually satisfy an algebraic equation of the n^{th} degree. Substituting values from (2), we obtain a similar relation[1] satisfied by the co-ordinates of the corresponding positions of Q, which therefore also lie on a surface of n^{th} degree. In particular a plane is transformed into a plane, and a quadric into a quadric.

The effect of a transformation may be considered geometrically in two ways :—

(i) As a change in the configuration of a geometrical diagram without change in the axes of reference. The change due to a linear transformation is of the kind known as a general homogeneous strain.

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[1] The relation cannot be of degree lower than the n^{th}, as an application of the reciprocal transformation shows.

(ii) As a change in the frame of reference ; in the case of the transformations with which we have to deal, the change is to a set of new axes through the same origin, the figure meanwhile remaining unaltered.

To take a simple example ; if the axes are rectangular, the effect of an orthogonal transformation may be considered

(i) As a rotation of the figure about the origin, or

(ii) As a change of axes to another rectangular set.

In this example the two points of view obviously represent two different aspects of the same fact, namely a change in the relative positions of the figure and the axes. In general the connection is less clear and we find more helpful now one now the other of the two ideas.

§ 2. *Application to Geometry of the Ellipsoid.*—A great deal of the geometry of the ellipsoid—including tangency conditions, polar properties, and elementary properties of enveloping cylinders and cones—can be deduced from the more obvious geometry of the sphere. The relation between the two surfaces is very like that between the ellipse and its auxiliary circle ; but while in two dimensions the transformation is generally considered as an orthogonal projection, in three dimensions it is more easily visualised as a homogeneous strain or "multiplication"—a uniform stretching of the space containing the figure in directions parallel to the co-ordinate axes—or as a change of scale, different for each of the directions of reference.

Example 1. *Properties of conjugate diameters.*—By the transformation

$$y_r = a_r x_r \quad [r = 1, 2, 3] \qquad \ldots.(3)$$

the sphere $\Sigma x^2 = 1$ is transformed into the ellipsoid $\Sigma \dfrac{y^2}{a^2} = 1$. Now any three mutually perpendicular lines through the centre of the sphere have the property, that the plane through any two of them bisects all chords parallel to the third, and, in particular, passes through the points of contact of tangents parallel to the third. We deduce at once similar properties for the corresponding sets of diameters of the ellipsoid—the conjugate properties.

Conversely[1] any set of conjugate diameters of the ellipsoid

[1] *Cf. Frost : Solid Geometry* (1886), p. 143.

corresponds to a set of diameters of the sphere which possess the conjugate properties, and are readily seen to be mutually perpendicular.

Example 2. *The equation of the ellipsoid when a set of conjugate diameters is taken as* (*oblique*) *axes.*—Let us carry the discussion a stage further. Referred to *any* set of mutually perpendicular lines through the centre as axes, the equation of the sphere is $\Sigma u^2 = 1$. To the axes of u_1, u_2, u_3, correspond, as we have just seen, a set of conjugate diameters of the ellipsoid ; and to the segments, whose measures, u_1, u_2, u_3, are the u-co-ordinates of a point P on the sphere, correspond segments whose measures, z_1, z_2, z_3, are the co-ordinates, referred to the conjugate diameters, of the point Q on the ellipsoid which corresponds to P. Since ratios of parallel segments are conserved, these quantities satisfy a relation of the form :

$$z_r = k_r u_r \quad [r = 1, 2, 3].$$

Thus the equation of the ellipsoid referred to a set of conjugate diameters as (oblique) axes is $\Sigma \frac{z^2}{k^2} = 1$ ....(4)

These examples require some space to write out in full, but once the central ideas have been grasped, such matters become so simple and natural as to make unnecessary algebra or "working" of any kind.

If we admit imaginary lines and transformations, we can deal similarly with the geometry of the hyperboloids, but here, except for those specially trained in the geometry of the unreal, the visualisation breaks down.

§ 3. *The Theory of Quadratic Forms.*—A quadratic form is an expression like $\sum_{r=1}^{r=3} \sum_{s=1}^{s=3} k_{rs} x_r x_s$, and may be denoted by $K(x, x)$. Using (2) we can express $K(x, x)$ in terms of y_1, y_2, y_3, and the new expression is a quadratic form in these variables, $K'(y, y)$. Suppose now that the geometry of the central conicoids has been fully developed either by the above methods or in any other way. The equation $K(x, x) = 1$ represents such a surface referred to three mutually perpendicular lines through the centre, and $K'(y, y) = 1$ represents the same conicoid after transformation.

Example 1. *Orthogonal transformation of a quadratic form into the sum of squares.*—It is known that, when the axes of reference are changed, so that the principal axes of the conicoid are taken as new axes of co-ordinates, the equation of the surface is of the form

$$a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 = 1 \qquad \ldots(5)$$

Since the principal axes are mutually perpendicular, the transformation is of the type known as orthogonal. This fact corresponds to the theorem: *A quadratic form in three variables may be transformed by an orthogonal transformation into the sum of three squares, each with a constant multiplier.*

In this example the working involved in a solution *ab initio* of the geometric problem is practically the same as in the algebraic one ; but each of the problems suggests a meaning for the various steps of the proof which throws fresh light on the other problem. Together they lead to the study of invariant lines and invariants of certain types of transformation. The conicoid concerned may, of course, be either an ellipsoid or a hyperboloid.

Example 2. *Non-orthogonal transformations to a sum of squares.*—Let any set of conjugate diameters of the conicoid $K(x, x) = 1$ be chosen as new (oblique) co-ordinate axes. Then the transformation which changes the axes reduces $K(x, x)$ to a sum of squares as in (4). This not only shows the possibility of transforming the quadratic form in an infinity of ways into a sum of squares, but provides a simple effective means of finding all possible ways of doing so.

Example 3. *Simultaneous transformation of two quadratic forms to sums of squares.*—Consider now two quadratic forms $K_1(x, x)$, $K_2(x, x)$ of which K_1 is always positive, non-zero, for real non-zero values of x_1, x_2, x_3. This means that any radius vector drawn from the origin meets the surface in real points, so that this surface is an ellipsoid E. Apply the transformation by means of which E is transformed[1] into a sphere S. The conicoid Q, corresponding to K_2, is by this transformation transformed into some other conicoid Q_1. Now S, Q_1 have one set of conjugate diameters coincident in direction, namely the principal axes of Q_1. Hence the corresponding lines before transformation form a set of common conjugate diameters of E, Q. Referred to this set of diameters as new (oblique) axes, the equations of E, Q have both on the left side a sum of squares. Thus we have proved that : *If* $K_1(x, x)$ *is*

[1] We first make the principal axes of E the new axes of reference, then apply a transformation like (3) of § 2.

always positive for non-zero values of the variables, $K_1(x, x)$, $K_2(x, x)$ can be transformed by the same transformation into sums of squares. The geometric ideas furnish a clue to the simplest method of finding the transformation in an arithmetic case.

§ 4. *A Theorem on the Transformation.*—Finally we give a geometric discussion of the theorem: *Any non-singular linear transformation can be exhibited as the result of three transformations, of which the first and third are orthogonal and the second a multiplication.* Consider any transformation (1), having a reciprocal (2). It is characterised by the fact that (2) transforms the points (1, 0, 0), (0, 1, 0), (0, 0, 1) into $B_r(b_{1r}, b_{2r}, b_{3r})$ $[r = 1, 2, 3]$, and (1) reverses this process. Take now the ellipsoid, centre O, through B_1, B_2, B_3 which has the lines OB_1, OB_2, OB_3 as a set of conjugate diameters. The existence of such an ellipsoid readily follows from equation (4) above. Apply to it the transformation of orthogonal type which makes the principal axes of the ellipsoid into the axes of co-ordinates. To the figure thus transformed apply the multiplication which transforms the ellipsoid into a sphere. We require that the lines corresponding to OB_1, OB_2, OB_3 be transformed into a right-handed triad of lines.[1] If necessary one of the coefficients of the multiplication should be taken negative in order to ensure this. Finally rotate the axes, by an orthogonal transformation, so that the new axes of co-ordinates are those lines, known to be mutually perpendicular, which correspond to OB_1, OB_2, OB_3 in the original figure. The effect has been to apply successively three transformations to bring about the same result as if (1) alone had been applied, and the theorem follows.

In this problem the geometry supplies not only a simple proof of existence, but an effective method of tackling an arithmetic case.

§ 5. *Bibliographical Note.*—The theorems of §§ 3, 4 are proved in text books on determinants such as *Bôcher's Introduction to Higher Algebra* or *Kowalewski's Determinanten-theorie.* The geometry is to be found in the standard books on analytical geometry of three dimensions such as those by *Salmon* and *Bell.* Applications to

[1] The line segments OA_1, OA_2, OA_3 form a right-handed triad, if the direction of rotation round the triangle $A_1A_2A_3$ is clockwise, when viewed from O. By a convention, the positive directions of three Cartesian axes of reference form a right-handed triad, unless in exceptional cases.

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geometry of the algebraic theory are discussed by *Bôcher* and *Salmon*, in many text-books on Projective Geometry like *Veblen and Young*, and in a *Cambridge Tract* by *Bromwich*.

The chief difference between the examples of the present note and the theory of the books mentioned, lies in the use of Cartesian co-ordinates in place of the more usual homogeneous co-ordinates. While the latter system is preferable in projective and advanced general work, Cartesian co-ordinates should surely not be entirely neglected, and indeed they have certain advantages for the present purpose. Their use is more elementary, and likely to be more familiar to the students for whose instruction the above examples are suggested. Also ideas of length and perpendicularity, and the geometrical interpretation of the orthogonal transformation are simpler in rectangular Cartesian co-ordinates than in any other. It will be found that all the examples discussed above are based on these ideas.

James Hyslop.

Conditions obtained by Multiplication of Determinants.

The condition that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ should represent two straight lines can be obtained very easily by considering the product

$$\begin{vmatrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{vmatrix} \begin{vmatrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{vmatrix},$$

which is equal to

$$\begin{vmatrix} 2ll', & lm' + l'm, & nl' + n'l \\ lm' + l'm, & 2mm', & mn' + m'n \\ nl' + n'l, & mn' + m'n, & 2nn' \end{vmatrix},$$

and is identically zero.

For if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is equivalent to

$$(lx + my + n)(l'x + m'y + n') = 0,$$

then $$\frac{ll'}{a} = \frac{mm'}{b} = \frac{mn'}{c} = \frac{mn' + m'n}{2f} = \frac{nl' + n'l}{2g} = \frac{lm' + l'm}{2h},$$