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NAVIER STOKES DERIVATIVE ESTIMATES IN THREE DIMENSIONS WITH BOUNDARY VALUES AND BODY FORCES

Dedicated to P.G. Rooney in celebration of many years of friendship

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ABSTRACT. For a vector solution $u(x,t)$ with finite energy of the Navier Stokes equations with body forces and boundary values on a region $\Omega \subseteq R^3$ for $t > 0$, conditions are established on the $L^{6/5}(\Omega)$ and $L^2(\Omega)$ norms of derivatives of the data that ensure the estimates $\|D_t^r D_x^s u\| \in L^{2(4r+2s-1)^{-1}}(0,T)$ and $\max_{x\in\Omega} |D_t^r D_x^s u| \in L^{(2r+s+1)^{-1}}(0,T)$, up to any given integer value of the weighted order $2r+s$, where r or $s = s_1+s_2+s_3 > 0$ and $0 < T < \infty$.

Table of Contents

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Chapter I. Non-homogeneous data

1. **Introduction.** Among the classical equations of applied mathematics and mathematical physics, the Navier Stokes equations in three space dimensions retain particular interest because the standard initial and boundary value problem for them has not been shown to be correctly posed in every respect. Indeed solutions with finite kinetic energy which develop singularities over time, and may be non-unique, are believed to exist. Although no completely satisfactory explicit example has yet been found, interesting cases that come very close to fulfilling all the conditions are discussed in [2,4,9,10,12]. The existence of long-time weak solutions in Hilbert space has been demonstrated [9], and these become smooth under a certain condition of small magnitudes [6,10,13]. The set of singular points in space-time has been shown to have Hausdorff dimension at most 1, and to have one-dimensional Hausdorff measure zero [2,12]. Solutions with singularities may be related to such physical motions as tornadoes.

In these circumstances interest will naturally attach to the general properties of solutions with finite energy. A first step of this kind was taken in [5] wherein the existence of energy-type estimates for higher space derivatives of solutions was shown for the case of a three-dimensional periodic parallelepiped or 3-torus. For the general initial value problem in three space dimensions, but with zero boundary values and body forces, higher order estimates for all space and time derivatives were found in [4] as follows:

Let $u(x,t)$ be a vector solution of the homogeneous Navier Stokes equations with finite kinetic energy on a three-dimensional region Ω, which vanishes on $\partial\Omega$ and is smooth except on a singular set of dimension 1 in space-time. Then the $L^2(\Omega)$ norm of $D_t^r D_x^s u$ is integrable to the power $2(4r+2s-1)^{-1}$ over every finite time interval $(0,T)$, where r or $s = s_1+s_2+s_3$ is a positive integer, and $\max_{x\in\Omega} |D_t^r D_x^s u|$ is integrable over $(0,T)$ to the power $(2r+s+1)^{-1}$, where $r,s = 0,1,2,\ldots$.

The present paper is concerned with the related higher derivative estimates, conditions, and results that apply when non-zero boundary values and body forces are introduced. The theorem is stated in § 4 below, but certain preliminary comments may be appropriate here. As shown in [4] the finiteness of the kinetic energy, or initial value norm $\|u\|_2$, with other data zero, is sufficient to ensure the higher derivative estimates of every order of derivatives, with the condition that the integrability over time decreases as the derivative order increases. For non-homogeneous boundary values and body forces, however, a cumulative sequence of hypotheses on corresponding derivatives of the data will be appropriate. This reflects the situation that, whereas initial values will exert their influence once for all at time zero, and then be left behind in the past, the boundary values and body forces, by later behaviour, or by developing singularities of some degree, can influence the later character of the solution to an extent not foreshadowed by earlier behaviour. This aspect will be considered again in the discussion of the main lemma on integrability of § 5.

2. **The Navier Stokes Equations.** Let x_i $(i = 1, 2, 3)$ denote Cartesian coordinates in R^3 and let t denote the time variable. Let $u_i(x, t)$ be the vector field of velocity components of a fluid flow, and $p(x, t)$ be the pressure. The constant viscosity coefficient is denoted by ν. Let $B_i(x, t)$ denote a vector field of imposed body forces. Then the nonhomogeneous Navier Stokes equations are

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + B_i(x, t) \tag{2.1}$$

where $i = 1, 2, 3$ and summation over $k = 1, 2, 3$ by the Einstein convention is understood for repeated indices.

The differential dx shall denote the volume element $dx_1 dx_2 dx_3$, while the Laplace operator in R^3 is denoted by $\Delta \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. We shall also assume that the equation of continuity, or incompressibility,

$$\operatorname{div} u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0 \tag{2.2}$$

holds in the above homogeneous form. The four equations for the unknowns u_i and p (the latter up to an additive constant) form a semi-linear elliptic-parabolic system.

Three initial conditions are appropriate:

$$u_i(x, 0) = u_{i0}(x) \tag{2.3}$$

where $u_0(x) = \{u_{i0}(x)\}$ is a given solenoidal vector field of integrable square on a given region Ω:

$$\|u_0\|_2^2 = \sum_{i=1}^{3} \int_\Omega |u_{i0}(x)|^2 \, dx < \infty. \tag{2.4}$$

We assume throughout the boundary $\partial\Omega$ satisfies a weak cone condition and is piecewise C^∞ with a finite number of edges or corners in any bounded subregion, and that $\partial\Omega$ is not too tightly coiled or layered at large distances as in (4.3) of [4] if Ω is unbounded. The boundary conditions shall be

$$u_i(x, t) = w_i(x, t) \tag{2.5}$$

for $x \in \partial\Omega$, where $w_i(x, t)$ is a given vector field defined on $\partial\Omega$ and in Ω, subject to conditions as stated below, and such that in Ω

$$\operatorname{div} w = \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = 0. \tag{2.6}$$

Taking the divergence of (2.1) we find

$$\Delta p = -\frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k} + \frac{\partial B_i}{\partial x_i} = -(u_k u_i)_{,ik} + B_{i,i} \tag{2.7}$$

where the subscript commas denote partial derivatives with respect to the indices following. Letting x approach the boundary we obtain the boundary relations for p:

$$\frac{\partial w_i}{\partial t} + w_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + B_i. \tag{2.8}$$

Let us set

$$u_i = v_i + w_i \tag{2.9}$$

so that on $\partial\Omega$ we have

$$v_i = 0. \tag{2.10}$$

Then also in Ω

$$v_{i,i} = u_{i,i} - w_{i,i} = 0 \tag{2.11}$$

and on $\partial\Omega$

$$w_k \frac{\partial v_i}{\partial x_k} = w_n \frac{\partial v_i}{\partial n}$$

where $n = \{n_i\}$ denotes the unit normal to $\partial\Omega$, and $\partial/\partial n$ the normal derivative. Also in view of (2.10) and (2.11) we have on $\partial\Omega$, with a suitable coordinate system as in [4, § 10],

$$\begin{aligned} n_i w_k \frac{\partial v_i}{\partial x_k} &= w_n \left(\frac{\partial v_n}{\partial n} - v_i \frac{\partial n_i}{\partial n} \right) = w_n \frac{\partial v_n}{\partial n} \\ &= -w_n \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) = 0 \end{aligned} \tag{2.12}$$

where n, x_1 and x_2 denote suitable coordinates locally normal and tangent to $\partial\Omega$, see [4, § 10]. Hence the normal component of (2.8) yields the Neumann type boundary condition

$$\frac{\partial p}{\partial n} = -\frac{\partial w_n}{\partial t} - n_i w_k \frac{\partial w_k}{\partial x_i} + \nu n_i \Delta v_i + \nu n_i \Delta w_i + B_n. \tag{2.13}$$

Estimates of the potentials arising as solutions of the boundary value problem (2.7) and (2.13) will be given in § 9 below. Here we merely note that the standard necessary condition for the consistency of this Neumann problem for p can easily be verified from the conditions listed above.

3. **Analytical Preliminaries.** The Lebesgue space $L^p(\Omega)$ will denote the vector (or sometimes scalar) functions on Ω with finite norm $\|u\|_p$, where

$$\|u\|_p^p = \int_\Omega \sum_{i=1}^{3} \left|u_i(x,t)\right|^p dx. \tag{3.1}$$

Throughout, these norms become functions of the time t. We set

$$(u,v) = \int_\Omega \sum_{i=1}^{3} u_i v_i \, dx \tag{3.2}$$

and observe by Hölder's inequality that

$$|(u, v)| \leq \|u\|_p \, \|v\|_q \tag{3.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1,\ p \geq 1, q \geq 1$ so that p and q are dual indices. As well, we employ Young's inequality [7, Theorem 37], equivalent to the theorem of arithmetic and geometric means,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{3.4}$$

where $a > 0,\ b > 0$. By an extension to a product of several variables, we have as well

$$a_1 \dots a_n \leq \sum_{j=1}^{n} \frac{1}{p_j} a_j^{p_j} \tag{3.5}$$

where $p_j = \frac{W}{w_j}$, $W = \sum_{j=1}^{n} w_j$. Here the weight $w_j > 0$ may be attributed to a_j so that each term on the right side has weight $p_j w_j = W$. As in [4] we also use the inequality

$$\sum_{j=1}^{n} a_j^p \leq n\Big(\sum_{j=1}^{n} a_j^{p/q}\Big)^q \tag{3.6}$$

where $a_j > 0, j = 1, \dots, n$ and $p > 0,\ q > 1$, easily shown by comparing with $n(\max a_j)^p$.

We shall frequently use the Sobolev inequality

$$\|w\|_q \leq C\big(\|\nabla w\|_p + \|w\|_p\big), \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{3} > 0 \tag{3.7}$$

where C is constant for each $p < 3$ [1]. Here ∇u denotes $\frac{\partial u_i}{\partial x_k}$ regarded as 9 components of a tensor in R^3. The cases, $p = 6/5, q = 2$ and $p = 2,\ q = 6$ are most frequently employed. For $q = \infty$ we use the inequality, see [1, p. 718; 4, § 2]

$$\max_{x \in \Omega} |u| = \|u\|_\infty \leq C\big(\|u\|_6^{1/2} \, \|\nabla u\|_6^{1/2} + \|u\|_6\big). \tag{3.8}$$

In the Hilbert space $L^2(\Omega)$ a vector field $w_i(x)$ can be expressed as a sum of gradient and solenoidal components:

$$w_i(x) = v_i(x) + \nabla_i \phi(x) \tag{3.9}$$

where $v_{i,i}(x) \equiv \operatorname{div} v = 0$ in Ω and $v_n = v_i n_i = 0$ on $\partial\Omega$. It follows that these two components are orthogonal with respect to the inner product (3.2) [9]. The solenoidal part $\tilde{\Delta} u_i$ of Δu_i is known as the solenoidal Laplacian or Stokes operator [9, p. 44], while the gradient part $\nabla_i f$, with

$$\Delta u_i = \tilde{\Delta} u_i + \nabla_i f, \tag{3.10}$$

defines the viscosity potential f corresponding to u_i, up to an additive constant [4]. Note that $\tilde{\Delta}$ is a globally supported operator on vectors: $\tilde{\Delta} u_i = (\tilde{\Delta} u)_i$ in contrast to the pointwise operator Δ on components u_i.

As shown in [1] there is a special Sobolev inequality of the form

$$\|D_x w\|_6 \le C\big(\|D_x^2 w\|_2 + \|D_x w\|_2\big), \tag{3.11}$$

where we have used D_x^s to denote the s^{th} order Cartesian derivative and $\|D_x^s w\|_p^p \equiv \sum_{s_1+s_2+s_3=s} \|D_1^{s_1} D_2^{s_2} D_3^{s_3} w\|_p^p$. Then (3.8) becomes

$$\max |w| = \|w\|_\infty \le C\big(\|D_x w\|_2^{1/2} \|D_x^2 w\|_2^{1/2} + \|D_x w\|_2 + \|w\|_2\big). \tag{3.12}$$

If (2.11) holds and v vanishes on $\partial\Omega$, we also note, using gradient notation

$$\|\nabla v\|_6 \le C\big(\|D^2 v\|_2 + \|Dv\|_2\big)$$

by (3.11)

$$\le C\big(\|\Delta v\|_2 + \|\nabla v\|_2\big)$$

by [9, p. 21]

$$\le C\big(\|\tilde{\Delta} v\|_2 + \|\nabla f\|_2 + \|\nabla v\|_2\big)$$

by (3.10)

$$\le C\big(\|\tilde{\Delta} v\|_2 + \|\tilde{\Delta} v\|_2^{1/2} \|\nabla v\|_2^{1/2} + \|\nabla v\|_2\big)$$

by [4, Lemma 2]. Hence

$$\|\nabla v\|_6 \le C\big(\|\tilde{\Delta} v\|_2 + \|\nabla v\|_2\big) \tag{3.13}$$

Similarly, if D_α^s denotes partial differentiation of order $s = \alpha_1 + \alpha_2$ with respect to tangential coordinates defined as in [4. Chapter III], it can be shown as in Lemma 1 of that paper that

$$\|\nabla D_\alpha^s v\|_6 \le C\big(\|\tilde{\Delta} D_\alpha^s v\|_2 + \sum_{\beta \le \alpha} \|\nabla D_\beta^m v\|_2\big)$$

where $\beta_1 + \beta_2 = m$, $B_i \le \alpha_i$.

4. **Statement of the Theorem.**

The conditions that a solution of (2.1)–(2.5) can be expected to satisfy are at best those of the theorem in [4] on initial values, for non-homogeneous data functions can not, a priori, be expected necessarily to reduce or remove singularities that can only be located, if they exist, by the construction of the solution itself. The problem now becomes the specification of conditions for boundary value and body force data that will at least preserve the same behaviour of the solutions. That this is possible is shown by the following

THEOREM. *Let $u_0 \in L^2(\Omega)$ and let r, s be non-negative integers, ρ an odd positive integer. Then if Ω is bounded,*

(a) if $\|w\|_2$, $\|D_x w\|_2 \in L^4(0,T)$, and $\|w_t\|_{6/5}$, $\|B\|_{6/5} \in L^2(0,T)$ then $\|u\|_2 \in L^\infty(0,T)$ and $\|\nabla u\|_2 \in L^2(0,T)$.

(b) if for a given odd value of $\rho > 1$ we also have, for $r \geq 0$, $s \geq 0$,

$$(4.1)\qquad \|D_t^{r+1}D_x^s w\|_{6/5} \in L^{\frac{2\rho}{4r+2s+1}}(0,T), \quad 2r+s \leq \frac{1}{2}(\rho-1)$$

$$(4.1)\qquad \|w\|_2 \in L^{2\rho}(0,T), \|D_x^s w\|_2 \in L^{\frac{2\rho}{2s-1}}(0,T), \quad 0 < s \leq \frac{1}{2}(\rho+1)$$

and

$$(4.3)\qquad \|D_t^r D_x^s B\|_{6/5} \in L^{\frac{2\rho}{4r+2s+1}}(0,T), \qquad 2r+s \leq \frac{1}{2}(\rho-1)$$

then

$$(4.4)\qquad \|D_t^r D_x^s u\|_2 \in L^{\frac{2}{4r+2s-1}}(0,T), \qquad 0 < 2r+s \leq \frac{1}{2}(\rho+1),$$

and

$$(4.5)\qquad \max|D_t^r D_x^s u| \in L^{\frac{1}{2r+s+1}}(0,T), \qquad 2r+s \leq \frac{1}{2}(\rho-3)$$

(c) if for all finite $p \geq 2$ we have for $r \geq 0, s \geq 0$

$$\|D_t^r D_x^s w\|_{6/5}, \|D_x^s w\|_2, \|D_t^r D_x^s B\|_{6/5} \in L^p(0,T),$$

then w and B are $C^\infty\big(\Omega \times (0,T)\big)$ and (4.4) and (4.5) hold for all $r, s \geq 0$. If Ω is unbounded, ρ should be replaced by $\rho+2$ in (4.1), (4.2) and (4.3) of (b).

Proof of this main result will occupy the rest of this paper, frequent reference being made to the calculations of [4].

Conditions (4.1) and (4.3) also imply certain bounds in $L^2(0,T)$. By (3.7) with $p = \frac{6}{5}$, we may write

$$(4.6)\qquad \|D_t^r D_x^s w\|_2 \leq C\{\|D_t^r D_x^{s+1} w\|_{6/5} + \|D_t^r D_x^s w\|_{6/5}\}$$

for $r > 0$, $s \geq 0$. By (4.1) the first term on the right side is in $L^{\frac{2\rho}{4r+2s-1}}(0,T)$ for $2r+s \leq \frac{1}{2}(\rho+1)$ while the second term is in $L^{\frac{2\rho}{4r+2s-3}}(0,T)$ for $2r+s \leq \frac{1}{2}(\rho+3)$ and consequently also in $L^{\frac{2\rho}{4r+2s-1}}(0,T)$ for the same values of r and s. Hence we find, with the given condition (4.2), that

$$(4.7)\qquad \|D_t^r D_x^s w\|_2 \in L^{\frac{2\rho}{4r+2s-1}}(0,T), \qquad 2r+s \leq \frac{1}{2}(\rho+1).$$

Similarly

$$(4.8)\qquad \|D_t^r D_x^s B\|_2 \leq C\{\|D_t^r D_x^{s+1} B\|_{6/5} + \|D_t^r D_x^s B\|_{6/5}\}$$

where $r \geq 0$, $s \geq 0$. Again, the first term is in $L^{\frac{2\rho}{4r+2s+3}}(0,T)$ for $2r+s \leq \frac{1}{2}(\rho-3)$ while the second term is in $L^{\frac{2\rho}{4r+2s+1}}(0,T)$ and hence also in $L^{\frac{2\rho}{4r+2s+3}}(0,T)$ for $2r+s \leq \frac{1}{2}(\rho-1)$. Thus we find

$$(4.9)\qquad \|D_t^r D_x^s B\|_2 \in L^{\frac{2\rho}{4r+2s+3}}(0,T), \qquad 2r+s \leq \frac{1}{2}(\rho-3).$$

Henceforth it will be convenient to regard (4.7) and (4.9) on the same footing as (4.1)–(4.3), the hypotheses of part (b) of the Theorem.

Although slightly different versions or groupings of the hypotheses and conclusions for derivatives of various orders are possible, we have chosen what appears to be the simplest grouping of partial derivatives, namely according to their weighted order $2r+s = 2r+s_1+s_2+s_3$. For each level $\rho = 1, 3, 5, \ldots$, all the conditions for the preceding weighted orders are understood to apply also, and are used in the reduction of the inequalities that forms the main part of the following proof. Thus when ρ and $s = s_1+s_2+s_3$ are given in (4.1)–(4.3) it suffices [7, Theorem 222] to consider only the highest value of r, namely $r = [\frac{1}{4}(\rho-1) - \frac{1}{2}s]$. Likewise for given ρ and r it suffices by the embedding theorems to consider only the three highest integer values of s for compact Ω. Although higher values of p appear for lower order derivatives as ρ increases, the entire scale from $L^2(0,T)$ to $L^\infty(0,T)$ is equivalent in the sense of embeddings to half of one time derivative only.

5. **An extended Integrability Lemma.** The method to be used below depends on a sequence of inequalities, one for each order of derivatives. With part 5) of the following lemma these can be used to deduce properties of integrability. The lemma, stated in full for completeness, is an extension to non-homogeneous inequalities of Lemma 3 of [4], see also [5] and [7, p. 114, 126 and 173]. Throughout assume all functions measurable: $Q(t)$ will denote a generic function in $L^1(0,T)$.

LEMMA 1. *Let $a > 1, p > 0, F(t) \geq F_0 > 0$, $F(t) \in L^p(0,T)$, $F(t)$ continuous where finite, $G(t) \geq 0$, $Q(t) \in L^1(0,T)$ and $Q(t) \geq 0$. Let $F'(t)$ be defined a.e. and satisfy*

$$F'(t) + G(t) \leq KF^{a+p}(t) + CF^a(t)Q(t). \tag{5.1}$$

Then

1) *$F^{1-a}(t)$ has bounded variation on $[0,T]$, with non-decreasing singular part.*
2) *The discontinuities at T_k of $F^{1-a}(t)$ are jumps up from value zero $\big(F(T_k)=\infty\big)$, and the sum $\sum_k F^{1-a}(T_k+0)$ is bounded.*
3) *As $t \longrightarrow T_k - 0$, $F(t) \geq y_k(t)^{(1-a-p)^{-1}}$ where $y_k(t)$ denotes the minimal retrograde solution of $y' = -(a+p-1)[K + CQ(t)y^{\frac{p}{a+p-1}}]$ which vanishes at T_k.*
4) *Also*

$$\frac{1}{a-1}\sum_k F^{1-a}(T_k+0) + \int_0^T \frac{G(t)}{F^a(t)}\,dt \leq K\int_0^T F(t)^p\,dt + C\int_0^T Q(t)\,dt + \frac{F^{1-a}(T)}{a-1} < \infty \tag{5.2}$$

so that $G(t) < C_1F^a(t)Q_1(t)$ where $Q_1(t) \in L^1(0,T)$.

5) *Hence $G(t) \in L^{\frac{p}{a+p}}(0,T)$ and*

$$\int_0^T G(t)^{\frac{p}{a+p}}\,dt \leq (K+1)\int_0^T F^p(t)\,dt + \frac{p}{a+p}\int_0^T Q(t)\,dt + \frac{p}{a+p}\frac{F_0^{1-a}}{a-1}. \tag{5.3}$$

6) Finally, $N(t) = CF^a(t)Q(t) \in L^{\frac{p}{a+p}}(0,T)$ *with*

$$\int_0^T N(t)^{\frac{p}{a+p}}\,dt \le \frac{a}{a+p}\int_0^T F(t)^p\,dt + \frac{pC}{a+p}\int_0^T Q(t)\,dt.$$

PROOF. Omitting the non-negative term $G(t)$ from the inequality, dividing by $F^a(t)$ and expressing the left side as a perfect differential we find

$$\frac{1}{a-1}\frac{d}{dt}F^{1-a}(t) \ge -KF^p(t) - CQ(t).$$

The right hand side being integrable over t, we see that the negative variation $N_F(t)$ of $F^{1-a}(t)$ [14, p. 18] is continuous and bounded over $[0,T]$. Hence the positive and total variations of $F^{1-a}(t)$ are also bounded over $[0,T]$. This proves 1).

Since $F(t)$ is continuous where finite, $F^{1-a}(t)$ can be discontinuous only at the level zero. As the range covers non-negative values only, the jump must be up; the sum of these jumps cannot exceed the positive variation of $F^{1-a}(t)$, which is bounded over $[0,T]$ as above. This verifies 2).

To establish 3), we again omit $G(t)$ from the main inequality and divide by $F^{a+p}(t)$ obtaining

$$\frac{d}{dt}F^{1-a-p}(t) \ge -(a+p-1)[K + CF^{-p}(t)Q(t)].$$

The comparison of $F^{1-a-p}(t)$ with $y_k(t)$ is then immediate and $F^{1-a-p}(t) \le y_k(t)$ for $t < T_k$. This proves 3).

Returning to the main inequality we divide by $F^a(t)$ and integrate over (T_{k-1}, T_k) where $T_0 = 0$. Extending the interval of integration on the right to T, where $T_k < T$ for all k, and summing over k, we find the inequality in 4). This gives an explicit estimate for the sum (see 2) above) and shows that the integral is convergent. Hence $G(t)F^{-a}(t) \in L^1(0,T)$, completing the proof of 4).

Now with Hölder's inequality 5) is established as follows:

$$\begin{aligned}\int_0^T G(t)^{\frac{p}{a+p}}\,dt &= \int_0^T \frac{G(t)^{\frac{p}{a+p}}}{F(t)^{\frac{ap}{a+p}}}F(t)^{\frac{ap}{a+p}}\,dt \\ &\le \left(\int_0^T F(t)^p\,dt\right)^{\frac{a}{a+p}}\left(\int_0^T \frac{G(t)}{F(t)^a}\,dt\right)^{\frac{p}{a+p}} \\ &\le \frac{a}{a+p}\int_0^T F(t)^p\,dt + \frac{p}{a+p}\int_0^T \frac{G(t)}{F(t)^a}\,dt\end{aligned}$$

by Young's inequality (3.4) with p replaced by $(a+p)/a$ and q by $(a+p)/p$. Finally

$$\begin{aligned}\int_0^T N(t)^{\frac{p}{a+p}}\,dt &\le C^{\frac{p}{a+p}}\int_0^T F(t)^{\frac{ap}{a+p}}Q(t)^{\frac{p}{a+p}}\,dt \\ &\le C^{\frac{p}{a+p}}\left(\int_0^T F(t)^p\,dt\right)^{\frac{a}{a+p}}\left(\int_0^T Q(t)\,dt\right)^{\frac{p}{a+p}} < \infty\end{aligned}$$

by Hölder's inequality once again, so implying 6) and completing the proof of Lemma 1.

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Note that when $C = 0$, part 3) goes back to [10, p. 224]. When $C \neq 0$, the singularity of $F(t)$ has the same order if $Q(t)$ is bounded, but may be lower if $Q(t)$ is unbounded as $t \longrightarrow T_k - 0$. Since the inequality (5.1) is not integrable over $(0, T)$ as it is given, the question might arise, why the hypothesis $N(t) \leq CF(t)^a Q(t)$ is appropriate. This is, however, the best condition possible if we are to be able to integrate after division by $F(t)^a$. If $F(t)$ should have a singularity, say at T_1, then a higher than integrable singularity becomes possible for $N(t)$ at T_1, since $Q(t)$ may also be singular at T_1. However we do not know in practice when singular instants may occur, so it is assumed not possible to predict such a coincidence of singularities of $F(t)$ and $N(t)$, as a hypothesis. If there is no singularity of $F(t)$ at a given time, then any singularity of $N(t)$ at that instant is restricted to be integrable. In the applications, this restriction of unpredictability also involves causality, for the occurrence of singular instants will be influenced in part by the prior behaviour of the impressed forces represented by $N(t) = CF^a(t)Q(t)$ in the inequality (5.1).

Chapter II. Time derivatives of the gradient and Stokes operators

6. **The sequence of inequalities.** With $u_i = v_i + w_i$ as in (2.9) the Navier Stokes momentum equations become

$$v_{i,t} + w_{i,t} + v_k v_{i,k} + w_k v_{i,k} + v_k w_{i,k} + w_k w_{i,k} = -p_{,i} + \nu \Delta v_i + \nu \Delta w_i + B_i(x, t). \tag{6.1}$$

We multiply by v_i, contract over index i and integrate over Ω, obtaining

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \int_\Omega v_i w_{i,t}\, dx + \int_\Omega v_i v_k v_{i,k}\, dx + \int_\Omega v_i w_k v_{i,k} dx \\ &\quad + \int_\Omega v_i v_k w_{i,k} dx + \int_\Omega v_i w_k w_{i,k} dx \\ &\quad = -\int_\Omega v_i p_{,i}\, dx + \nu \int_\Omega v_i \Delta v_i\, dx + \nu \int_\Omega v_i \Delta w_i\, dx + \int_\Omega v_i B_i\, dx. \end{aligned} \tag{6.2}$$

In view of (2.2) and (2.6) we have $v_{k,k} = u_{k,k} - w_{k,k} = 0$ so the third term on the left is

$$\int_\Omega v_i v_k v_{i,k}\, dx = \frac{1}{2}\int_\Omega v_k (v_i^2)_{,k}\, dx = -\frac{1}{2}\int_\Omega v_{k,k}(v_i^2)\, dx = 0 \tag{6.3}$$

and the fourth term on the left of (6.2) likewise vanishes.

The second term on the left is estimated by

$$\begin{aligned} \left|\int_\Omega v_i w_{i,t}\, dx\right| &\leq \|v\|_6\, \|w_t\|_{6/5} \leq C\|\nabla v\|_2\, \|w_t\|_{6/5} \\ &\leq \frac{\nu}{8}\|\nabla v\|_2^2 + K_1\|w_t\|_{6/5}^2 \end{aligned} \tag{6.4}$$

in view of Sobolev's and Young's inequalities.

Now $\int_\Omega v_i p_{,i} dx = 0$ by the orthogonality of v_i to gradient fields, while

$$\left|\int_\Omega v_i \Delta w_i dx\right| = \left|\int_\Omega \nabla v_i \nabla w_i dx\right| \leq \frac{\nu}{8}\|\nabla v\|_2^2 + K_2\|D_x w\|_2^2$$

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and

$$\left|\int_\Omega v_i B_i dx\right| \le \|v\|_6 \, \|B\|_{6/5}$$

$$(6.5) \qquad \le C\|\nabla v\|_2 \, \|B\|_{6/5}$$

$$\le \frac{\nu}{8}\|\nabla v\|_2^2 + K_3\|B\|_{6/5}^2.$$

Also

$$\int_\Omega v_i \Delta v_i \, dx = -\int_\Omega (\nabla v)^2 \, dx = -\|\nabla v\|_2^2$$

and

$$\left|\int_\Omega v_i v_k w_{i,k} \, dx\right| \le \|v\|_6 \, \|v\|_3 \, \|D_x w\|_2$$

$$\le C\|\nabla v\|_2 \, \|v\|_6^{1/2}\|v\|_2^{1/2}\|D_x w\|_2$$

$$(6.6) \qquad \le C\|\nabla v\|_2^{3/2} \, \|v\|_2^{1/2}\|D_x w\|_2$$

$$\le \frac{\nu}{16}\|\nabla v\|_2^2 + K_4\|v\|_2^2\|D_x w\|_2^4$$

by Young's inequality with exponents $\frac{4}{3}$ and 4. Similarly,

$$(6.7) \qquad \left|\int_\Omega v_i w_k w_{i,k} \, dx\right| \le \|v\|_3 \, \|w\|_6 \, \|D_x w\|_2$$

$$\le C\|v\|_6^{1/2} \, \|v\|_2^{1/2}(\|D_x w\|_2 + \|w\|_2)\|D_x w\|_2$$

$$\le C\|\nabla v\|_2^{1/2} \, \|v\|_2^{1/2}(\|D_x w\|_2^2 + \|w\|_2^2)$$

$$\le \frac{\nu}{16}\|\nabla v\|_2^2 + K_5\|v\|_2^{2/3}(\|D_x w\|_2^{8/3} + \|w\|_2^{8/3})$$

$$\le \frac{\nu}{16}\|\nabla v\|_2^2 + K_6\Big(\|v\|_2^2 + \|D_x w\|_2^4 + \|w\|_2^4\Big)$$

where Young's inequality has been used twice, with exponents 4 and $\frac{4}{3}$, then with exponents 3 and $\frac{3}{2}$. Two integrals containing $v_i v_{i,k} = \frac{1}{2}(v_i)^2_{,k}$ vanish, by (2.6) and (2.11). Assembling these estimates, we find, after multiplication by 2 and certain cancellations

$$(6.8) \qquad \frac{d}{dt}\|v\|_2^2 + \nu\|\nabla v\|_2^2 \le C\|v\|_2^2(\|D_x w\|_2^4 + 1) + N_1(t)$$

where

$$(6.9) \qquad N_1(t) = C\big\{\|D_x w\|_2^2 + \|D_x w\|_2^4 + \|w\|_2^4 + \|w_t\|_{6/5}^2 + \|B\|_{6/5}^2\big\}.$$

By hypothesis (a) $C\|D_x w\|_2^4 \in L^1(0,T)$. Hence the integrating factor $\exp\big(-C\int_0^t(\|D_x w\|_2^4 + 1)\,dt'\big)$ is bounded above and below by positive constants:

$$(6.10) \qquad 0 < A_1 < \exp\Big(-\int_0^t C(\|D_x w\|_2^4 + 1)\,dt'\Big) < A_2$$

for $0 \le t \le T$.

We may omit the second term on the left of (6.8) and find an explicit bound for the resulting first order linear differential inequality for $\|v\|_2^2$:

$$
\begin{aligned}
\|v\|_2^2(t) \le \|v_0\|_2^2 \exp\Big(C\int_0^t (\|D_x w\|_2^4+1)\,dt'\Big) \\
+\int_0^t \exp\Big(C\int_\tau^t (\|D_x w\|_2^4+1)\,dt'\Big)N_1(\tau)\,d\tau \\
\le A_1^{-1}\|v_0\|_2^2+\frac{A_2}{A_1}\int_0^t N_1(\tau)\,d\tau.
\end{aligned} \tag{6.11}
$$

By hypothesis (a), $N_1(\tau)\in L^1(0,T)$ so this expression is uniformly bounded on $0\le t\le T$. Hence $\|v\|_2(t) < K_0$, say, in this range.

Returning to the inequality (6.8) we can now integrate it directly, and so find

$$
\begin{aligned}
\|v\|_2^2(T)+\nu\int_0^T \|\nabla v\|_2^2\,dt \le CK_0\int_0^T(\|D_x w\|_2^4+1)\,dt+\int_0^T N_1(t')\,dt'+\|v_0\|_2^2 \\
\le K_7<\infty.
\end{aligned} \tag{6.12}
$$

Hence $\|\nabla v\|_2\in L^2(0,T)$ as required. This completes the case (a) of the Theorem.

To establish the second inequality, multiply (6.1) by $\tilde\Delta v_i$ and integrate; obtaining

$$
\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2+\int_\Omega \tilde\Delta v_i w_{i,t}\,dx+\int_\Omega \tilde\Delta v_i v_k v_{i,k}\,dx \\
+\int_\Omega \tilde\Delta v_i w_k v_{i,k}\,dx+\int_\Omega \tilde\Delta v_i v_k w_{i,k}\,dx+\int_\Omega \tilde\Delta v_i w_k w_{i,k}\,dx \\
=-\int_\Omega \tilde\Delta v_i p_{,i}\,dx+\nu\int_\Omega \tilde\Delta v_i \Delta v_i\,dx+\nu\int_\Omega \tilde\Delta v_i\Delta w_i\,dx+\int_\Omega \tilde\Delta v_i B_i\,dx.
\end{aligned} \tag{6.13}
$$

Detailed reductions of the terms are as follows:

$$
\Big|\int_\Omega \tilde\Delta v_i w_{i,t}\,dx\Big| \le \|\tilde\Delta v\|_2\|w_t\|_2 \le \frac{\nu}{16}\|\tilde\Delta v\|_2^2+C\|w_t\|_2^2
$$

$$
\begin{aligned}
\Big|\int_\Omega \tilde\Delta v_i v_k v_{i,k}\,dx\Big| &\le \|\tilde\Delta v\|_2\|v\|_6\|\nabla v\|_3 \\
&\le C\|\tilde\Delta v\|_2\|\nabla v\|_2\|\nabla v\|_2^{1/2}\|\nabla v\|_6^{1/2} \\
&\le C\|\tilde\Delta v\|_2^{3/2}\|\nabla v\|_2^{3/2}+C\|\tilde\Delta v\|_2\|\nabla v\|_2^2 \\
&\le \frac{\nu}{16}\|\tilde\Delta v\|_2^2+C\big\{\|\nabla v\|_2^6+\|\nabla v\|_2^4\big\}
\end{aligned}
$$

where we have used (2.20) of [4] and Young's inequality with indices 4 and $\frac{4}{3}$, 2 and 2,

$$
\begin{aligned}
\Big|\int_\Omega \tilde\Delta v_i w_k v_{i,k}\,dx\Big| &\le \|\tilde\Delta v\|_2\|w\|_6\|\nabla v\|_3 \\
&\le C\|\tilde\Delta v\|_2(\|\tilde\Delta v\|_2^{1/2}+\|\nabla v\|_2^{1/2})(\|D_x w\|_2+\|w\|_2)\|\nabla v\|_2^{1/2} \\
&\le \frac{\nu}{16}\|\tilde\Delta v\|_2^2+K(\|D_x w\|_2^4+\|w\|_2^4+\|D_x w\|_2^2+\|w\|_2^2)\|\nabla v\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int_\Omega \tilde\Delta v_i v_k w_{i,k}\,dx\Big| &\le \|\tilde\Delta v\|_2\|v\|_6\|D_x w\|_3 \\
&\le C\|\tilde\Delta v\|_2\|\nabla v\|_2\|D_x w\|_2^{1/2}(\|D_x^2 w\|_2^{1/2}+\|D_x w\|_2^{1/2}) \\
&\le \frac{\nu}{16}\|\tilde\Delta v\|_2^2+K\|\nabla v\|_2^2\|D_x w\|_2(\|D_x^2 w\|_2+\|D_x w\|_2)
\end{aligned}
$$

$$\begin{aligned}\left|\int_\Omega \tilde{\Delta} v_i w_k w_{i,k}\,dx\right| &\le \|\tilde{\Delta} v\|_2\|w\|_6\|D_x w\|_3\\ &\le C\|\tilde{\Delta} v\|_2\|D_x w\|_2^{1/2}(\|D_x w\|_2+\|w\|_2)(\|D_x^2 w\|_2^{1/2}+\|D_x w\|_2^{1/2})\\ &\le \frac{\nu}{16}\|\tilde{\Delta} v\|_2^2+K(\|D_x w\|_2^3+\|w\|_2^3)(\|D_x^2 w\|_2+\|D_x w\|_2)\end{aligned}$$

$$\int_\Omega \tilde{\Delta} v_i p_{,i}\,dx = 0$$

$$\left|\int_\Omega \tilde{\Delta} v_i \Delta w_i\,dx\right| \le \frac{\nu}{16}\|\tilde{\Delta} v\|_2^2 + C\|\Delta w\|_2^2$$

$$\left|\int_\Omega \tilde{\Delta} v_i B_i\,dx\right| \le \frac{\nu}{16}\|\tilde{\Delta} v\|_2^2 + C\|B\|_2^2$$

Collecting terms, we find

$$\begin{aligned}&\frac{d}{dt}\|\nabla v\|_2^2+\frac{9}{8}\nu\|\tilde{\Delta} v\|_2^2\\ &\quad\le K\big\{\|\nabla v\|_2^6+\|w_t\|_2^2+(\|D_x w\|_2^4+\|w\|_2^4+\|D_x w\|_2^2+\|w\|_2^2)\|\nabla v\|_2^2\\ &\qquad+\|\nabla v\|_2^2\|D_x w\|_2\|D_x^2 w\|_2+\|D_x w\|_2^3\|D_x^2 w\|_2\\ &\qquad+\|D_x^2 w\|_2^2+\|B\|_2^2\big\}.\end{aligned}$$

Since

$$\|\nabla v\|_2^2\|D_x w\|_2\|D_x^2 w\|_2 \le \frac{1}{3}\|\nabla v\|_2^6+\frac{1}{6}\|D_x w\|_2^6+\frac{1}{2}\|D_x^2 w\|_2^2$$

and

$$\|D_x w\|_2^3\|D_x^2 w\|_2 \le \frac{1}{2}\|D_x w\|_2^6+\frac{1}{2}\|D_x^2 w\|_2^2$$

we find

$$\frac{d}{dt}\|\nabla v\|_2^2+\frac{9}{8}\nu\|\tilde{\Delta} v\|_2^2 \le K\|\nabla v\|_2^6+N_3(t) \tag{6.14}$$

where

$$N_3(t)=C\big\{\|w_t\|_2^2+\|B\|_2^2+\|D_x^2 w\|_2^2+\|D_x w\|_2^6+\|w\|_2^6+\|D_x w\|_2^3+\|w\|_2^3+1\big\} \tag{6.15}$$

Again, multiplying (6.1) by $v_{i,t}$ and integrating, we have

$$\begin{aligned}&\|v_t\|_2^2+\int_\Omega v_{i,t}w_{i,t}\,dx\int_\Omega v_{i,t}v_k v_{i,k}\,dx+\int_\Omega v_{i,t}w_k v_{i,k}\,dx\\ &\quad+\int_\Omega v_{i,t}v_k w_{i,k}\,dx+\int_\Omega v_{i,t}w_k w_{i,k}\,dx\\ &\quad=-\int_\Omega v_{i,t}p_{,i}\,dx+\nu\int_\Omega v_{i,t}\Delta v_i\,dx+\nu\int_\Omega v_{i,t}\Delta w_i\,dx+\int_\Omega v_{i,t}B_i\,dx.\end{aligned} \tag{6.16}$$

In detail, we find

$$\left|\int_\Omega v_{i,t}w_{i,t}\,dx\right| \le \|v_t\|_2\|w_t\|_2 \le \frac{1}{16}\|v_t\|_2^2+K\|w_t\|_2^2$$

$$\left|\int_\Omega v_{i,t}v_kv_{i,k}\,dx\right| \le \|v_t\|_2\|v\|_6\|\nabla v\|_3$$
$$\le C\|v_t\|_2\|\nabla v\|_2^{3/2}\|\tilde\Delta v\|_2^{1/2}$$
$$\le \frac{1}{16}\|v_t\|_2^2 + K\|\nabla v\|_2^3\|\tilde\Delta v\|_2$$
$$\le \frac{1}{16}\|v_t\|_2^2 + \frac{\nu}{32}\|\tilde\Delta v\|_2^2 + K\|\nabla v\|_2^6$$

$$\left|\int_\Omega v_{i,t}w_kv_{i,k}\,dx\right| \le \|v_t\|_2\|w\|_6\|\nabla v\|_3$$
$$\le C\|v_t\|_2(\|D_xw\|_2+\|w\|_2)\|\nabla v\|_2^{1/2}\|\tilde\Delta v\|_2^{1/2}$$
$$\le \frac{1}{16}\|v_t\|_2^2 + \frac{\nu}{32}\|\tilde\Delta v\|_2^2 + K\left\{\|\nabla v\|_2^6 + \|D_xw\|_2^6 + \|w\|_2^6\right\}$$

by Young's inequality for four factors, with two small coefficients, and exponents 2, 6, 12 and 4,

$$\left|\int_\Omega v_{i,t}v_kw_{i,k}\,dx\right| \le \|v_t\|_2\|v\|_6\|D_xw\|_3$$
$$\le C\|v_t\|_2\|\nabla v\|_2\|D_xw\|_2^{1/2}\|D_x^2w\|_2^{1/2}$$
$$\le \frac{1}{16}\|v_t\|_2^2 + K\left\{\|D_x^2w\|_2^2 + \|\nabla v\|_2^6 + \|D_xw\|_2^6\right\}$$

$$\left|\int_\Omega v_{i,t}w_kw_{i,k}\,dx\right| \le \|v_t\|_2\|w\|_6\|D_xw\|_3$$
$$\le C\|v_t\|_2(\|D_xw\|_2^{3/2} + \|w\|_2^{3/2})(\|D_x^2w\|_2^{1/2} + \|D_xw\|_2^{1/2})$$
$$\le \frac{1}{16}\|v_t\|_2^2 + K\left\{\|D_x^2w\|_2^2 + \|D_xw\|_2^6 + \|w\|_2^6)\right\}$$

$$\int_\Omega v_{i,t}p_{,i}\,dx = 0$$
$$\int_\Omega v_{i,t}\Delta v_i\,dx = -\int_\Omega \nabla v_{i,t}\nabla v_i\,dx = -\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2$$
$$\left|\int_\Omega v_{i,t}\Delta w_i\,dx\right| \le \|v_t\|_2\|D_x^2w\|_2 \le \frac{1}{16}\|v_t\|_2^2 + K\|D_x^2w\|_2^2$$
$$\left|\int_\Omega v_{i,t}B_i\,dx\right| \le \|v_t\|_2\|B\|_2 \le \frac{1}{16}\|v_t\|_2^2 + K\|B\|_2^2$$

Assembling these inequalities, we find after multiplying by 2,

$$\nu\frac{d}{dt}\|\nabla v\|_2^2 + \|v_t\|_2^2 \le K\|\nabla v\|_2^6 + \frac{\nu}{8}\|\tilde\Delta v\|_2^2 + N_{32}(t) \tag{6.17}$$

where

$$N_{32}(t) = K\left\{\|w_t\|_2^2 + \|D_x^2w\|_2^2 + \|D_xw\|_2^6 + \|w\|_2^6 + \|B\|_2^2\right\} \le CN_3(t). \tag{6.18}$$

As in (8.1) of [4] we now add (6.14) and (6.17) together with the result that after cancellation of a term in $\|\tilde{\Delta} v\|_2^2$,

$$(6.19) \qquad (1+\nu)\frac{d}{dt}\|\nabla v\|_2^2 + \nu\|\tilde{\Delta} v\|_2^2 + \|v_t\|_2^2 \le K\|\nabla v\|_2^6 + CN_3(t).$$

By hypothesis for $\rho = 3$, we have $N_3(t) \in L^1(0,T)$. We write $F_3(t) = 1 + \|\nabla v\|_2^2$ and $G_3(t) = \nu\|\tilde{\Delta} v\|_2^2 + \|v_t\|_2^2$ as in [4, § 8]. Observing that $N_3(t) < CF_3(t)^2 Q(t)$ where $Q(t) \in L^1(0,T)$ we see that the hypotheses of Lemma 1 hold with $p = 1, a = 2$. Hence the conclusion $G_3(t) \in L^{1/3}(0,T)$ is also valid. Consequently $\|\tilde{\Delta} v\|_2$ and $\|v_t\|_2 \in L^{2/3}(0,T)$ as desired. This in turn, together with the similar hypotheses on w which are satisfied with much to spare imply $\|\tilde{\Delta} u\|_2, \|u_t\|_2 \in L^{2/3}(0,T)$.

For the second group of three inequalities we differentiate (6.1) once with respect to t, obtaining

$$(6.20) \qquad \begin{aligned} &v_{i,tt} + w_{i,tt} + v_{k,t}v_{i,k} + v_k v_{i,kt} + w_{k,t}v_{i,k} + w_k v_{i,kt} \\ &\quad + v_{k,t}w_{i,k} + v_k w_{i,kt} + w_{k,t}w_{i,k} + w_k w_{i,kt} \\ &\qquad = -p_{,it} + \nu\Delta v_{i,t} + \nu\Delta w_{i,t} + B_{i,t}(x,t). \end{aligned}$$

First multiply by $v_{i,t}$ and integrate, finding

$$(6.21) \qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_t\|_2^2 + \int_\Omega v_{i,t}w_{i,tt}\,dx + \int_\Omega v_{i,t}v_{k,t}v_{i,k}\,dx \\ &\quad + \int_\Omega v_{i,t}v_k v_{i,kt}\,dx + \int_\Omega v_{i,t}w_{k,t}v_{i,k}\,dx \\ &\quad + \int_\Omega v_{i,t}w_k v_{i,kt}\,dx + \int_\Omega v_{i,t}v_{k,t}w_{i,k}\,dx \\ &\quad + \int_\Omega v_{i,t}v_k w_{i,kt}\,dx + \int_\Omega v_{i,t}w_{k,t}w_{i,k}\,dx + \int_\Omega v_{i,t}w_k w_{i,kt}\,dx \\ &\qquad = -\int_\Omega v_{i,t}p_{,it}\,dx + \nu\int_\Omega v_{i,t}\Delta v_{i,t}\,dx \\ &\qquad\quad + \nu\int_\Omega v_{i,t}\Delta w_{i,t}\,dx + \int_\Omega v_{i,t}B_{i,t}\,dx. \end{aligned}$$

Using $x^a < 1 + x^b, 0 < a < b, x > 0$ to retain highest powers only, we find

$$\begin{aligned} \left|\int_\Omega v_{i,t}w_{i,tt}\,dx\right| &\le \|v_t\|_6\|w_{tt}\|_{6/5} \\ &\le C\|\nabla v_t\|_2\|w_{tt}\|_{6/5} \\ &\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|w_{tt}\|_{6/5}^2 \end{aligned}$$

$$\left|\int_\Omega v_{i,t}v_{k,t}v_{i,k}\,dx\right| \le \|v_t\|_4^2\|\nabla v\|_2$$
$$\le C\|\nabla v_t\|_2^{3/2}\|v_t\|_2^{1/2}\|\nabla v\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|v_t\|_2^2\|\nabla v\|_2^4$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|v_t\|_2^{10/3} + \|\nabla v\|_2^{10}\right)$$

$$\int_\Omega v_{i,t}v_kv_{i,kt}\,dx = \int_\Omega v_k\frac{1}{2}(v_{i,t})^2_{,k}\,dx$$
$$= -\int_\Omega v_{k,k}\frac{1}{2}(v_{i,t})^2\,dx = 0$$

$$\int_\Omega v_{i,t}w_kv_{i,kt}\,dx = \int_\Omega w_k\frac{1}{2}(v_{i,t})^2_{,k}\,dx$$
$$= -\int_\Omega w_{k,k}\frac{1}{2}(v_{i,t})^2\,dx = 0$$

$$\left|\int_\Omega v_{i,t}w_{k,t}v_{i,k}\,dx\right| \le \|v_t\|_6\|w_t\|_3\|\nabla v\|_2$$
$$\le C\|\nabla v_t\|_2(\|w_t\|_2^{1/2}\|D_xw_t\|_2^{1/2} + \|w_t\|_2)\|\nabla v\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + \|w_t\|_2(\|D_xw_t\|_2 + \|w_t\|_2)\|\nabla v\|_2^2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|D_xw_t\|_2^2 + \|w_t\|_2^{10/3} + \|\nabla v\|_2^{10} + 1\right)$$

$$\left|\int_\Omega v_{i,t}v_{k,t}w_{i,k}\,dx\right| \le \|v_t\|_4^2\|D_xw\|_2$$
$$\le C\|\nabla v_t\|_2^{3/2}\|v_t\|_2^{1/2}\|D_xw\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|v_t\|_2^2\|D_xw\|_2^4$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|v_t\|_2^{10/3} + \|D_xw\|_2^{10}\right)$$

$$\left|\int_\Omega v_{i,t}v_kw_{i,kt}\,dx\right| \le \|v_t\|_3\|v\|_6\|D_xw_t\|_2$$
$$\le C\|\nabla v_t\|_2^{1/2}\|v_t\|_2^{1/2}\|\nabla v\|_2\|D_xw_t\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|v_t\|_2^{2/3}\|\nabla v\|_2^{4/3}\|D_xw_t\|_2^{4/3}$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|v_t\|_2^{10/3} + \|\nabla v\|_2^{10} + \|D_xw_t\|_2^2\right)$$

$$\left|\int_\Omega v_{i,t}w_{k,t}w_{i,k}\,dx\right| \le \|v_t\|_6\|w_t\|_3\|D_xw\|_2$$
$$\le C\|\nabla v_t\|_2\|w_t\|_2^{1/2}(\|D_xw_t\|_2^{1/2} + \|w_t\|_2^{1/2})\|D_xw\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|w_t\|_2(\|D_xw_t\|_2 + \|w_t\|_2)\|D_xw\|_2^2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|D_xw_t\|_2^2 + \|w_t\|_2^{10/3} + \|D_xw\|_2^{10} + 1\right)$$

$$\left|\int_\Omega v_{i,t} w_k w_{i,kt}\,dx\right| \le \|v_t\|_3 \|w\|_6 \|D_x w_t\|_2$$
$$\le C\|\nabla v_t\|_2^{1/2}\|v_t\|_2^{1/2}(\|D_x w\|_2 + \|w\|_2)\|D_x w_t\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|v_t\|_2^{2/3}(\|D_x w\|_2^{4/3} + \|w\|_2^{4/3})\|D_x w_t\|_2^{4/3}$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\left(\|D_x w_t\|_2^2 + \|v_t\|_2^{10/3} + \|D_x w\|_2^{10} + \|w\|_2^{10} + 1\right)$$

$$\int_\Omega v_{i,t} p_{,it}\,dx = 0$$

$$\int_\Omega v_{i,t}\Delta v_{i,t}\,dx = -\int_\Omega (\nabla v_{i,t})^2\,dx = -\|\nabla v_t\|_2^2$$

$$\left|\int_\Omega v_{i,t}\Delta w_{i,t}\,dx\right| = \left|-\int_\Omega \nabla v_{i,t} D_x w_{i,t}\,dx\right|$$
$$\le \|\nabla v_t\|_2 \|D_x w_t\|_2$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|D_x w_t\|_2^2$$

$$\left|\int_\Omega v_{i,t} B_{i,t}\,dx\right| \le \|v_t\|_6 \|B_t\|_{6/5}$$
$$\le C\|\nabla v_t\|_2 \|B_t\|_{6/5}$$
$$\le \frac{\nu}{32}\|\nabla v_t\|_2^2 + K\|B_t\|_{6/5}^2$$

Combining the results of these calculations we obtain the inequality

$$\begin{aligned}(6.22)\quad \frac{d}{dt}\|v_t\|_2^2 + \nu\|\nabla v_t\|_2^2 \le K\Big\{&\|v_t\|_2^{10/3} + \|\nabla v\|_2^{10} \\ &+ \|D_x w_t\|_2^2 + \|w_t\|_2^{10/3} + \|D_x w\|_2^{10} + \|w\|_2^{10} \\ &+ \|w_{tt}\|_{6/5}^2 + \|B_t\|_{6/5}^2 + 1\Big\}\end{aligned}$$

Now multiply (6.14) by $3\|\nabla v\|_2^4$ and add the resulting expressions to (6.22), obtaining

$$\begin{aligned}&\frac{d}{dt}(\|v_t\|_2^2 + \|\nabla v\|_2^6) + \nu\left\{\|\nabla v_t\|_2^2 + 3\|\nabla v\|_2^4\|\tilde{\Delta} v\|_2^2\right\} \\ (6.23)\quad &\le K\left\{\|v_t\|_2^{10/3} + \|\nabla v\|_2^{10}\right\} + M_5(t) + 3\|\nabla v\|_2^4 N_3(t) \\ &\le K\left(\|v_t\|_2^2 + \|\nabla v\|_2^6\right)^{5/3} + M_5(t) + 3\|\nabla v\|_2^4 N_3(t)\end{aligned}$$

where

$$\begin{aligned}M_5(t) = K\Big\{&\|D_x w_t\|_2^2 + \|w_t\|_2^{10/3} + \|D_x^2 w\|_2^{10/3} + \|D_x w\|_2^{10} + \|w\|_2^{10} \\ (6.24)\quad &+ \|w_{tt}\|_{6/5}^2 + \|B_t\|_{6/5}^2 + \|B\|_2^{10/3} + 1\Big\}.\end{aligned}$$

As in (8.2) of [4] we set

$$F_5(t) = 1 + \|v_t\|_2^2 + \|\nabla v\|_2^6$$

and

$$G_5(t) = \|\nabla v_t\|_2^2 + 3\|\nabla v\|_2^4\|\tilde{\Delta} v\|_2^2$$

By the hypotheses of the theorem for $\rho = 5$ we have $M_5(t) \in L^1(0,t)$ and we define

$$N_5(t) = M_5(t) + 3\|\nabla v\|_2^4 N_3(t).$$

Thus

$$\begin{aligned} N_5(t) &< F_5^{4/3}(t)Q(t) + 3F_5^{4/3}(t)Q(t) \\ &< F_5^{4/3}(t)Q(t) \end{aligned}$$

where $Q(t) \in L^1(0,T)$ as required in the integrability lemma with $p = \frac{1}{3}$, $a = \frac{4}{3}$. The lemma then applies to (6.23) and we conclude $G_5(t) \in L^{\frac{1}{5}}(0,T)$ so that $\|\nabla v_t\|_2 \in L^{\frac{2}{5}}(0,T)$.

The next inequality is found if we multiply (6.20) by $\tilde{\Delta} v_{i,t}$ and then integrate, obtaining

$$\begin{aligned} &\int_\Omega \tilde{\Delta} v_{i,t} v_{i,tt}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} w_{i,tt}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} v_{k,t} v_{i,k}\,dx \\ &\quad + \int_\Omega \tilde{\Delta} v_{i,t} v_k v_{ik,t}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} w_{k,t} v_{i,k}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} w_k v_{i,kt}\,dx \\ &\quad + \int_\Omega \tilde{\Delta} v_{i,t} v_{k,t} w_{i,k}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} v_k w_{i,kt}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} w_{k,t} w_{i,k}\,dx \\ &\quad + \int_\Omega \tilde{\Delta} v_{i,t} w_k w_{i,kt}\,dx \\ &\quad = -\int_\Omega \tilde{\Delta} v_{i,t} p_{,it}\,dx + \nu \int_\Omega \tilde{\Delta} v_{i,t} \Delta v_{i,t}\,dx + \nu \int_\Omega \tilde{\Delta} v_{i,t} \Delta w_{i,t}\,dx + \int_\Omega \tilde{\Delta} v_{i,t} B_{i,t}\,dx \end{aligned} \tag{6.25}$$

The various terms are transformed as follows, by orthogonality and inequalities:

$$\begin{aligned} \int_\Omega \tilde{\Delta} v_{i,t} v_{i,tt}\,dx &= \int_\Omega \Delta v_{i,t} v_{i,tt}\,dx \\ &= -\int_\Omega \nabla v_{i,t} \nabla v_{i,tt}\,dx \\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla v_t\|_2^2 \end{aligned}$$

$$\begin{aligned} \left|\int_\Omega \tilde{\Delta} v_{i,t} w_{i,tt}\,dx\right| &\le \|\tilde{\Delta} v_t\|_2 \|w_{tt}\|_2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|w_{tt}\|_2^2 \end{aligned}$$

$$\begin{aligned} \left|\int_\Omega \tilde{\Delta} v_{i,t} v_{k,t} v_{i,k}\,dx\right| &\le \|\tilde{\Delta} v_t\|_2 \max|v_t|\,\|\nabla v\|_2 \\ &\le C\|\tilde{\Delta} v_t\|_2^{3/2}\|\nabla v_t\|_2^{1/2}\|\nabla v\|_2 + C\|\tilde{\Delta} v_t\|_2\|\nabla v_t\|_2\|\nabla v\|_2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v_t\|_2^2(\|\nabla v\|_2^4 + \|\nabla v\|_2^2) \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\Big(\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\Big) \end{aligned}$$

$$\left|\int_\Omega \tilde{\Delta} v_{i,t} v_k v_{i,kt}\, dx\right| \le \|\tilde{\Delta} v_t\|_2 \|v\|_6 \|\nabla v_t\|_3$$
$$\le C\|\tilde{\Delta} v_t\|_2 \|\nabla v\|_2 \|\nabla v_t\|_2^{1/2}(\|\tilde{\Delta} v_t\|_2^{1/2} + \|\nabla v_t\|_2^{1/2})$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v_t\|_2^2\left(\|\nabla v\|_2^4 + \|\nabla v\|_2^2\right)$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\right)$$

$$\left|\int_\Omega \tilde{\Delta} v_{i,t} w_{k,t} v_{i,k}\, dx\right|$$
$$\le \|\tilde{\Delta} v_t\|_2 \max|w_t| \, \|\nabla v\|_2$$
$$\le C\|\tilde{\Delta} v_t\|_2 \|(\|D_x w_t\|_2^{1/2}\|D_x^2 w_t\|_2^{1/2} + \|D_x w_t\|_2 + \|w_t\|_2)\|\nabla v\|_2$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K(\|D_x w_t\|_2 \|D_x^2 w_t\|_2 + \|D_x w_t\|_2^2 + \|w_t\|_2^2)\|\nabla v\|_2^2$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|w_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\right)$$

$$\left|\int_\Omega \tilde{\Delta} v_{i,t} w_k v_{i,kt}\, dx\right| \le \|\tilde{\Delta} v_t\|_2 \|w\|_6 \|\nabla v_t\|_3$$
$$\le C\|\tilde{\Delta} v_t\|_2(\|D_x w\|_2 + \|w\|_2)\|\nabla v_t\|_2^{1/2}(\|\tilde{\Delta} v_t\|_2^{1/2} + \|\nabla v_t\|_2^{1/2})$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K(\|D_x w\|_2^4 + \|w\|_2^4 + 1)\|\nabla v_t\|_2^2$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|\nabla v_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} + 1\right)$$

$$\left|\int_\Omega \tilde{\Delta} v_{i,t} v_{k,t} w_{i,k}\, dx\right| \le \|\tilde{\Delta} v_t\|_2 \max|v_t| \, \|D_x w\|_2$$
$$\le C\|\tilde{\Delta} v_t\|_2^{3/2}\|\nabla v_t\|_2^{1/2}\|D_x w\|_2 + C\|\tilde{\Delta} v_t\|_2\|\nabla v_t\|_2\|D_x w\|_2$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v_t\|_2^2\left(\|D_x w\|_2^4 + 1\right)$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|\nabla v_t\|_2^{14/5} + \|D_x w\|_2^{14} + 1\right)$$

$$\left|\int_\Omega \tilde{\Delta} v_{i,t} v_k w_{i,kt}\, dx\right|$$
$$\le \|\tilde{\Delta} v_t\|_2 \|v\|_6 \|D_x w_t\|_3$$
$$\le C\|\tilde{\Delta} v_t\|_2 \|\nabla v\|_2(\|D_x w_t\|_2^{1/2} + \|w_t\|_2^{1/2})\|D_x^2 w_t\|_2^{1/2}$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v\|_2^2(\|D_x w_t\|_2 + \|w_t\|_2)\|D_x^2 w_t\|_2$$
$$\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|w_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\right)$$

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$$\begin{aligned}\left|\int_\Omega \tilde{\Delta} v_{i,t} w_{k,t} w_{i,k}\, dx\right| &\le \|\tilde{\Delta} v_t\|_2 \max |w_t|\, \|D_x w\|_2 \\ &\le C\|\tilde{\Delta} v_t\|_2 (\|D_x w_t\|_2^{1/2} \|D_x^2 w_t\|_2^{1/2} + \|D_x w_t\|_2 + \|w_t\|_2)\|D_x w\|_2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K(\|D_x w_t\|_2 \|D_x^2 w_t\|_2 + \|D_x w_t\|_2^2 + \|w_t\|_2^2)\|D_x w\|_2^2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|w_t\|_2^{14/5} + \|D_x w\|_2^{14} + 1\right)\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega \tilde{\Delta} v_{i,t} w_k w_{i,kt}\, dx\right| &\le \|\tilde{\Delta} v_t\|_2 \|w\|_6 \|D_x w_t\|_3 \\ &\le C\|\tilde{\Delta} v_t\|_2 (\|D_x w\|_2 + \|w_t\|_2)(\|D_x w_t\|_2^{1/2}\|D_x^2 w_t\|_2^{1/2} + \|D_x w_t\|) \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K(\|D_x w\|_2^2 + \|w\|_2^2)\|D_x w_t\|_2(\|D_x^2 w_t\|_2 + \|D_x w_t\|_2) \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\left(\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} + 1\right)\end{aligned}$$

$$\int_\Omega \tilde{\Delta} v_{i,t} p_{,it}\, dx = 0$$

$$\int_\Omega \tilde{\Delta} v_{i,t} \Delta v_{i,t}\, dx = \int_\Omega \left(\tilde{\Delta} v_{i,t}\right)^2 dx = \|\tilde{\Delta} v_t\|_2^2$$

$$\begin{aligned}\int_\Omega \tilde{\Delta} v_{i,t} \Delta w_{,t}\, dx &= \int_\Omega \tilde{\Delta} v_{i,t} D_x^2 w_{i,t}\, dx \\ &\le \|\tilde{\Delta} v_t\|_2 \|D_x^2 w_{,t}\|_2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|D_x^2 w_t\|_2^2\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega \tilde{\Delta} v_{i,t} B_{i,t}\, dx\right| &\le \|\tilde{\Delta} v_t\|_2 \|B_t\|_2 \\ &\le \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|B_t\|_2^2\end{aligned}$$

When combined these calculations lead to the inequality

$$\begin{aligned}\frac{d}{dt}\|\nabla v_t\|_2^2 + \frac{3}{2}\nu\|\tilde{\Delta} v_t\|_2^2 \le K\Big\{&\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} \\ &+ \|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} \\ &+ \|w_{tt}\|_2^2 + \|B_t\|_2^2 + 1\Big\}.\end{aligned} \tag{6.26}$$

As in [4, § 7] we shall combine this inequality with a companion inequality of the same singular index obtained when (6.20) is multiplied by $v_{i,tt}$ and integrated. Thus we obtain

$$\begin{aligned}&\int_\Omega (v_{i,tt})^2\, dx + \int_\Omega v_{i,tt} w_{i,tt}\, dx + \int_\Omega v_{i,tt} v_{k,t} v_{i,k}\, dx \\ &+ \int_\Omega v_{i,tt} v_k v_{i,kt}\, dx + \int_\Omega v_{i,tt} w_{k,t} v_{i,k}\, dx + \int_\Omega v_{i,tt} w_k v_{i,kt}\, dx \\ &+ \int_\Omega v_{i,tt} v_{k,t} w_{i,k}\, dx + \int_\Omega v_{i,tt} v_k w_{i,kt}\, dx + \int_\Omega v_{i,tt} w_{k,t} w_{i,k}\, dx\end{aligned} \tag{6.27}$$

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$$+\int_\Omega v_{i,tt} w_k w_{i,kt}\, dx$$
$$= -\int_\Omega v_{i,tt} p_{,it}\, dx + \nu \int_\Omega v_{i,tt} \Delta v_{i,t}\, dx + \nu \int_\Omega v_{i,tt} \Delta w_{i,t}\, dx$$
$$+ \int_\Omega v_{i,tt} B_{i,t}\, dx.$$

These various terms should be transformed as follows; the first term on the left being obviously in the desired form already:

$$\begin{aligned}\left|\int_\Omega v_{i,tt} w_{i,tt}\, dx\right| &\le \|v_{tt}\|_2 \|w_{tt}\|_2 \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + K\|w_{tt}\|_2^2\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega v_{i,tt} v_{k,t} v_{i,k}\, dx\right| &\le \|v_{tt}\|_2 \max|v_t|\ \|\nabla v\|_2 \\ &\le C\|v_{tt}\|_2(\|\nabla v_t\|_2^{1/2}\|\tilde{\Delta} v_t\|_2^{1/2} + \|\nabla v_t\|_2)\|\nabla v\|_2 \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + K_1(\|\nabla v_t\|_2\|\tilde{\Delta} v_t\|_2\| + \|\nabla v_t\|_2^2)\|\nabla v\|_2^2 \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v_t\|_2^2(\|\nabla v\|_2^4 + \|\nabla v\|_2^2) \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\big(\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\big)\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega v_{i,tt} v_k v_{i,kt}\, dx\right| &\le \|v_{tt}\|_2 \|v\|_6 \|\nabla v_t\|_3 \\ &\le C\|v_{tt}\|_2\|\nabla v\|_2(\|\nabla v_t\|_2^{1/2}\|\tilde{\Delta} v_t\|_2^{1/2} + \|\nabla v_t\|_2) \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + K_1\|\nabla v\|_2^2(\|\nabla v_t\|_2\|\tilde{\Delta} v_t\|_2 + \|\nabla v_t\|_2^2) \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\big(\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\big)\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega v_{i,tt} w_{k,t} v_{i,k}\, dx\right| &\le \|v_{tt}\|_2 \max|w_t|\ \|\nabla v\|_2 \\ &\le C\|v_{tt}\|_2(\|D_x w_t\|_2^{1/2}\|D_x^2 w_t\|_2^{1/2} + \|D_x w_t\|_2 + \|w_t\|_2)\|\nabla v\|_2 \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + K_1(\|D_x w_t\|_2\|D_x^2 w_t\|_2 + \|D_x w_t\|_2^2 + \|w_t\|_2^2)\|\nabla v\|_2^2 \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + K\big(\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|w_t\|_2^{14/5} + \|\nabla v\|_2^{14} + 1\big)\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega v_{i,tt} w_k v_{i,kt}\, dx\right| &\le \|v_{tt}\|_2 \|w\|_6 \|\nabla v_t\|_3 \\ &\le C\|v_{tt}\|_2(\|D_x w\|_2 + \|w\|_2)(\|\nabla v_t\|_2^{1/2}\|\tilde{\Delta} v_t\|_2^{1/2} + \|\nabla v_t\|_2) \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\|\nabla v_t\|_2^2(\|D_x w\|_2^4 + \|w\|_2^4 + 1) \\ &\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde{\Delta} v_t\|_2^2 + K\big(\|\nabla v_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} + 1\big)\end{aligned}$$

$$\left|\int_\Omega v_{i,tt}v_{k,t}w_{i,k}\,dx\right| \le \|v_{tt}\|_2 \max|v_t|\ \|D_xw\|_2$$
$$\le C\|v_{tt}\|_2(\|\nabla v_t\|_2^{1/2}\|\tilde\Delta v_t\|_2^{1/2} + \|\nabla v_t\|_2)\|D_xw\|_2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + \frac{\nu}{64}\|\tilde\Delta v_t\|_2^2 + K\big(\|\nabla v_t\|_2^{14/5} + \|D_xw\|_2^{14} + 1\big)$$

$$\left|\int_\Omega v_{i,tt}v_k w_{i,kt}\,dx\right| \le \|v_{tt}\|_2\|v\|_6\|D_xw_t\|_3$$
$$\le C\|v_{tt}\|_2\|\nabla v\|_2(\|D_xw_t\|_2^{1/2}\|D_x^2w_t\|_2^{1/2} + \|D_xw_t\|_2)$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K(\|D_x^2w_t\|_2\|D_xw_t\|_2 + \|D_xw_t\|_2^2)\|\nabla v\|_2^2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K\big(\|\nabla v\|_2^{14} + \|D_x^2w_t\|_2^2 + \|D_xw_t\|_2^{14/5} + 1\big)$$

$$\left|\int_\Omega v_{i,tt}w_{k,t}w_{i,k}\,dx\right| \le \|v_{tt}\|_2 \max|w_t|\ \|D_xw\|_2$$
$$\le C\|v_{tt}\|_2(\|D_xw_t\|_2^{1/2}\|D_x^2w_t\|_2^{1/2} + \|D_xw_t\|_2 + \|w_t\|_2)\|D_xw\|_2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K_1(\|D_xw_t\|_2\|D_x^2w_t\|_2 + \|D_xw_t\|_2^2 + \|w_t\|_2^2)\|D_xw\|_2^2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K\big(\|D_x^2w_t\|_2^2 + \|D_xw_t\|_2^{14/5} + \|w_t\|_2^{14/5} + \|D_xw\|_2^{14} + 1\big)$$

$$\left|\int_\Omega v_{i,tt}w_k w_{i,kt}\,dx\right| \le \|v_{tt}\|_2\|w\|_6\|D_xw_t\|_3$$
$$\le C\|v_{tt}\|_2(\|D_xw\|_2 + \|w\|_2)(\|D_xw_t\|_2^{1/2}\|D_x^2w_t\|_2^{1/2} + \|D_xw_t\|_2)$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K(\|D_xw\|_2^2 + \|w\|_2^2)(\|D_xw_t\|_2\|D_x^2w_t\|_2 + \|D_xw_t\|_2^2)$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K\big(\|D_x^2w_t\|_2^2 + \|D_xw_t\|_2^{14/5} + \|D_xw\|_2^{14} + \|w\|_2^{14}\big)$$

$$\int_\Omega v_{i,tt}p_{,it}\,dx = 0$$

$$\int_\Omega v_{i,tt}\Delta v_{i,t}\,dx = -\int_\Omega \nabla v_{i,tt}\nabla v_{i,t}\,dx = -\frac{1}{2}\frac{d}{dt}\|\nabla v_t\|_2^2$$

$$\int_\Omega v_{i,tt}\Delta w_{i,t}\,dx = \int_\Omega v_{i,tt}D_x^2w_{i,t}\,dx \le \|v_{tt}\|_2\|D_x^2w_t\|_2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K\|D_x^2w_t\|_2^2$$

$$\int_\Omega v_{i,tt}B_{i,t}\,dx$$
$$\le \|v_{tt}\|_2\|B_t\|_2$$
$$\le \frac{1}{32}\|v_{tt}\|_2^2 + K\|B_t\|_2^2$$

From these calculations we are led to the inequality

$$(6.28)\quad \begin{aligned} \nu\frac{d}{dt}\|\nabla v_t\|_2^2 + \|v_{tt}\|_2^2 \le K\Big\{&\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + \|D_x^2 w_t\|_2^2 \\ &+ \|D_x w_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} + \|w_{tt}\|_2^2 + \|B_t\|_2^2\Big\} \\ &+ \frac{\nu}{2}\|\tilde{\Delta} v_t\|_2^2. \end{aligned}$$

By adding together (6.26) and (6.28) and cancelling a term on the right side in $\|\tilde{\Delta} v_t\|_2^2$, we find

$$(6.29)\quad \begin{aligned} (1+\nu)\frac{d}{dt}\|\nabla v_t\|_2^2 &+ \nu\|\tilde{\Delta} v_t\|_2^2 + \|v_{tt}\|_2^2 \\ &\le K\Big\{\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14} + \|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} \\ &+ \|w\|_2^{14} + \|D_x w\|_2^{14} + \|w_{tt}\|_2^2 + \|B_t\|_2^2 + 1\Big\}. \end{aligned}$$

We now multiply inequality (6.14) by $5\|\nabla v\|_2^8$ and add it to (6.29), thus finding

$$(6.30)\quad \begin{aligned} &\frac{d}{dt}\Big\{(1+\nu)\|\nabla v_t\|_2^2 + \|\nabla v\|_2^{10}\Big\} + \nu\|\tilde{\Delta} v_t\|_2^2 + \|v_{tt}\|_2^2 + 5\|\nabla v\|_2^8\|\tilde{\Delta} v\|_2^2 \\ &\le K\Big\{\|\nabla v_t\|_2^{14/5} + \|\nabla v\|_2^{14}\Big\} + M_7(t) + 5\|\nabla v\|_2^8 N_3(t) \\ &\le K\Big(\|\nabla v_t\|_2^2 + \|\nabla v\|_2^{10}\Big)^{7/5} + M_7(t) + 5\|\nabla v\|_2^8 N_3(t) \end{aligned}$$

where

$$(6.31)\quad M_7(t) = K\Big\{\|D_x^2 w_t\|_2^2 + \|D_x w_t\|_2^{14/5} + \|D_x w\|_2^{14} + \|w\|_2^{14} + \|w_{tt}\|_2^2 + \|B_t\|_2^2 + 1\Big\}.$$

Again following (8.4) of [4] we set

$$\begin{aligned} F_7(t) &= 1 + (1+\nu)\|\nabla v_t\|_2^2 + \|\nabla v\|_2^{10} \\ G_7(t) &= \nu\|\tilde{\Delta} v_t\|_2^2 + \|v_{tt}\|_2^2 + 5\|\nabla v\|_2^8\|\tilde{\Delta} v\|_2^2. \end{aligned}$$

By the hypotheses of the theorem for $\rho = 7$ we have $M_7(t) \in L^1(0,T)$ and we now define

$$N_7(t) = M_7(t) + 5\|\nabla v\|_2^8\|N_3(t).$$

Thus

$$N_7(t) = F_7^{6/5}(t)Q(t) + 5F_7^{6/5}(t)Q(t) < F_7^{6/5}(t)Q(t)$$

where $Q(t) \in L^1(0,T)$ as required in the lemma on integrability with $p = \frac{1}{5}$, $a = \frac{6}{5}$. The lemma can now be seen to apply to (6.30) and we may conclude: $G_7(t) \in L^{1/7}(0,T)$. Hence both $\|\tilde{\Delta} v_t\|_2$ and $\|v_{tt}\|_2$ lie in $L^{2/7}(0,T)$.

As in [4, § 8] an induction on the order of time derivatives can now be started.

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7. **Inequalities for Higher Orders of Time Derivatives.** The first six inequalities illustrate and initiate a cycle of relationships among space derivatives of orders 0, 1 and 2 that leads to the desired estimates. The induction process to be described now extends these relationships to time derivatives of higher order. We follow the method of [4, § 7,8] as modified in the preceding section.

The momentum equations (6.1) differentiated r times with respect to t take the form

$$
\begin{aligned}
D_t^{r+1}v_i + D_t^{r+1}w_i + \sum_{j=0}^{r}\binom{r}{j}\big\{D_t^j v_k D_t^{r-j}v_{i,k} + D_t^j w_k D_t^{r-j}v_{i,k}\\
+ D_t^j v_k D_t^{r-j}w_{i,k} + D_t^j w_k D_t^{r-j}w_{i,k}\big\}\\
= -D_t^r p_{,i} + \nu D_t^r \Delta v_i + \nu D_t^r \Delta w_i + D_t^r B_i.
\end{aligned}
\tag{7.1}
$$

Multiply by $D_t^r v_i$ and integrate over Ω, obtaining after some routine calculations

$$
\begin{aligned}
&\frac{1}{2}D_t\|D_t^r v\|_2^2 + \nu\|D_t^r\nabla v\|_2^2\\
&= -\sum_{j=0}^{r}\binom{r}{j}\int_\Omega D_t^r v_i\big\{D_t^j v_k D_t^{r-j}v_{i,k} + D_t^j w_k D_t^{r-j}v_{i,k}\\
&\quad + D_t^j v_k D_t^{r-j}w_{i,k} + D_t^j w_k D_t^{r-j}w_{i,k}\big\}\,dx\\
&\quad - \int_\Omega D_t^r v_i D_t^{r+1}w_i\,dx - \nu\int_\Omega D_t^r\nabla v_i D_t^r D_x w_i\,dx + \int_\Omega D_t^r v_i D_t^r B_i\,dx
\end{aligned}
\tag{7.2}
$$

The last three terms on the right side can be transformed as follows:

$$
\begin{aligned}
\left|\int_\Omega D_t^r v_i D_t^{r+1} w_i\,dx\right| &\le \|D_t^r v\|_6\|D_t^{r+1}w\|_{6/5}\\
&\le C\|D_t^r\nabla v\|_2\|D_t^{r+1}w\|_{6/5}\\
&\le \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2 + K\|D_t^{r+1}w\|_{6/5}^2\\
\left|\int_\Omega D_t^r\nabla v_i D_t^r D_x w_i\,dx\right| &\le \|D_t^r\nabla v\|_2\|D_t^r D_x w\|_2\\
&\le \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2 + K\|D_t^r D_x w\|_2^2\\
\left|\int_\Omega D_t^r v_i D_t^r B_i\,dx\right| &\le \|D_t^r v\|_6\|D_t^r B\|_{6/5}\\
&\le C\|D_t^r\nabla v\|_2\|D_t^r B\|_{6/5}\\
&\le \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2 + K\|D_t^r B\|_{6/5}^2
\end{aligned}
$$

Certain terms with $j = 0$ in the sum will vanish by the divergence property:

$$
\begin{aligned}
\int_\Omega D_t^r v_i v_k D_t^r v_{i,k}\,dx &= \frac{1}{2}\int_\Omega v_k\big(D_t^r v_i\big)^2_{,k}\,dx\\
&= -\frac{1}{2}\int_\Omega v_{k,k}\big(D_t^r v_i\big)^2\,dx = 0
\end{aligned}
$$

and likewise

$$
\int_\Omega D_t^r v_i w_k D_t^r v_{i,k}\,dx = -\frac{1}{2}\int_\Omega w_{k,k}\big(D_t^r v_i\big)^2\,dx = 0.
$$

However there are now two terms with $j = 0$ that do not vanish, namely

$$\begin{aligned}\left|\int_\Omega D_t^r v_i v_k D_t^r w_{i,k}\,dx\right| &\le \|D_t^r v\|_3 \|v\|_6 \|D_t^r D_x w\|_2 \\ &\le C(\|D_t^r v\|_2^{1/2}\|D_t^r \nabla v\|_2^{1/2} + \|D_t^r v\|_2)\|\nabla v\|_2 \|D_t^r D_x w\|_2 \\ &\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K\|D_t^r v\|_2^{2/3}\|\nabla v\|_2^{4/3}\|D_t^r D_x w\|_2^{4/3} \\ &\qquad + K\|D_t^r v\|_2 \|\nabla v\|_2 \|D_t^r D_x w\|_2.\end{aligned}$$

and likewise

$$\begin{aligned}&\left|\int_\Omega D_t^r v_i w_k D_t^r w_{i,k}\,dx\right| \\ &\quad\le \frac{r}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K\|D_t^r v\|_2^{\frac{2}{3}}(\|D_x w\|_2^{\frac{4}{3}} + \|w\|_2^{\frac{4}{3}})\|D_t^r D_x w\|_2^{\frac{4}{3}} \\ &\qquad + K\|D_t v\|_2 \|D_x w\|_2 \|D_t^r D_x w\|_2.\end{aligned}$$

For the terms with $0 < j < r$ we follow the estimate (7.14) of [2] in this way:

$$\begin{aligned}&\left|\int_\Omega D_t^r v_i D_t^j v_k D_i^{r-j} v_{i,k}\,dx\right| \\ &\quad\le \|D_t^r v\|_6 \|D_t^j v\|_3 \|D_t^{r-j}\nabla v\|_2 \\ &\quad\le C\|D_t^r \nabla v\|_2 (\|D_t^j v\|_2^{1/2}\|D_t^j \nabla v\|_2^{1/2} + \|D_t^j v\|_2)\|D_t^{r-j}\nabla v\|_2 \\ &\quad\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K(\|D_t^j v\|_2 \|D_t^j \nabla v\|_2 + \|D_t^j v\|_2^2)\|D_t^{r-j}\nabla v\|_2^2\end{aligned}$$

and likewise

$$\begin{aligned}&\left|\int_\Omega D_t^r v_i D_t^j w_k D_t^{r-j} v_{i,k}\,dx\right| \\ &\quad\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K(\|D_t^j w\|_2 \|D_t^j D_x w\|_2 + \|D_t^j w\|_2^2)\|D_t^{r-j}\nabla v\|_2^2\end{aligned}$$

$$\begin{aligned}&\left|\int_\Omega D_t^r v_i D_t^j v_k D_t^{r-j} w_{i,k}\,dx\right| \\ &\quad\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K(\|D_t^j v\|_2 \|D_t^j \nabla v\|_2 + \|D_t^j v\|_2^2)\|D_t^{r-j} D_x w\|_2^2\end{aligned}$$

$$\begin{aligned}&\left|\int_\Omega D_t^r v_i D_t^j w_k D_t^{r-j} w_{i,k}\,dx\right| \\ &\quad\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K(\|D_t^j w\|_2 \|D_t^j D_x w\|_2 + \|D_t^j w\|_2^2)\|D_t^{r-j} D_x w\|_2^2\end{aligned}$$

Finally the terms with $j = r$ become

$$\begin{aligned}\left|\int_\Omega D_t^r v_i D_t^r v_k v_{i,k}\,dx\right| &\le \|D_t^r v\|_4^2 \|\nabla v\|_2 \\ &\le C\|D_t^r v\|_2^{1/2}\|D_t^r \nabla v\|_2^{3/2}\|\nabla v\|_2 \\ &\le \frac{\nu}{2^{2r+4}}\|D_t^r \nabla v\|_2^2 + K\|D_t^r v\|_2^2 \|\nabla v\|_2^4\end{aligned}$$

$$\left|\int_\Omega D_t^r v_i D_t^r w_k v_{i,k}\,dx\right| \le \|D_t^r v\|_6 \|D_t^r w\|_3 \|\nabla v\|_2$$

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$$\leq C\|D_t^r\nabla v\|_2(\|D_t^r w\|_2^{1/2}\|D_t^r D_x w\|_2^{1/2}+\|D_t^r w\|_2)\|\nabla v\|_2$$

$$\leq \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2+K(\|D_t^r w\|_2\|D_t^r D_x w\|_2+\|D_t^r w\|_2^2)\|\nabla v\|_2^2$$

$$\left|\int_\Omega D_t^r v_i D_t^r v_k w_{i,k}\,dx\right| \leq \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2+K\|D_t^r v\|_2^2\|D_x w\|_2^4$$

$$\left|\int_\Omega D_t^r v_i D_t^r w_k w_{i,k}\,dx\right| \leq \frac{\nu}{2^{2r+4}}\|D_t^r\nabla v\|_2^2+K(\|D_t^r w\|_2\|D_t^r D_x w\|_2+\|D_t^r w\|_2^2)\|D_x w\|_2^2$$

Assembling these calculations, we find after certain cancellations of terms containing $\|D_t^r\nabla v\|_2^2$ the following inequality:

$$\begin{aligned}
&D_t\|D_t^r v\|_2^2+\nu\|D_t^r\nabla v\|_2^2\\
&\quad\leq C\Big\{\|D_t^r v\|_2^2\big(\|\nabla v\|_2^4+\|D_x w\|_2^4+\|w\|_2^{4/3}\big)\\
&\qquad+\|D_t^r v\|_2^{2/3}\|D_t^r D_x w\|_2^{4/3}\big(\|\nabla v\|_2^{4/3}+\|D_x w\|_2^{4/3}\big)\\
&\qquad+\sum_{j=1}^{r-1}\|D_t^j v\|_2\|D_t^j\nabla v\|_2\big(\|D_t^{r-j}\nabla v\|_2^2+\|D_t^{r-j}D_x w\|_2^2\big)\\
&\qquad+\sum_{j=1}^{r-1}\|D_t^j w\|_2\|D_t^j D_x w\|_2\big(\|D_t^{r-j}\nabla v\|_2^2+\|D_t^{r-j}D_x w\|_2^2\big)\\
&\qquad+\big(\|D_t^r w\|_2\|D_t^r D_x w\|_2+\|D_t^r w\|_2^2\big)\big(\|\nabla v\|_2^2+\|D_x w\|_2^2\big)\\
&\qquad+\|D_t^{r+1}w\|_{6/5}^2+\|D_t^r D_x w\|_2^2+\|D_t^r B\|_{6/5}^2+1\Big\}.
\end{aligned}\tag{7.3}$$

For the purposes of calculation, it is convenient at this stage to attribute to derivatives of w the same formal singular index (reciprocal power of integrability) as derivatives of v: thus $\|\nabla v\|_2$ and $\|D_x w\|_2$ have formal singular index $\frac{1}{2}$, $\|D_t^r D_x^s v\|_2$ and $\|D_t^r D_x^s w\|_2$ have formal singular index $\frac{1}{2}(4r+2s-1)$. The highest singular index of each such term on the right side of (7.3) is then the sum of the indices of its factors and is $4r+1$. Young's inequality in the form (3.5) then applies with the singular index playing the role of the weight w for each factor. Resolving each product of terms on the right side of (7.3) into a sum of powers of the factors, we find (7.3) takes the form

$$\begin{aligned}
D_t\|D_t^r v\|_2^2+\nu\|D_t^r\nabla v\|_2^2 \leq C\Big\{&\sum_{j=1}^{r}\|D_t^j v\|_2^{\frac{8r+2}{4j-1}}\\
&+\sum_{j=0}^{r-1}\|D_t^j\nabla v\|_2^{\frac{8r+2}{4j+1}}+\sum_{j=1}^{r}\|D_t^j w\|_2^{\frac{8r+2}{4j-1}}\\
&+\sum_{j=0}^{r}\|D_t^j D_x w\|_2^{\frac{8r+2}{4j+1}}+\|D_t^{r+1}w\|_{6/5}^2\\
&+\|D_t^r B\|_{6/5}^2+\|w\|_2^{8r+2}+1\Big\}.
\end{aligned}\tag{7.4}$$

Note particularly the incorporation of the term $\|D_t^r D_x w\|_2^2$ in the last sum on the right hand side; the corresponding term in v is now present only on the left side.

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We now write this inequality in the form

$$(7.5)\qquad D_t\|D_t^r v\|_2^2 + \nu\|D_t^r\nabla v\|_2^2 \le C\Big\{\sum_{j=1}^{r}\|D_t^j v\|_2^{\frac{8r+2}{4j-1}} + \sum_{j=0}^{r-1}\|D_t^j\nabla v\|_2^{\frac{8r+2}{4j+1}}\Big\} + M_{4r+1}(t)$$

where

(7.6)

$$M_{4r+1}(t) = C\Big(\sum_{j=1}^{r}\|D_t^j w\|_2^{\frac{8r+2}{4j-1}} + \sum_{j=0}^{r}\|D_t^j D_x w\|_2^{\frac{8r+2}{4j+1}} + \|D_t^{r+1}w\|_{6/5}^2 + \|w\|_2^{8r+2} + \|D_t^r B\|_{6/5}^2 + 1\Big)$$

and C denotes the constant in (7.5).

Note that the hypotheses of the Theorem for $\rho = 4r+1$ imply $M_{4r+1}(t) \in L^1(0,T)$.

Next multiply (7.1) by $D_t^r\tilde{\Delta}v_i$ and integrate over Ω: we obtain after routine calculations

$$(7.7)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|D_t^r\nabla v\|_2^2 + \nu\|D_t^r\tilde{\Delta}v\|_2^2 \\ &= \int_\Omega D_t^r\tilde{\Delta}v_i D_t^{r+1}w_i\,dx - \int_\Omega D_t^r\tilde{\Delta}v_i D_t^r\Delta w_i\,dx - \int_\Omega D_t^r\tilde{\Delta}v_i D_t^r B_i\,dx \\ &\quad + \sum_{j=0}^{r}\binom{r}{j}\int_\Omega D_t^r\tilde{\Delta}v_i\big(D_t^j v_k D_t^{r-j}v_{i,k} + D_t^j w_k D_t^{r-j}v_{i,k} \\ &\quad + D_t^j v_k D_t^{r-j}w_{i,k} + D_t^j w_k D_t^{r-j}w_{i,k}\big)\,dx \end{aligned}$$

The various terms may be treated as shown:

$$\begin{aligned}\Big|\int_\Omega D_t^r\tilde{\Delta}v_i D_t^{r+1}w_i\,dx\Big| &\le \|D_t^r\tilde{\Delta}v_i\|_2\|D_t^{r+1}w\|_2 \\ &\le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta}v_i\|_2^2 + K\|D_t^{r+1}w\|_2^2\end{aligned}$$

$$\begin{aligned}\Big|\int_\Omega D_t^r\tilde{\Delta}v_i D_t^r\Delta w_i\,dx\Big| &\le \Big|\int_\Omega D_t^r\tilde{\Delta}v_i D_t^r D_x^2 w_i\,dx\Big| \\ &\le \|D_t^r\tilde{\Delta}v_i\|_2\|D_t^r D_x^2 w\|_2 \\ &\le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta}v\|_2^2 + K\|D_t^r D_x^2 w\|_2^2\end{aligned}$$

$$\begin{aligned}\Big|\int_\Omega D_t^r\tilde{\Delta}v_i D_t^r B_i\,dx\Big| &\le \|D_t^r\tilde{\Delta}v\|_2\|D_t^r B\|_2 \\ &\le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta}v\|_2^2 + K\|D_t^r B\|_2^2.\end{aligned}$$

For $j \le \left[\frac{r}{2}\right]$ (where $[x]$ denotes the greatest integer less than or equal to x) we write

$$\begin{aligned}&\Big|\int_\Omega D_t^r\tilde{\Delta}v_i D_t^j v_k D_t^{r-j}v_{i,k}\,dx\Big| \\ &\le \|D_t^r\tilde{\Delta}v\|_2\|D_t^j v\|_6\|D_t^{r-j}\nabla v\|_3 \\ &\le C\|D_t^r\tilde{\Delta}v\|_2\|D_t^j\nabla v\|_2(\|D_t^{r-j}\nabla v\|_2^{1/2}\|D_t^{r-j}\tilde{\Delta}v\|_2^{1/2} + \|D_t^{r-j}\nabla v\|_2) \\ &\le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta}v\|_2^2 + K\|D_t^j\nabla v\|_2^2(\|D_t^{r-j}\nabla v\|_2\|D_t^{r-j}\tilde{\Delta}v\|_2 + \|D_t^{r-j}\nabla v\|_2^2)\end{aligned}$$

The corresponding term with $\left[\frac{r}{2}\right] < j \le r$ is handled differently:

$$\begin{aligned}
&\left|\int_\Omega D_t^r\tilde{\Delta} v_i D_t^j v_k D_t^{r-j} v_{i,k}\,dx\right| \\
&\quad \le \|D_t^r\tilde{\Delta} v\|_2 \max|D_t^j v|\,\|D_t^{r-j}\nabla v\|_2 \\
&\quad \le C\|D_t^r\tilde{\Delta} v\|_2(\|D_t^j\nabla v\|_2^{1/2}\|D_t^j\tilde{\Delta} v\|_2^{1/2} + \|D_t^j\nabla v\|_2)\|D_t^{r-j}\nabla v\|_2 \\
&\quad \le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta} v\|_2^2 + K(\|D_t^j\nabla v\|_2\|D_t^j\tilde{\Delta} v\|_2 + \|D_t^j\nabla v\|_2^2)\|D_t^{r-j}\nabla v\|_2^2.
\end{aligned}$$

Because the involutory change of the summation index j into $r-j$ converts each of these expressions into the other, the summation can be reduced to the first half range provided all terms are doubled. The sums with mixed v and w factors behave similarly but with a *crossover:* for $0 \le j \le \left[\frac{r}{2}\right]$

$$\begin{aligned}
&\left|\int_\Omega D_t^r\tilde{\Delta} v_i D_t^j w_k D_t^{r-j} v_{i,k}\,dx\right| \\
&\quad \le \|D_t^r\tilde{\Delta} v\|_2\|D_t^j w\|_6\|D_t^{r-j}\nabla v\|_3 \\
&\quad \le C\|D_t^r\tilde{\Delta} v\|_2(\|D_t^j D_x w\|_2 + \|D_t^j w\|_2)\|D_t^{r-j}\nabla v\|_2^{1/2}\|D_t^j\tilde{\Delta} v\|_2^{1/2} \\
&\quad \le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta} v\|_2^2 + K(\|D_t^j D_x w\|_2^2 + \|D_t^r w\|_2^2)\|D_t^{r-j}\nabla v\|_2\|D_t^{r-j}\tilde{\Delta} v\|_2.
\end{aligned}$$

The corresponding leading term for $\left[\frac{r}{2}\right] < j \le r$ is in the complementary term

$$\begin{aligned}
&\left|\int_\Omega D_t^r\tilde{\Delta} v_i D_t^j v_k D_t^{r-j} w_{i,k}\,dx\right| \\
&\quad \le \|D_t^r\tilde{\Delta} v\|_2\ \max|D_t^j v|\,\|D_t^{r-j}D_x w\|_2 \\
&\quad \le C\|D_t^r\tilde{\Delta} v\|_2\|D_t^{r-j}D_x w\|_2(\|D_t^j\nabla v\|_2^{1/2}\|D_t^j\tilde{\Delta} v\|_2^{1/2} + \|D_t^j\nabla v\|_2) \\
&\quad \le \frac{\nu}{2^{2r+5}}\|D_t^r\tilde{\Delta} v\|_2^2 + K\|D_t^{r-j}D_x w\|_2^2(\|D_t^j\nabla v\|_2\|D_t^j\tilde{\Delta} v\|_2 + \|D_t^j\nabla v\|_2^2)
\end{aligned}$$

which resembles the preceding when j is mapped into $r-j$ and vice versa.

Consequently all leading terms in the four sums on the right side of (7.7) can be expressed as sums over the lower half-range $0 \le j \le \left[\frac{r}{2}\right]$. We thus find after multiplying by 2, and cancelling a large group of terms containing $\|D_t^2\tilde{\Delta} v\|_2^2$,

$$\begin{aligned}
&D_t\|D_t^r\nabla v\|_2^2 + \frac{3}{2}\nu\|D_t^r\tilde{\Delta} v\|_2^2 \\
&\quad \le C\Big\{\sum_{j=0}^{[\frac{r}{2}]}\big(\|D_t^j\nabla v\|_2^2 + \|D_t^j D_x w\|_2^2 + \|D_t^j w\|_2^2\big)\|D_t^{r-j}\nabla v\|_2\|D_t^{r-j}\tilde{\Delta} v\|_2 \\
&\qquad + \sum_{j=0}^{[\frac{r}{2}]}\big(\|D_t^j\nabla v\|_2^2 + \|D_t^j D_x w\|_2^2 + \|D_t^j w\|_2^2\big) \\
&\qquad\quad \cdot\big(\|D_t^{r-j}D_x w\|_2\|D_t^{r-j}D_x^2 w\|_2 + \|D_t^{r-j}D_x w\|_2^2\big) \\
&\qquad + \|D_t^{r+1}w\|_2^2 + \|D_t^r D_x^2 w\|_2^2 + \|D_t^r B\|_2^2 + 1\Big\}.
\end{aligned} \tag{7.8}$$

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Whereas all the terms containing w can now be expressed simply as powers of the norms involved, by use of Young's inequality, the terms in $\|D_t^r\tilde{\Delta}v\|_2$ must be treated more delicately, as in [4, § 7,8]. Thus we express the term with $j = 0$ in the first sum in (7.8) as follows, where γ is a real number with $0 < \gamma < 2$ that will be specified later:

$$\begin{aligned}(7.9)\quad &\left(\|\nabla v\|_2^2 + \|D_x w\|_2^2 + \|w\|_2^2\right)\|D_t^r\nabla v\|_2\|D_t^r\tilde{\Delta}v\|_2^4 \\ &\leq \frac{1}{4}(2-\gamma)\nu\|D_t^r\tilde{\Delta}v\|_2^2 + K\|D_t^r\nabla v\|_2^2\left(\|\nabla v\|_2^4 + \|D_x w\|_2^4 + \|w\|_2^4\right).\end{aligned}$$

The remaining terms with $1 \leq j \leq \left[\frac{r}{2}\right]$ of the first sum are written in the form

(7.10)

$$\begin{aligned}&\left(\|D_t^j\nabla v\|_2^2 + \|D_t^jD_x w\|_2^2 + \|D_t^j w\|_2^2\right)\|D_t^{r-j}\nabla v\|_2^{\frac{4r-8j+1}{4(r-j)+1}} \times \|D_t^{r-j}\nabla v\|_2^{\frac{4j}{4(r-j)+1}}\|D_j^{r-j}\tilde{\Delta}v\|_2 \\ &\leq C_{rj}\left(\|D_t^j\nabla v\|_2^{\frac{8r+6}{4j+1}} + \|D_t^jD_x w\|_2^{\frac{8r+6}{4j+1}} + \|D_t^j w\|_2^{\frac{8r+6}{4j+1}} + \|D_t^{r-j}\nabla v\|_2^{\frac{8r+6}{4(r-j)+1}}\right) \\ &\quad + \frac{1}{4r^3C}\|D_t^{r-j}\nabla v\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde{\Delta}v\|_2^2.\end{aligned}$$

Here the constant C_{rj} depends on r and j, the C in the denominator is the constant C in (7.8), and the last term has the index $8r + 6$. With these precautions we can now write (7.8) in the following special form:

$$\begin{aligned}&D_t\|D_t^r\nabla v\|_2^2 + \left(\nu + \frac{\gamma\nu}{4}\right)\|D_t^r\tilde{\Delta}v\|_2^2 \\ &\leq C\Big\{\|D_t^r\nabla v\|_2^{\frac{8r+6}{4r+1}} + \|\nabla v\|_2^{8r+6} + \|D_x w\|_2^{8r+6} \\ &\quad + \sum_{j=1}^{[\frac{r}{2}]}\Big(\|D_t^j\nabla v\|_2^{\frac{8r+6}{4j+1}} + \|D_t^jD_x w\|_2^{\frac{8r+6}{4j+1}} + \|D_t^j w\|_2^{\frac{8r+6}{4j+1}} \\ (7.11)\quad &\quad + \|D_t^{r-j}\nabla v\|_2^{\frac{8r+6}{4(r-j)+1}} + \|D_t^{r-j}D_x w\|_2^{\frac{8r+6}{4(r-j)+1}} + \|D_t^{r-j}D_x^2 w\|_2^{\frac{8r+6}{4(r-j)+3}}\Big) \\ &\quad + \|D_t^{r+1}w\|_2^2 + \|D_t^rD_x^2 w\|_2^2 + \|D_t^rB\|_2^2 + 1\Big\} \\ &\quad + \frac{1}{4r^3}\sum_{j=0}^{[\frac{r}{2}]}\|D_t^{r-j}\nabla v\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde{\Delta}v\|_2^2.\end{aligned}$$

To be combined with (7.11) is another inequality formed by multiplying (7.1) by $D_t^{r+1}v_i$ and integrating over Ω. This takes the form, after routine calculations:

$$\begin{aligned}(7.12)\quad &\frac{\nu}{2}D_t\|D_t^r\nabla v\|_2^2 + \|D_t^{r+1}v\|_2^2 \\ &= -\int_\Omega D_t^{r+1}v_i\sum_{j=0}^{r}\binom{r}{j}\Big\{D_t^jv_kD_t^{r-j}v_{i,k} + D_t^jw_kD_t^{r-j}v_{i,k} \\ &\quad + D_t^jv_kD_t^{r-j}w_{i,k} + D_t^jw_kD_t^{r-j}w_{i,k}\Big\}\,dx \\ &\quad + \int_\Omega D_t^{r+1}v_i\left(-D_t^rp_{,i} + \nu D_t^r\Delta v_i + \nu D_t^r\Delta w_i + D_t^rB_i\right)dx.\end{aligned}$$

Now $\int_\Omega D_t^{r+1} v_i D_t^r p_{,i}\, dx = 0$ by orthogonality, since $D_t^{r+1} v_i$ is solenoidal and vanishes on $\partial\Omega$, and $D_t^r p_{,i}$ is a gradient. Also

$$\begin{aligned}
\left|\int_\Omega D_t^{r+1} v_i D_t^r \Delta v_i\, dx\right| &= \left|\int_\Omega D_t^{r+1} v_i D_t^r \tilde{\Delta} v_i\, dx\right| \\
&\leq \|D_t^{r+1} v\|_2 \|D_t^r \tilde{\Delta} v\|_2 \\
&\leq \frac{\nu(2-\gamma)}{2^{2r+5}} \|D_t^{r+1} v\|_2^2 + K\|D_t^r \tilde{\Delta} v\|_2^2, \\
\left|\int_\Omega D_t^{r+1} v_i D_t^r \Delta w_i\, dx\right| &= \left|\int_\Omega D_t^{r+1} v_i D_t^r D_x^2 w_i\, dx\right| \\
&\leq \|D_t^{r+1} v\|_2 \|D_t^r D_x^2 w\|_2 \\
&\leq \frac{\nu(2-\gamma)}{2^{2r+5}} \|D_t^{r+1} v\|_2^2 + K\|D_t^r D_x^2 w\|_2^2, \\
\left|\int_\Omega D_t^{r+1} v_i D_t^r B_i\, dx\right| &= \|D_t^{r+1} v\|_2 \|D_t^r B\|_2 \\
&\leq \frac{\nu(2-\gamma)}{2^{2r+5}} \|D_t^{r+1} v\|_2^2 + K\|D_t^r B\|_2^2.
\end{aligned}$$

Since $D_t^{r+1} v$ is solenoidal vanishing on $\partial\Omega$, we can remove the gradient part of the other factor in the first two of this last group. This could also, if desired, be done for the last of these terms, retaining only $\tilde{B}_i$ where $B_i = \tilde{B}_i + \nabla\beta$.

Again, we have

$$\begin{aligned}
&\left|\sum_{j=0}^{r} \binom{r}{j} \int_\Omega D_t^{r+1} v_i D_t^j v_k D_t^{r-j} v_{i,k}\, dx\right| \\
&\leq C\Big(\sum_{j=0}^{[\frac{r}{2}]} \|D_t^{r+1} v\|_2 \|D_t^j v\|_6 \|D_t^{r-j} \nabla v\|_3 \\
&\qquad + \sum_{j=[\frac{r}{2}+1]}^{r} \|D_t^{r+1} v\|_2 \max |D_t^j v|\, \|D_t^{r-j} \nabla v\|_2\Big) \qquad (7.13) \\
&\leq C\|D_t^{r+1} v\|_2 \sum_{j=0}^{[\frac{r}{2}]} \|D_t^j \nabla v\|_2 (\|D_t^{r-j} \nabla v\|_2^{1/2} \|D_t^{r-j} \tilde{\Delta} v\|_2^{1/2} + \|D_t^{r-j} \nabla v\|_2) \\
&\leq \frac{2-\gamma}{2^6} \|D_t^{r+1} v\|_2^2 + C\sum_{j=0}^{[\frac{r}{2}]} \|D_t^j \nabla v\|_2^2 (\|D_t^{r-j} \nabla v\|_2 \|D_t^{r-j} \tilde{\Delta} v\|_2 + \|D_t^{r-j} \nabla v\|_2^2\|)
\end{aligned}$$

while

$$\begin{aligned}
&\left|\sum_{j=0}^{r} \binom{r}{j} \int_\Omega D_t^{r+1} v_i \big[D_t^j w_k D_t^{r-j} v_{i,k} + D_t^j v_k D_t^{r-j} w_{i,k}\big]\, dx\right| \qquad (7.14) \\
&\leq \quad C\Big|\sum_{j=0}^{[\frac{r}{2}]} \|D_t^j \nabla v\|_2^2 \big(\|D_t^{r-j} D_x w\|_2 \|D_t^{r-j} D_x^2 w\|_2 + \|D_t^{r-j} D_x w\|_2^2\big) \\
&\qquad + \sum_{j=0}^{[\frac{r}{2}]} \|D_t^j \nabla w\|_2^2 (\|D_t^{r-j} \nabla v\|_2 \|D_t^{r-j} \tilde{\Delta} v\|_2 + \|D_t^{r-j} \nabla v\|_2^2)\Big| \\
&\qquad + \frac{2-\gamma}{2^5} \|D_t^{r+1} v\|_2^2.
\end{aligned}$$

Combining these formulas together with a version of (7.13) in which w replaces v in all but the first factor $D_t^{r+1}v_i$, we find using (7.8) in conjunction with the foregoing calculations,

$$
\begin{aligned}
&\nu D_t\|D_t^r\nabla v\|_2^2+\gamma\|D_t^{r+1}v\|_2^2\\
&\le \nu\|D_t^r\tilde\Delta v\|_2^2+C\Big\{\|D_t^r\nabla v\|_2^{\frac{8r+6}{4r+1}}+\|\nabla v\|_2^{8r+6}+\|D_xw\|_2^{8r+6}+\|w\|_2^{8r+6}\\
&\quad+\sum_{j=1}^{[\frac{r}{2}]}\Big(\|D_t^j\nabla v\|_2^{\frac{8r+6}{4j+1}}+\|D_t^jD_xw\|_2^{\frac{8r+6}{4j+1}}+\|D_t^jw\|_2^{\frac{8r+6}{4j-1}}\\
&\quad+\|D_t^{r-j}\nabla v\|_2^{\frac{8r+6}{4(r-j)+1}}+\|D_t^{r-j}D_xw\|_2^{\frac{8r+6}{4(r-j)+1}}+\|D_t^{r-j}D_x^2w\|_2^{\frac{8r+6}{4(r-j)+3}}\Big)\\
&\quad+\|D_t^{r+1}w\|_2^2+\|D_t^rD_x^2w\|_2^2+\|D_t^rB\|_2^2+1\Big\}\\
&\quad+\frac{1}{4r^3}\sum_{j=1}^{[\frac{r}{2}]}\|D_t^{r-j}\nabla v\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde\Delta v\|_2^2.
\end{aligned}
\tag{7.15}
$$

Now we add (7.11) and (7.15) to obtain the inequality

$$
\begin{aligned}
&(1+\nu)D_t\|D_t^r\nabla v\|_2^2+\gamma\Big(\frac{\nu}{4}\|D_t^r\tilde\Delta v\|_2^2+\|D_t^{r+1}v\|_2^2\Big)\\
&\qquad\le C\Big\{\|D_t^r\nabla v\|_2^{\frac{8r+6}{4r+1}}+\|\nabla v\|_2^{8r+6}\\
&\qquad\quad+\sum_{j=1}^{[\frac{r}{2}]}\Big(\|D_t^j\nabla v\|_2^{\frac{8r+6}{4j+1}}\\
&\qquad\quad+\|D_t^{r-j}\nabla v\|_2^{\frac{8r+6}{4(r-j)+1}}\Big)\Big\}\\
&\qquad\quad+\frac{1}{4r^3}\sum_{j=0}^{[\frac{r}{2}]}\|D_t^{r-j}\nabla v\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde\Delta v\|_2\\
&\qquad\quad+M_{4r+3}(t)
\end{aligned}
\tag{7.16}
$$

where

$$
\begin{aligned}
M_{4r+3}(t)=C\Big(&\|D_xw\|_2^{8r+6}+\sum_{j=1}^{[\frac{r}{2}]}\|D_t^jD_xw\|_2^{\frac{8r+6}{4j+1}}+\|D_t^jw\|_2^{\frac{8r+6}{4j-1}}\\
&+\sum_{j=0}^{[\frac{r}{2}]}\Big(\|D_t^{r-j}D_xw\|_2^{\frac{8r+6}{4(r-j)+1}}+\|D_t^{r-j}D_x^2w\|_2^{\frac{8r+6}{4(r-j)+3}}\Big)\\
&+\|D_t^{r+1}w\|_2^2+\|D_t^rD_x^2w\|_2^2+\|D_t^rB\|_2^2+\|w\|_2^{8r+6}+1\Big)
\end{aligned}
\tag{7.17}
$$

and C denotes the constant in (7.16). We observe that the hypotheses of the Theorem for $\rho=4r+3$ imply that $M_{4r+3}(t)\in L^1(0,T)$.

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8. **The Induction on Orders of Time Derivatives.** The result of the Theorem has been shown in § 6 for $r \leq 1$ and $s \leq 2$, that is, for the derivative norms $\|v\|_2, \|\nabla v\|_2$, $\|\tilde{\Delta} v\|_2$ and hence $\|D_x^2 v\|_2$; $\|v_t\|_2, \|\nabla v_t\|_2$ and $\|D_x v_t\|_2$, together with $\|v_{tt}\|_2$. Using these results as a starting point, we now set up the induction on r, assuming the result of the Theorem holds for a given r and demonstrating its truth for $r+1$. Thus we assume $\|D_t^h \nabla v\|_2 \in L^{2(4h+1)^{-1}}(0,T)$ and $\|D_t^h \tilde{\Delta} v\|_2, \|D_t^{h+1} v\|_2 \in L^{2(4h+3)^{-1}}(0,T)$ for $h = 0,1,2,\ldots, r-1$.

As in [4, § 8] we define recursively for $r = 2,3\ldots,$ functions

$$F_{4r+1}(t) = \|D_t^r v\|_2^2 + \frac{4r-7}{4r-1} F_{4r-1}^{\frac{4r-1}{4r-3}}(t) + F_{4r-3}^{\frac{4r-1}{4r-5}}(t) \tag{8.1}$$

$$G_{4r+1}(t) = \nu \|D_t^r \nabla v\|_2^2 + F_{4r-1}^{\frac{2}{4r-3}}(t) G_{4r-1}(t) + F_{4r-3}^{\frac{4}{4r-5}}(t) G_{4r-3}(t), \tag{8.2}$$

and now also

$$N_{4r+1}(t) = M_{4r+1}(t) + F_{4r-1}^{\frac{2}{4r-3}}(t) N_{4r-1}(t) + \frac{4r-1}{4r-5} F_{4r-3}^{\frac{4}{4r-5}}(t) N_{4r-3}(t). \tag{8.3}$$

Likewise, for the second stage necessary at this induction step, we also define

$$F_{4r+3}(t) = (1+\nu)\|D_t^r \nabla v\|_2^2 + F_{4r-1}^{\frac{4r+1}{4r-3}}(t) \tag{8.4}$$

$$G_{4r+3}(t) = \frac{\nu}{4}\|D_t^r \tilde{\Delta} v\|_2^2 + \|D_t^{r+1} v\|_2^2 + \frac{4r+1}{4r-3} F_{4r-1}^{\frac{4}{4r-3}}(t) G_{4r-1}(t) \tag{8.5}$$

$$N_{4r+3}(t) = M_{4r+3}(t) + \frac{4r+1}{4r-3} F_{4r-1}^{\frac{4}{4r-3}}(t) N_{4r-1}(t) \tag{8.6}$$

As in [4, § 8] the singular index of $F_q(t)$ is in general $q-2$, and the singular index of $G_q(t)$ is q, for every odd positive integer q. The definitions for $q = 3,5,7$ are given in § 6.

To show that $N_{4r+1}(t)$ and $N_{4r+3}(t)$ satisfy the condition $N(t) \leq CF(t)^a Q(t)$ of the Integrability Lemma is a necessary part of the induction step. This was initiated in § 6 for $r = 1, s = 0$ and $s = 1$. To establish this result for $N_{4r+1}(t)$ we note that by the induction hypothesis $a = 4r/4r-1$,

$$N_{4r-3}(t) = F_{4r-3}(t)^{\frac{4r-4}{4r-5}} Q(t)$$

and

$$N_{4r-1}(t) = F_{4r-1}(t)^{\frac{4r-2}{4r-3}} Q(t).$$

Hence by (8.1)

$$\begin{aligned} N_{4r+1}(t) &\leq Q(t) + F_{4r-3}^{\frac{4}{4r-5}}(t)\big(F_{4r-3}(t)^{\frac{4r-4}{4r-5}} Q(t)\big) \\ &\quad + F_{4r-3}^{\frac{2}{4r-3}}(t)\big(F_{4r-3}(t)^{\frac{4r-2}{4r-5}} Q(t)\big) \\ &\leq Q(t)\Big[1 + F_{4r-3}(t)^{\frac{4r}{4r-5}} + F_{4r-1}(t)^{\frac{4r}{4r-3}}\Big] \\ &\leq Q(t)\Big[1 + F_{4r+1}(t)^{\frac{4r-5}{4r-1}\cdot\frac{4r}{4r-5}} + F_{4r+1}(t)^{\frac{4r-3}{4r-1}\cdot\frac{4r}{4r-3}}\Big] \\ &\leq Q(t)\cdot F_{4r+1}(t)^{\frac{4r}{4r-1}}. \end{aligned} \tag{8.7}$$

To show the same induction result for $N_{4r+3}(t)$ we have $a = \frac{4r+2}{4r+1}$ and by (8.4)

$$
\begin{aligned}
N_{4r+3}(t) \leq & Q(t)\Big[1 + c \cdot F_{4r-1}(t)^{\frac{4}{4r-3}} \cdot F_{4r-1}(t)^{\frac{4r-2}{4r-3}}\Big] \\
= & Q(t)\Big[1 + c \cdot F_{4r-1}(t)^{\frac{4r+2}{4r-3}}\Big] \\
\leq & Q(t)\Big[1 + c \cdot F_{4r+3}(t)^{\frac{4r-3}{4r+1} \cdot \frac{4r+2}{4r-3}}\Big] \\
= & Q(t) F_{4r+3}(t)^{\frac{4r+2}{4r+1}}
\end{aligned} \tag{8.8}
$$

Thus the induction on r is completed for this condition in the lemma.

The basic inequalities for the preceding stage of the induction are

$$F'_{4r-3}(t) + G_{4r-3}(t) \leq CF_{4r-3}(t)^{\frac{4r-3}{4r-5}} + N_{4r-3}(t) \tag{8.9}$$

and

$$F'_{4r-1}(t) + \frac{4r-3}{4r-7} G_{4r-1}(t) \leq CF_{4r-1}(t)^{\frac{4r-1}{4r-3}} + N_{4r-1}(t) \tag{8.10}$$

so we regard these inequalities as established. Multiply (8.9) by $\frac{4r-1}{4r-5} F_{4r-3}^{\frac{4}{4r-5}}(t)$ and (8.10) by $\frac{4r-7}{4r-3} F_{4r-1}(t)^{\frac{2}{4r-3}}$ and add both to (7.5). Thus we obtain

$$
\begin{aligned}
& D_t\Big[\|D_t^r v\|_2^2 + \frac{4r-7}{4r-1} F_{4r-1}(t)^{\frac{4r-1}{4r-3}} + F_{4r-3}(t)^{\frac{4r-1}{4r-5}}\Big] \\
& \quad + \nu \|D_t^r \nabla v\|_2^2 + F_{4r-1}(t)^{\frac{2}{4r-3}} G_{4r-1}(t) + \frac{4r-1}{4r-5} F_{4r-3}(t)^{\frac{4}{4r-5}} G_{4r-3}(t) \\
& \leq C\Big[\sum_{j=1}^{r} \|D_t^j v\|_2^{\frac{8r+2}{4j-1}} + \sum_{j=0}^{r-1} \|D_t^j \nabla v\|_2^{\frac{8r+2}{4j+1}} + F_{4r-3}(t)^{\frac{4r+1}{4r-5}} + F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\Big] \\
& \quad + M_{4r+1}(t) + \frac{4r-1}{4r-5} F_{4r-3}(t)^{\frac{4}{4r-3}} N_{4r-3}(t) + \frac{4r-7}{4r-3} F_{4r-1}(t)^{\frac{2}{4r-5}} N_{4r-1}(t)
\end{aligned} \tag{8.11}
$$

The last three terms on the right are together less than $N_{4r+1}(t)$.

By means of the relations

$$
\begin{aligned}
\|D_t^j v\|_2^{\frac{8r+2}{4j-1}} & \leq F_{4j+1}(t)^{\frac{4r+1}{4j-1}} \leq F_{4j+5}(t)^{\frac{4r+1}{4j+3}} \\
& \leq \cdots \leq F_{4r-3}(t)^{\frac{4r+1}{4r-5}}, \quad j = 1, \ldots, r-1,
\end{aligned}
$$

and

$$
\begin{aligned}
\|D_t^j \nabla v\|_2^{\frac{8r+2}{4j+1}} & \leq F_{4j+3}(t)^{\frac{4r+1}{4j+1}} \leq F_{4j+7}(t)^{\frac{4r+1}{4j+5}} \\
& \leq \cdots \leq F_{4r-1}(t)^{\frac{4r+1}{4r-3}}, \quad j = 0, 1, \ldots, r-1
\end{aligned}
$$

we can show the right hand side of (8.11) is bounded above by

$$
\begin{aligned}
& C[\|D_t^r v\|_2^{\frac{8r+2}{4r-1}} + F_{4r-3}(t)^{\frac{4r+1}{4r-5}} + \frac{4r-7}{4r-1} F_{4r-1}(t)^{\frac{4r+1}{4r-3}}] + N_{4r+1}(t) \\
& \leq C[\|D_t^r v\|_2^2 + F_{4r-3}(t)^{\frac{4r-1}{4r-5}} + \frac{4r-7}{4r-1} F_{4r-1}(t)^{\frac{4r-1}{4r-5}}]^{\frac{4r+1}{4r-1}} + N_{4r+1}(t) \\
& = CF_{4r+1}(t)^{\frac{4r+1}{4r-1}} + N_{4r+1}(t).
\end{aligned} \tag{8.12}
$$

Thus we have shown that

$$F'_{4r+1}(t) + G_{4r+1}(t) \leq CF_{4r+1}(t)^{\frac{4r+1}{4r-1}} + N_{4r+1}(t). \tag{8.13}$$

which is the counterpart of (8.9) at the next higher value of r. Observing that $F_{4r+1}(t) \in L^{\frac{1}{4r-1}}(0,T)$ by hypotheses of the induction for $\rho = 4r-1$, we see that the hypotheses of the Integrability Lemma for $F_{4r+1}(t)$ and $N_{4r+1}(t)$ are fulfilled with $p = \frac{1}{4r-1}$, $a+p = \frac{4r+1}{4r-1}$ so $a = 1 + \frac{1}{4r-1}$. Hence $\frac{p}{a+p} = \frac{1}{4r+1}$ so $G_{4r+1}(t) \in L^{\frac{1}{4r+1}}(0,T)$ and $\|D_t^r \nabla u\|_2$ must belong to $L^{\frac{2}{4r+1}}(0,T)$. This now also shows that $F_{4r+3}(t) \in L^{\frac{1}{4r+1}}(0,T)$.

To establish the corresponding results for $\|D_t^r \tilde{\Delta} v\|_2$ and $\|D_t^{r+1} v\|_2^2$ we multiply (8.10) by $\frac{4r+1}{4r-3} F_{4r-1}(t)^{\frac{4}{4r-3}}$ and add to (7.16) wherein γ is set equal to $\frac{4r+1}{4r-3}$. This leads to

$$\begin{aligned}
&D_t\Big[(1+\nu)\|D_t^r \nabla v\|_2^2 + F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\Big] \\
&\quad + \frac{4r+1}{4r-3}\Big\{\frac{\nu}{4}\|D_t^r \tilde{\Delta} v\|_2^2 + \|D_t^{r+1} v\|_2^2\Big\} + \frac{4r+1}{4r-7} F_{4r-1}(t)^{\frac{4}{4r-3}} G_{4r-1}(t) \\
&\quad \leq C\Big\{\|D_t^r \nabla v\|_2^{\frac{8r+6}{4r+1}} + \|\nabla v\|_2^{8r+6} + \sum_{j=1}^{[\frac{r}{2}]}(\|D_t^j \nabla v\|_2^{\frac{8r+6}{4j+1}} + \|D_t^{r-j} \nabla v\|_2^{\frac{8r+6}{4(r-j)+1}})\Big\} \\
&\qquad + \frac{1}{4r^3}\sum_{j=0}^{[\frac{r}{2}]} \|D_t^{r-j} \nabla v\|_2^{\frac{8j}{4(r-j)+1}} \|D_t^{r-j} \tilde{\Delta} v\|_2^2 + M_{4r+3}(t) \\
&\qquad + \frac{4r+1}{4r-3} F_{4r-1}(t)^{\frac{4}{4r-3}}\Big[CF_{4r-1}(t)^{\frac{4r-1}{4r-3}} + N_{4r-1}(t)\Big] \\
&\qquad + F_{4r-1}(t)^{\frac{4r+3}{4r-3}} + M_{4r+3}(t).
\end{aligned} \tag{8.14}$$

Thus the right hand side of (8.14) is bounded above by

$$\begin{aligned}
&C\Big[\|D_t^r \nabla v\|_2^{\frac{8r+6}{4r+1}} + \|\nabla v\|_2^{8r+6} + F_{4r-1}(t)^{\frac{4r+3}{4r-3}} + \sum_{j=1}^{[\frac{r}{2}]}\Big(\|D_t^j \nabla v\|_2^{\frac{8r+6}{4j+1}} + \|D_t^{r-j} \nabla v\|_2^{\frac{8r+6}{4(r-j)+1}}\Big)\Big] \\
&\quad + \frac{1}{4r^3}\nu \sum_{j=1}^{[\frac{r}{2}]} \|D_t^{r-j} \nabla v\|_2^{\frac{8j}{4(r-j)+1}} \|D_t^{r-j} \tilde{\Delta} v\|_2^2 + N_{4r+3}(t)
\end{aligned} \tag{8.15}$$

Again, we have the properties

$$\begin{aligned}
\|D_t^j \nabla v\|_2^{\frac{8r+6}{4j+1}} &\leq F_{4j+3}(t)^{\frac{4r+3}{4j+1}} \leq F_{4j+7}(t)^{\frac{4r+3}{4j+5}} \\
&\leq F_{4j+11}(t)^{\frac{4r+3}{4j+9}} \leq \cdots \leq F_{4r-1}(t)^{\frac{4r+3}{4r-3}}
\end{aligned} \tag{8.16}$$

and

$$\begin{aligned}
&\nu\|D_t^{r-j} \nabla v\|_2^{\frac{8j}{4(r-j)+1}} \|D_t^{r-j} \tilde{\Delta} v\|_2^2 \\
&\leq \|D_t^{r-j} \nabla v\|_2^{\frac{8(j-1)}{4(r-j)+1}} F_{4(r-j)+3}(t)^{\frac{4}{4(r-j)+1}} G_{4(r-j)+3}(t) \\
&\leq \|D_t^{r-j} \nabla v\|_2^{\frac{8(j-2)}{4(r-j)+1}} F_{4(r-j)+7}(t)^{\frac{4}{4(r-j)+5}} G_{4(r-j)+7}(t) \\
&\leq \cdots \leq \|D_t^{r-j} \nabla v\|_2^{\frac{8}{4(r-j)+1}} F_{4r-5}(t)^{\frac{4}{4r-7}} G_{4r-5}(t) \\
&\leq F_{4r-1}(t)^{\frac{4}{4r-3}} G_{4r-1}(t)
\end{aligned} \tag{8.17}$$

Hence the terms containing $\|D_t^{r-j}\tilde{\Delta}v\|_2$ in (8.15) can be majorized by small multiples of $F_{4r-1}(t)^{\frac{4}{4r-5}}G_{4r-1}(t)$ and so cancelled against a small part of the corresponding term on the left side in (8.14). Choosing this small multiple as indicated in (8.15) and using the inequality

$$\left[\frac{r}{2}\right]\frac{1}{4r^3} < \frac{4r+1}{4r-7} - \left(\frac{4r+1}{4r-3}\right)^2, \qquad r = 2,3,\dots$$

we now find from (8.14) and (8.15)

$$\begin{aligned}(8.18)\qquad & D_t\left[(1+\nu)\|D_t^r v\|_2^2 + F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right] \\ &+ \frac{4r+1}{4r-3}\left\{\frac{\nu}{4}\|D_t^r\tilde{\Delta}v\|_2^2 + \|D_t^{r+1}v\|_2^2 + \frac{4r+1}{4r-3}F_{4r-1}^{\frac{4}{4r-3}}(t)G_{4r-1}(t)\right\} \\ &\le C\left[(1+\nu)\|D_t^r\nabla v\|_2^2 + F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right]^{\frac{4r+3}{4r+1}} + N_{4r+3}(t)\end{aligned}$$

This establishes the second basic inequality (8.10) for the next induction step, in the form

$$(8.19)\qquad F'_{4r+3}(t) + \frac{4r+1}{4r-3}G_{4r+3}(t) \le CF_{4r+3}(t)^{\frac{4r+3}{4r+1}} + N_{4r+3}(t)$$

By the hypotheses of the Theorem for $\rho = 4r+3$, we see that $M_{4r+3}(t)$ as given by (7.17) is in $L^1(0,T)$. Hence, by (8.8), $N_{4r+3}(t)$ satisfies the conditions of the Integrability Lemma. Thus the lemma applies to (8.19) with $p = \frac{1}{4r+1}$ and $a = 1 + \frac{1}{4r+1}$. It follows that $G_{4r+3}(t) \in L^{\frac{1}{4r+3}}(0,T)$ and thus $\|D_t^r\tilde{\Delta}v\|_2$ and $\|D_t^{r+1}v\|_2 \in L^{\frac{2}{4r+3}}(0,T)$.

Now let f denote the viscosity potential of v, which is a solenoidal vector field vanishing on $\partial\Omega$, as in (2.18) of [4]. As in Lemma 1 of [4] we have the estimate

$$(8.20)\qquad \|\nabla f\|_2^2 \le C\|\nabla v\|_2\left(\|\tilde{\Delta}v\|_2 + \|\nabla v\|_2\right)$$

and its time derivative analogues

$$(8.21)\qquad \|D_t^r\nabla f\|_2^2 \le C\|D_t^r\nabla v\|_2\left(\|D_t^r\tilde{\Delta}v\|_2 + \|D_t^r\nabla v\|_2\right).$$

Hence $\|D_t^r\nabla f\|_2 \in L^{\frac{1}{2r+1}}(0,T)$ which follows from the preceding results. By an estimate of Ladyzhenskaya [6, p.21] we have

$$\begin{aligned}(8.22)\qquad \|D_t^r D_i D_j v\|_2^2 &\le C\left(\|D_t^r\Delta v\|_2^2 + \|D_t^r\nabla v\|_2^2\right) \\ &\le C\left(\|D_t^r\tilde{\Delta}v\|_2^2 + \|D_t^r\nabla f\|_2^2 + \|D_t^r\nabla v\|_2^2\right) \in L^{\frac{1}{4r+3}}(0,T)\end{aligned}$$

This establishes the result $\|D_t^r D_x^2 v\|_2 \in L^{\frac{2}{4r+3}}(0,T)$ and shows that the result of the Theorem is valid for v, u, and their first and second order space derivatives, together with all orders of time derivatives of these quantities; i.e. for $r = 0,1,2,\dots$ and $s = 0,1,2$.

As in [4, § 9] we also have by (3.12) and the hypotheses of the Theorem for $\rho = 3$,

$$(8.23)\qquad \max_{x\in\Omega}|u| \le C\left(\|\nabla u\|_2^{1/2}\|D_x^2 u\|_2^{1/2} + \|u\|_2\right) \in L^1(0,T)$$

while for $r = 1,2,\dots$ and the hypotheses of the Theorem for $\rho = 4r+3$,

$$(8.24)\qquad \max_{x\in\Omega}|D_t^r u| \le C\left(\|D_t^r\nabla u\|_2^{1/2}\|D_t^r D_x^2 u\|_2^{1/2} + \|D_t^r u\|_2\right) \in L^{\frac{1}{2r+1}}(0,T)$$

This establishes the maximum norm result of the Theorem for $s = 0$.

Chapter III. Estimates for Tangential and Normal Derivatives

9. **The Pressure Potentials.** To estimate space derivatives of order higher than the second, we again make use of tangential coordinate systems as introduced in [2, § 10]. Substituting (2.9) into (2.7) and (2.8) we find, in view of (2.6)

$$\Delta p = -v_{i,k}v_{k.i} - 2v_{i,k}w_{k,i} - w_{i,k}w_{k,i} + B_{i,i}. \tag{9.1}$$

Taking only the normal component of (2.8) while noting that v vanishes on $\partial\Omega$, as well as (2.12)

$$\begin{aligned}\frac{\partial p}{\partial n} = &- w_k\frac{\partial w_i}{\partial x_k}n_i + \nu n_i\Delta v_i + \nu n_i\Delta w_i \\ &- w_{i,t}n_i + B_in_i.\end{aligned} \tag{9.2}$$

Hence we may write

$$p = b_1 + b_2 + 2b_3 + b_4 + \nu f_1 + \nu f_2 + f_3 \tag{9.3}$$

where the various pressure terms indicated are defined as follows. Noting (2.12) we set

$$\begin{aligned}
&\Delta b_1 = -v_{i,k}v_{k,i}; &&\frac{\partial b_1}{\partial n} = 0\\
&\Delta b_2 = -w_{i,k}w_{k,i}; &&\frac{\partial b_2}{\partial n} = -w_k\frac{\partial w_i}{\partial x_k}n_i\\
&\Delta b_3 = -v_{i,k}w_{k,i}; &&\frac{\partial b_3}{\partial n} = 0\\
&\Delta b_4 = B_{i,i}(x,t); &&\frac{\partial b_4}{\partial n} = B_in_i\\
&\Delta f_1 = 0; &&\frac{\partial f_1}{\partial n} = n_i\Delta v_i\\
&\Delta f_2 = 0; &&\frac{\partial f_2}{\partial n} = n_i\Delta w_i\\
&\Delta f_3 = 0; &&\frac{\partial f_3}{\partial n} = -w_{i,t}n_i
\end{aligned} \tag{9.4}$$

and observe that each pressure term is defined up to a constant which can be specified by setting the average value over Ω to be zero if Ω is bounded. It can be shown that each of the seven listed Neumann problems has data that satisfy the necessary condition $\int_\Omega \Delta u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,ds$ and we leave the verification to the reader.

Observe that b_1 and f_1 satisfy exactly the same conditions as b and f in (11.1), (11.2), (5.1) and (5.2) of [4], except that v now replaces u in these conditions. Hence we may adopt for b_1 and f_1 the results of Lemmas 7 and 8 of [4] and their time derivative versions as in Lemma 8r there, with v of course replacing u in these estimates. Similarly, replacing u by w in Lemma 7 of [4], we obtain the estimate for f_2.

To estimate b_2 we proceed as follows, using the commutation formula (10.17) of [4]

$$
\begin{aligned}
\|\nabla D^s_\alpha b_2\|_2^2 &= \int_{\partial\Omega} D^s_\alpha b_2 \frac{\partial D^s_\alpha b_2}{\partial n}\, dS - \int_\Omega D^s_\alpha b_2 \Delta D^s_\alpha b_2 \, dx \\
&= -\frac{1}{2}\int_{\partial\Omega} D^s_\alpha b_2 D^s_\alpha w_k \frac{\partial w_i}{\partial x_k} n_i \, dS - \int_\Omega D^s_\alpha b_2 \Big\{ \sum_{j=0}^{s} \overline{A}^\alpha_\beta D^j_\beta \Delta b_2 \\
&\quad + \sum_{\beta<\alpha} \overline{B}^\alpha_\beta D_3 D^j_\beta b_2 + \sum_{\beta\le\alpha} \overline{C}^\alpha_\beta D_3 D^j_\beta b_2 + \sum_{\beta<\alpha} \overline{E}^\alpha_\beta D^j_\beta \big(D_1^2 + D_2^2\big) b_2 \Big\} dx \\
&= -\frac{1}{2}\int_\Omega \big(\nabla_i D^s_\alpha b_2\big) D^s_\alpha \Big(w_k \frac{\partial w_i}{\partial x_k}\Big) dx - \frac{1}{2}\int_\Omega D^\alpha_s b_2 \nabla_i D^\alpha_s \Big(w_k \frac{\partial w_i}{\partial x_k}\Big) dx \qquad (9.5)\\
&\quad + \int_\Omega D^s_\alpha b_2 \Big\{ \sum_{j=0}^{s} \overline{A}^\alpha_\beta D^j_\beta \big(w_{i,k} w_{k,i}\big) + \sum_{\beta<\alpha} \overline{B}^\alpha_\beta D_3 D^j_\beta b_2 + \sum_{\beta\le\alpha} \overline{C}^\alpha_\beta D^j_\beta b_2 \\
&\quad + \sum_{\beta<\alpha} \overline{E}^\alpha_\beta D^j_\beta \big(D_1^2 + D_2^2\big) b_2 \Big\} \, dx
\end{aligned}
$$

The permutation of derivatives in the second integral deserves some comment. We have the commutation formulae

$$
D^s_\alpha \nabla_i w = \nabla_i D^s_\alpha w + \sum_{j=0}^{s-1} g_j \nabla_i D^j_\beta w \qquad (9.6)
$$

$$
\nabla_i D^s_\alpha w = D^s_\alpha \nabla_i w + \sum_{j=0}^{s-1} \bar{g}_j D^j_\beta \nabla_i w \qquad (9.7)
$$

where $\nabla_i = h_i D_i$. The coefficients $g_j, \bar{g}_j$ are composed of positive or negative integer multiples of derivatives of $\log h_i$ which are independent of r near the pole and hence bounded and smooth everywhere. Thus

$$
\begin{aligned}
\nabla_i D^s_\alpha \Big(w_k \frac{\partial w_i}{\partial x_k}\Big) &= D^s_\alpha \nabla_i \Big(w_k \frac{\partial w_i}{\partial x_k}\Big) + \sum_{j=0}^{s-1} \bar{g}_j D^j_\alpha \nabla_i \Big(w_k \frac{\partial w_i}{\partial x_k}\Big) \\
&= D^s_\alpha \Big(\frac{\partial w_k}{\partial x_i}\frac{\partial w_i}{\partial x_k}\Big) + \sum_{j=0}^{s-1} \bar{g}_j D^j_\alpha \Big(\frac{\partial w_k}{\partial x_i}\frac{\partial w_i}{\partial x_k}\Big) \qquad (9.8)\\
&= \sum_{j=0}^{s} \binom{s}{j} D^j_\alpha \frac{\partial w_k}{\partial x_i} D^{s-j}_\beta \frac{\partial w_i}{\partial x_k} \\
&\quad + \sum_{j=0}^{s-1} \bar{g}_j \sum_{\ell=0}^{j} \binom{j}{\ell} D^\ell_\beta \frac{\partial w_k}{\partial x_i} D^{j-\ell}_\beta \frac{\partial w_i}{\partial x_k}
\end{aligned}
$$

Expanding the expression by means of (9.6), we see that the integral of the second term on the right hand side of (9.5) is bounded in magnitude by an expression

$$
\|D^s_\alpha b\|_6 \cdot C \sum_{j=0}^{s} \sum_{\ell=0}^{[\frac{j}{2}]} \|\nabla_i D^\ell_\alpha w\|_2 \|\nabla_i D^{j-\ell}_\alpha w\|_3 \qquad (9.9)
$$

Thus, in the same way as in (11.18) of [2]

$$
\begin{aligned}
&\Big|\int_\Omega D^s_\alpha b_2\cdot\nabla_i D^s_\alpha\Big(w_k\frac{\partial w_i}{\partial x_k}\Big)dx\Big|\\
&\qquad\le C\|\nabla D^s_\alpha b_2\|_2\sum_{j=0}^{s}\sum_{\ell=0}^{[\frac{j}{2}]}\|D_xD^\ell_\alpha w\|_2\|\nabla D^{j-\ell}_\alpha w\|_3\\
&\qquad\le\frac14\|\nabla D^s_\alpha b_2\|_2^2+\sum_{j=0}^{s}\sum_{\ell=0}^{[\frac{j}{2}]}\Big\{\|D_xD^\ell_\alpha w\|_2^2\Big(\|D_xD^{j-\ell}_\alpha w\|_2\|D^2_xD^{j-\ell}_\alpha w\|_2\\
&\qquad\quad+\|\nabla D^{j-\ell}_\alpha w\|_2^2\Big)\Big\}
\end{aligned}
$$

A reduction similar to a part of the foregoing shows that the term in (9.5) containing coefficients $\overline{A}^\alpha_\beta$ has a similar bound which can thus be combined with the preceding by adjustment of the constant C. The terms with coefficients $\overline{B}^\alpha_\beta$, $\overline{C}^\alpha_\beta$ and $\overline{E}^\alpha_\beta$ are treated as in the proof of Lemma 7 of [4].

Finally, since by (9.6) we have

$$
\begin{aligned}
D^s_\alpha\Big(w_k\frac{\partial w_i}{\partial x_k}\Big)&=\sum_{j=0}^{s}\binom{s}{j}D^j_\beta w_kD^{s-j}_\beta\frac{\partial w_i}{\partial x_k}\\
&=\sum_{j=0}^{s}\binom{s}{j}D^j_\beta w_k\sum_{\ell=0}^{s-j}g_\ell\nabla_kD^\ell_\beta w_i
\end{aligned}
$$

it follows that the first integral on the right side of (9.5) is bounded as follows by (9.6):

$$
\begin{aligned}
&\Big|\int_\Omega \nabla_iD^s_\alpha b_2\cdot D^s_\alpha\Big(w_k\frac{\partial w_i}{\partial x_k}\Big)\,dx\Big|\\
&\quad\le\|\nabla_iD^s_\alpha b_2\|_2\|D^s_\alpha\Big(w_k\frac{\partial w_i}{\partial x_k}\Big)\|_2\\
&\quad\le C\|\nabla D^s_\alpha b_2\|_2\Big\{\sum_{0\le j+\ell\le s,\ell\ge\frac{s}{2}}\|D^j_\beta w_k\|_6\|D_xD^\ell_\beta w_i\|_3\\
&\qquad+\sum_{\substack{0\le j+\ell\le s\\0\le\ell<[\frac{s}{2}]}}\max_{x\in\Omega}|D^j_\beta w_k|\,\|D_xD^\ell_\beta w\|_2\Big\}\\
&\quad\le C\|\nabla D^s_\alpha b_2\|_2\Big\{\sum_{\substack{j+\ell\le s\\ \ell\ge[\frac{s}{2}]}}\|D_xD^j_\beta w\|_2\Big(\|D_xD^\ell_\beta w\|_2^{1/2}\|D^2_xD^\ell_\beta w\|_2^{1/2}+\|D_xB^\ell_\beta w\|_2\Big)\\
&\qquad+\sum_{\substack{j+\ell\le s\\0\le\ell<[\frac{s}{2}]}}\Big(\|D_xD^j_\beta w\|_2^{1/2}\|D^2_xD^j_\beta w\|_2^{1/2}+\|D_xD^j_\beta w\|_2+\|D^j_\beta w\|_2\Big)\cdot\|D_xD^\ell_\beta w\|_2\Big\}\\
&\quad\le\frac14\|\nabla D^s_\alpha b_2\|_2^2+C\sum_{\substack{0\le j+\ell\le s\\ \ell\ge[\frac{s}{2}]}}\|D_xD^j_\beta w\|_2^2\Big(\|D_xD^\ell_\beta w\|_2\|D^2_xD^\ell_\beta w\|_2+\|D_xD^\ell_\beta w\|_2^2\Big)\\
&\qquad+C\sum_{\substack{0\le j+\ell\le s\\0\le\ell<[\frac{s}{2}]}}\Big(\|D_xD^j_\beta w\|_2\|D^2_xD^j_\beta w\|_2+\|D_xD^j_\beta w\|_2^2+\|D^j_\beta w\|_2^2\Big)\|D_xD^\ell_\beta w\|_2^2
\end{aligned}
$$

Assembling these results, we find that (9.5) becomes after cancellations,

$$\begin{aligned}\|\nabla D_\alpha^s b_2\|_2^2 \le\ & C\sum_{j=0}^{s}\sum_{\ell=0}^{[\frac{j}{2}]} \|D_x D_\alpha^\ell w\|_2^2 \\ & \times \left(\|D_x D_\alpha^{j-\ell} w\|_2 \|D_x^2 D_\alpha^{j-\ell} w\|_2 + \|D_x D_\alpha^{j-\ell} w\|_2^2 + \|D_\alpha^{j-l} w\|_2^2\right) \\ & + C_1 \sum_{\ell=0}^{s-1} \|\nabla D_\alpha^\ell b_2\|_2^2 \end{aligned} \tag{9.10}$$

By evaluation successively for $s = 0, 1, 2, 3, \dots$ and successive substitution of the earlier formulas into (9.10), we find as in Lemmas 7 and 8 of [4]

LEMMA 2. *For $s = 0, 1, 2, \dots$ we have*

$$\begin{aligned}\|\nabla D_\alpha^s b_2\|_2^2 \le C\sum_{j=0}^{s}\sum_{\ell=0}^{[\frac{j}{2}]} \|D_x D_\alpha^\ell w\|_2^2 \Big(&\|D_x D_\alpha^{j-\ell} w\|_2 \|D_x^2 D_\alpha^{j-\ell} w\|_2 \\ &+ \|D_x D_\alpha^{j-\ell} w\|_2^2 + \|D_\alpha^{j-l} w\|_2^2\Big)\end{aligned} \tag{9.11}$$

As in Lemma 8 of [4] we state the corresponding form of the result for time derivatives of order r without detailed calculation:

LEMMA 2R. *For $r, s = 0, 1, 2, 3, \dots$ we have*

$$\begin{aligned}&\|D_t^r \nabla D_\alpha^s b_2\|_2^2 \\ &\quad \le C\sum_{h=0}^{r}\sum_{j=0}^{s}\sum_{\ell=0,\ell\le s}^{[\frac{j}{2}+r-2h]} \|D_t^h D_x D_\alpha^\ell w\|_2^2 \\ &\quad \Big(\|D_t^{r-h} D_x D_\alpha^{j-\ell} w\|_2 \|D_t^{r-h} D_x^2 D_\alpha^{j-\ell} w\|_2 + \|D_t^{r-h} D_x D_\alpha^{j-\ell} w\|_2^2 \\ &\quad + \|D_t^{r-h} D_\alpha^{j-l} w\|_2^2\Big)\end{aligned} \tag{9.12}$$

Here C depends on r and s but not on w.

By entirely similar derivations we obtain for b_3 the estimates in

LEMMA 3. *For $s = 0, 1, 2, \dots$ we have*

$$\begin{aligned}\|\nabla D_\alpha^s b_3\|_2^2 \le C\sum_{j=0}^{s}\sum_{\ell=0}^{[\frac{j}{2}]} \Big\{&\|D_x D_\alpha^\ell w\|_2^2 \Big(\|\nabla D_\alpha^{j-\ell} v\|_2 \|\tilde{\Delta} D_\alpha^{j-\ell} v\|_2 + \sum_{\beta\le j-\ell} \|\nabla_\beta v\|_2^2\Big) \\ &+ \|\nabla D_\alpha^\ell v\|_2^2 \Big(\|D_x D_\alpha^{j-\ell} w\|_2 \|D_x^2 D_\alpha^{j-\ell} w\|_2 + \|D_x D_\alpha^{j-\ell} w\|_2^2 + \|D_\alpha^{j-l} w\|_2^2\Big)\Big\}\end{aligned} \tag{9.13}$$

and

LEMMA 3R. *For $r, s = 0, 1, 2, \dots$ we have*

$$\|D_t^r \nabla D_\alpha^s b_3\|_2^2$$

$$\le C\sum_{j=0}^{r}\sum_{j=0}^{s}\sum_{\ell=0,\ell\le s}^{[\frac{j}{2}+r-2h]}\Big\{\|D_t^h\nabla D_\alpha^\ell w\|_2^2$$

$$(\|D_t^{r-h}\nabla D_\alpha^{j-\ell}v\|_2\|D_t^{r-h}\tilde{\Delta}D_\alpha^{j-\ell}v\|_2+\sum_{\alpha\le\beta}\|D_t^{r-h}\nabla D_\beta^{r-\ell}v\|_2^2) \tag{9.14}$$

$$+\|D_t^h\nabla D_\alpha^\ell v\|_2^2\big(\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2\|D_t^{r-h}D_x^2D_\alpha^{j-\ell}w\|_2$$

$$+\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2^2+\|D_t^{r-h}D_\alpha^{j-l}w\|_2^2\big)\Big\}$$

where C depends on r, s but not on v or w.

To estimate b_4 we have

$$\|\nabla D_\alpha^s b_4\|_2^2=\int_{\partial\Omega}D_\alpha^s b_4\frac{\partial D_\alpha^s b_4}{\partial n}\,dS-\int_\Omega D_\alpha^s b_4\Delta D_\alpha^s b_4\,dx$$

$$=\int_{\partial\Omega}D_\alpha^s b_4 D_\alpha^s B_i n_i\,dS-\int_\Omega D_\alpha^s b_4\Big(D_\alpha^s\Delta b_4+\sum_{\beta<\alpha}\overline{A}_\beta^\alpha D_\beta^j\Delta b_4$$

$$+\sum_{\beta<\alpha}\overline{B}_\beta^\alpha D_3D_\beta^j b_4+\sum_{\beta\le\alpha}\overline{C}_\beta^\alpha D_\beta^j b_4+\sum_{\beta<\alpha}\overline{E}_\beta^\alpha D_\beta^j(D_1^2+D_2^2)b_4\Big)\,dx$$

$$=\int_\Omega\nabla_iD_\alpha^s b_4\cdot D_\alpha^s B_i\,dx+\int_\Omega D_\alpha^s b_4\cdot\nabla_iD_\alpha^s B_i\,dx$$

$$-\int_\Omega D_\alpha^s b_4\Big\{D_\alpha^s B_{i,i}+\sum_{\beta<\alpha}\overline{A}_\beta^\alpha D_\beta^j B_{i,i}+\sum_{\beta<\alpha}\overline{B}_\beta^\alpha D_3D_\beta^j b_4$$

$$+\sum_{\beta\le\alpha}\overline{C}_\beta^\alpha D_\beta^j b_4+\sum_{\beta<\alpha}\overline{E}_\beta^\alpha D_\beta^j(D_1^2+D_2^2)b_4\Big\}\,dx \tag{9.15}$$

$$\le\|\nabla D_\alpha^s b_4\|_2\|D_\alpha^s B\|_2+\|D_\alpha^s b_4\|_6\|\nabla D_\alpha^s B\|_{6/5}$$

$$+\|D_\alpha^s b_4\|_6\Big\{\|D_\alpha^s\nabla_iB_i\|_{6/5}+C\sum_{j=0}^{s-1}\|D_\alpha^j\nabla_iB_i\|_{6/5}\Big\}$$

$$+C\sum_{j=0}^{s-1}\|\nabla D_\alpha^j b_4\|_2^2$$

$$\le\|\nabla D_\alpha^s b_4\|_2\Big\{\|D_\alpha^s B\|_2+C\|\nabla D_\alpha^s B\|_{6/5}+C\sum_{j=0}^{s-1}\|\nabla D_\alpha^j B\|_{6/5}\Big\}$$

$$+\sum_{j=0}^{s-1}\|\nabla D_j^\alpha b_4\|_2^2$$

where we have again used (9.6) and (9.7) to interchange orders of ∇ and D_α^j, as well as calculations of commutator terms like those leading to Lemma 2.

Hence by Young's inequality once more we find

$$\|\nabla D_\alpha^s b_4\|_2^2\le C\Big\{\|D_\alpha^s B\|_2^2+\sum_{j=0}^{s}\|\nabla D_\alpha^s B\|_{6/5}^2\Big\}+C\sum_{j=0}^{s-1}\|\nabla D_\alpha^j b_4\|_2^2. \tag{9.16}$$

By (4.8) this leads as before to

LEMMA 4. *For $s = 0, 1, 2, \ldots$ we have*

$$\|\nabla D^s_\alpha b_4\|_2^2 \le C\{\|B\|^2_{6/5} + \sum_{j=0}^{s} \|\nabla D^j_\alpha B\|^2_{6/5}\}. \tag{9.17}$$

Since the definition of b_4 is linear in B, we can, as is the case with Lemma 7 of [4] and with f, simply insert the symbolic factor D^r_t in every term of this estimate to obtain the general case.

The final estimate is for f_3. We have

$$\|\nabla D^s_\alpha f_3\|_2^2 = \int_{\partial\Omega} D^s_\alpha f_3 \frac{\partial D^s_\alpha f_3}{\partial n}\, dS - 0$$

in view of $\Delta f_3 = 0$. Substituting the boundary condition, we find the surface integral becomes

$$-\int_{\partial\Omega} D^s_\alpha f_3 D^s_\alpha w_{i,t} n_i\, dS = -\int_\Omega \nabla_i D^s_\alpha f_3 D^s_\alpha w_{i,t}\, dx - \int_\Omega D^s_\alpha f_3 \nabla_i D^s_\alpha w_{i,t}\, dx$$

From (9.7) we see that the second integral equals a sum of expressions the first of which vanishes because $w_{i,i} = 0$. Estimating by (3.3) and (3.7), we find

LEMMA 5.

$$\|\nabla D^s_\alpha f_3\|_2 \le C \sum_{j=0}^{s} \|D^j_\beta w_{i,t}\|_2 + C \sum_{j=0}^{s-1} \|D_x D^j_\beta w_{i,t}\|_{6/5}. \tag{9.18}$$

The corresponding formula for time derivatives is also valid.

10. **Tangential Derivative Inequalities.** To construct inequalities for the estimation of $D^r_t D^s_\alpha v$ we set $u = v + w$ in (2.1) and differentiate r times with respect to t, s_α times with respect to ξ_α where $s = s_1 + s_2$, and so obtain, after commuting Δ and D^s_α,

$$\begin{aligned} D^{r+1}_t D^s_\alpha(v_i + w_i) &+ \sum_{r=0}^{r}\sum_{j=0}^{s} \binom{r}{h}\binom{s}{j} D^h_t D^j_\alpha (v_k + w_k) D^{r-h}_t D^{s-j}_\alpha (v_{i,k} + w_{i,k}) \\ &= -D^r_t D^s_\alpha (b_{1,i} + b_{2,i} + 2b_{3,i} + b_{4,i} + \nu f_{1,i} + \nu f_{2,i} + f_{3,i}) \\ &\quad + \nu D^r_t \Delta D^s_\alpha (v_i + w_i) + \nu \sum_{j=0}^{s-1} A^\alpha_\beta D^r_t \Delta D^j_\beta (v_i + w_i) \\ &\quad + \nu \sum_{j=0}^{s-1} B^\alpha_\beta D^r_t D_3 D^j_\beta (v_i + w_i) + \nu \sum_{j=1}^{s} C^\alpha_\beta D^r_t D^j_\beta (v_i + w_i) \\ &\quad + \nu \sum_{j=0}^{s-1} E^\alpha_\beta D^r_t D^j_\beta (D_1^2 + D_2^2)(v_i + w_i) + D^r_t D^s_\alpha B_i. \end{aligned} \tag{10.1}$$

For the first inequality at this level, multiply by $D_t^r D_\alpha^s v_i$, contract over i, integrate over Ω and integrate by parts the terms containing the Laplacian. This yields

(10.2)
$$\begin{aligned}
&\frac{1}{2}D_t\|D_t^r D_\alpha^s v\|_2^2 + \int_\Omega D_t^r D_\alpha^s v_i D_t^{r+1} D_\alpha^s w_i.\ dx + \nu\|D_t^r \nabla D_\alpha^s v\|_2^2 \\
&\quad = -\int_\Omega D_t^r D_\alpha^s v_i \sum_{h,j=0}^{r,s} \binom{r}{h}\binom{s}{j} D_t^h D_\alpha^j (v_k + w_k) D_t^{r-h} D_\alpha^{s-j}(v_{i,k} + w_{i,k})\, dx \\
&\qquad - \int_\Omega D_t^r D_\alpha^s v_i D_t^r D_\alpha^s (b_{1,i} + b_{2,i} + 2b_{3,i} + b_{4,i} + \nu f_{1,i} + \nu f_{2,i} + f_{3,i})\, dx \\
&\qquad - \nu \int_\Omega \nabla D_t^r D_\alpha^s v_i (D_t^r \nabla D_\alpha^s w_i + \sum_{j=0}^{s-1} A_\beta^\alpha D_t^r \nabla D_\beta^j (v_i + w_i))\, dx \\
&\qquad + \nu \int_\Omega D_t^r D_\alpha^s v_i (\sum_{j=0}^{s-1} B_\beta^\alpha D_t^r D_3 D_\beta^j (v_i + w_i) + \sum_{j=0}^{s} C_\beta^\alpha D_t^r D_\beta^j (v_i + w_i))\, dx \\
&\qquad + \nu \int_\Omega D_t^r D_\alpha^s v_i \sum_{j=0}^{s-1} E_\beta^\alpha D_t^r D_\beta^j (D_1^2 + D_2^2)(v_i + w_i)\, dx \\
&\qquad + \int_\Omega D_t^r D_\alpha^s v_i D_t^r D_\alpha^s B_i\, dx
\end{aligned}$$

Let us denote the first three integrals on the right side as I_1, I_2, I_3, and the terms with B, C and E coefficients I_4, I_5 and I_6 respectively, the last term being I_7. In I_1 we treat the products in the order $v\nabla\, v$, $w\nabla\, v$, $vD_x w$ and $wD_x w$ respectively; thus I_{13hj} will denote the integral term over $v\nabla\, w$ with given values of h and j. We note that I_{1100} and I_{1200} vanish, as in preceding cases, by the divergence theorem, after permutation of ∇_k and D_α^s in the third factor. By (9.6) the lower order terms arising from this commutation take the form

$$\begin{aligned}
&\left|\sum_{j=0}^{s-1} g_k \int_\Omega D_t^h D_\alpha^s v_i (v_k + w_k) D_t^r \nabla_k D_\alpha^j v_i\, dx\right| \\
&\quad \leq C\|D_t^r D_\alpha^s v\|_2 \big\{ (\|\nabla\, v\|_2^{\frac{1}{2}} \|\tilde{\Delta} v\|_2^{\frac{1}{2}} + \|\nabla\, v\|_2 + \|D_x w\|_2^{\frac{1}{2}} \|D_x^2 w\|_2^{\frac{1}{2}} + \|D_x w\|_2) \\
&\quad \times \|D_t^r \nabla D_\alpha^{s-1} v\|_2 \qquad (10.3) \\
&\quad + \sum_{j=0}^{s-2} (\|\nabla\, v\|_2 + \|D_x w\|_2)(\|D_t^r \nabla D_\alpha^j v\|_2^{\frac{1}{2}} \|D_t^r \tilde{\Delta} D_\alpha^j v\|_2^{\frac{1}{2}} + \sum_{\beta \leq \alpha} \|D_t^r \nabla D_\beta^j v\|_2) \big\}.
\end{aligned}$$

In the third and fourth terms of the same group we find, after a similar permutation, and using (3.12)

$$\begin{aligned}
&\left|\int_\Omega \sum_{v=0}^{s} g_j D_t^r D_\alpha^s v_i (v_k + w_k) D_t^r \nabla_k D_\alpha^j w_i\, dx\right| \\
&\leq C\|D_t^h D_\alpha^s v\|_2 \{ \|\nabla\, v\|_2^{\frac{1}{2}} \|\tilde{\Delta} v\|_2^{\frac{1}{2}} + \|D_x w\|_2^{\frac{1}{2}} \|D_x^2 w\|_2^{\frac{1}{2}} + \|D_x w\|_2 \} \|D_t^r \nabla D_\alpha^s w\|_2 \\
&\quad + \sum_{j=0}^{s-1} (\|\nabla\, v\|_2 + \|D_x w\|_2)(\|D_t^r \nabla D_\alpha^j w\|_2^{\frac{1}{2}} \|D_t^r D_x^2 D_\alpha^j w\|_2^{\frac{1}{2}} + \|D_t^r D_x D_\alpha^j w\|_2)
\end{aligned}$$

Other terms of the double sum I_1 will be estimated as in § 12 of [4], and are accompanied by lower order terms that need not be treated explicitly on every occasion.

The 7 terms arising from I_2 are also treated as in § 12 of [4] and may then be substituted with the aid of Lemmas 7 and 8r of [4] and Lemmas 2r, 3r, 4 and 5 of the preceding § 10.

The first term in I_3 may be bounded:

$$
\begin{aligned}
|I_{31}| &= \left| \int_\Omega D_t^r \nabla D_\alpha^s v_i D_t^r \nabla D_\alpha^s w_i \, dx \right| \\
&\le \|D_t^r \nabla D_\alpha^s v\|_2 \|D_t^r \nabla D_\alpha^s w\|_2 \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \nabla D_\alpha^s v\|_2^2 + C \|D_t^r D_x D_\alpha^s w\|_2^2 .
\end{aligned} \tag{10.4}
$$

The remaining terms in I_3 are bounded by

$$
\begin{aligned}
&C \sum_{j=0}^{s-1} \|D_t^r \nabla D_\alpha^s v\|_2 (\|D_t^r \nabla D_\alpha^j v\|_2 + \|D_t^r D_x D_\alpha^j w\|_2) \\
&\qquad \le \frac{\nu}{2^{2r+s+5}} \|D_t^r \nabla D_\alpha^s v\|_2^2 \\
&\qquad\quad + C \sum_{j=0}^{s-1} (\|D_t^r \nabla D_\alpha^j v\|_2^2 + \|D_t^r D_x D_\alpha^j w\|_2^2).
\end{aligned} \tag{10.5}
$$

Also

$$
\begin{aligned}
|I_4| &\le C \sum_{j=0}^{s-1} \int_\Omega |D_t^r \nabla D_\alpha^s v_i| |D_t^r \nabla D_\beta^j (v_i + w_i)| \, dx \\
&\le C \sum_{j=0}^{s-1} \|D_t^r \nabla D_\alpha^s v\|_2 \big(\|D_t^r \nabla D_\beta^j v\|_2 + \|D_t^r D_x D_\beta^j w\|_2 \big) \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \nabla D_\alpha^s v\|_2^2 + C \sum_{j=0}^{s-1} \big(\|D_t^r \nabla D_\beta^j v\|_2^2 + \|D_t^r D_x D_\beta^j w\|_2^2 \big)
\end{aligned}
$$

while I_5, I_6 have similar bounds.

Finally,

$$
\begin{aligned}
|I_7| &\le \|D_t^r D_\alpha^s v\|_6 \|D_t^r D_\alpha^s B\|_{6/5} \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \nabla D_\alpha^s v\|_2^2 + C \|D_t^r D_\alpha^s B\|_{6/5}^2
\end{aligned} \tag{10.7}
$$

Likewise, the additional term on the left side of (10.2) can be estimated as follows:

$$
\left| \int_\Omega D_t^r D_\alpha^s v_i D_t^{r+1} D_\alpha^s w_i \, dx \right| \le \frac{\nu}{2^{2r+s+5}} \|D_t^r \nabla D_\alpha^s v\|_2^2 + C \|D_t^{r+1} D_\alpha^s w\|_{6/5}^2 . \tag{10.8}
$$

The companion inequality is now formed by multiplying (10.1) by $D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i$ after changing s into $s-1$ throughout, then integrating over Ω. After routine calculations we find

$$
\begin{aligned}
&\frac{1}{2} D_t \|D_t^r \nabla D_\alpha^{s-1} v\|_2^2 + \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i D_t^{r+1} D_\alpha^{s-1} w_i \, dx + \nu \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2^2 \\
&\quad = - \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i \sum_{h,j=0}^{r,s-1} \binom{r}{h} \binom{s-1}{j} D_t^h D_\alpha^j (v_k + w_k) D_t^{r-h} D_\alpha^{s-j-1} (v_{i,k} + w_{i,k}) \, dx
\end{aligned}
$$

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$$
\begin{aligned}
&- \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i D_t^r D_\alpha^{s-1}(b_{1,i} + b_{2,i} + 2b_{3,i} + b_{4,i} + \nu f_{1,i} + \nu f_{2,i} + f_{3,i})\, dx \\
&+ \nu \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i \{ D_t^r \Delta D_\alpha^{s-1} w_i + \sum_{j=0}^{s-2} A_\beta^\alpha D_t^r \Delta D_\beta^j (v_i + w_i)\}\, dx \\
&+ \nu \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i \{ \sum_{j=0}^{s-2} B_\beta^\alpha D_t^r D_3 D_\beta^j (v_i + w_i) + \sum_{j=0}^{s-1} C_\beta^\alpha D_t^r D_\beta^j (v_i + w_i)\}\, dx \\
&+ \nu \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i \sum_{j=0}^{s-2} E_\beta^\alpha D_t^r D_\beta^j (D_1^2 + D_2^2)(v_i + w_i)\, dx \\
&+ \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i D_t^r D_\alpha^{s-1} B_i\, dx
\end{aligned} \tag{10.9}
$$

As before denote the first three integrals by J_1, J_2 and J_3, the terms with B, C and E coefficients by J_4, J_5 and J_6, and the last integral by J_7.

As in § 12 of [4], we apply the $L^6(\Omega)$ norm to the factor $D_t^h D_\alpha^j v_k$ or $D_t^h D_\alpha^j w_k$ in J_1, when $2h + j < r + \frac{s}{2}$, otherwise we apply to it the $L^\infty(\Omega)$ norm; the resulting expression being as in (12.13) of [4] where $u = v$ in the first factors, and $u = v + w$ in the second and third factors of each term.

The seven terms in J_2 are also treated according to Lemmas 7 and 8r of [4] and Lemmas 2r 3r 4 and 5 of the preceding § 9.

In J_3 we note that the Δ operators on the second factors can be estimated, as in (12.17) of [4], by Stokes and gradient terms, the latter yielding zero for the first term, in which orthogonality of gradients and solenoidal fields of L can be invoked.

For J_4 we have

$$
\begin{aligned}
|J_4| &\le C\nu \int_\Omega |D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i| \sum_{j=0}^{s-2} |D_t^r \nabla D_\beta^j (v_i + w_i)|\, dx \\
&\le C \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2 \sum_{j=0}^{s-2} (\|D_t^r \nabla D_\beta^j v\|_2 + \|D_t^r D_x D_\beta^j w\|_2) \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2^2 \\
&\quad + C \sum_{j=0}^{s-2} \big(\|D_t^r \nabla D_\beta^j v\|_2^2 + \|D_t^r D_x D_\beta^j w\|_2^2\big).
\end{aligned} \tag{10.10}
$$

Similar estimates hold for J_5 and J_6 while for J_7 we find

$$
\begin{aligned}
|J_7| &\le \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2 \|D_t^r D_\alpha^{s-1} B\|_2 \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2^2 + C \|D_t^r D_\alpha^{s-1} B\|_2^2.
\end{aligned} \tag{10.11}
$$

Also the additional term on the left side of (10.9) can be estimated:

$$
\begin{aligned}
\left| \int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} v_i D_t^{r+1} D_\alpha^{s-1} w_i\, dx \right| &\le \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2 \|D_t^{r+1} D_\alpha^{s-1} w\|_2 \\
&\le \frac{\nu}{2^{2r+s+5}} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2^2 + C \|D_t^{r+1} D_\alpha^{s-1} w\|_2^2.
\end{aligned} \tag{10.12}
$$

Adding together (10.2) and (10.9) we find after these reductions and multiplying by 2,

$$
\begin{aligned}
&D_t\{\|D_t^rD_\alpha^s v\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}v\|_2^2+\frac{3}{2}\nu\{\|D_t^r\nabla D_\alpha^s v\|_2^2+\|D_t^r\tilde\Delta D_\alpha^{s-1}v\|_2^2\}\\
&\le C\Big\{(\|D_t^rD_\alpha^s v\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}v\|_2^2)\\
&\qquad\times(\|\nabla v\|_2^4+\|D_xw\|_2^4+\|\nabla v\|_2\|\tilde\Delta v\|_2+\|D_xw\|_2\|D_x^2w\|_2+1)\\
&\quad+\sideset{}{'}\sum_{\substack{h,\ell,j\ge0\\ 2h+\frac{j}{2}\ge r+\frac{\ell}{2}}}^{r,s,s-\ell}\|D_t^hD_\alpha^j v\|_2\|D_t^h\nabla D_\alpha^j v\|_2(\|D_t^{r-h}\nabla D_\alpha^\ell v\|_2^2+\|D_t^{r-h}D_xD_\alpha^\ell w\|_2^2)\\
&\quad+\sideset{}{'}\sum_{\substack{h,\ell,j\ge0\\ 2h+\frac{\ell}{2}>r+\frac{j}{2}}}^{r,s-1,s-\ell-1}(\|D_t^h\nabla D_\alpha^\ell v\|_2\|D_t^h\tilde\Delta D_\alpha^\ell v\|_2+\sum_{\beta\le\alpha}\|D_t^h\nabla D_\beta^\ell v\|_2^2)\\
&\qquad\times(\|D_t^{r-h}\nabla D_\alpha^j v\|_2^2+\|D_t^{r-h}D_xD_\alpha^j w\|_2^2) \qquad\qquad (10.13)\\
&\quad+\sum_{h,j=0}^{r,s-1}\sum_{\ell=0,\ell\le s}^{[j/2+r-2h]}(\|D_t^h\nabla D_\alpha^\ell v\|_2^2+\|D_t^hD_xD_\alpha^\ell w\|_2^2)\\
&\qquad\times(\|D_t^{r-h}\nabla D_\alpha^{j-\ell}v\|_2\|D_t^{r-h}\tilde\Delta D_\alpha^{j-\ell}v\|_2+\sum_{\beta\le\alpha}\|D_t^{r-h}\nabla D_\beta^{j-\ell}v\|_2^2)\\
&\quad+\sum_{j=0}^{s-1}\|D_t^r\nabla D_\alpha^j v\|_2(\|D_t^r\tilde\Delta D_\alpha^j v\|_2+\sum_{\beta\le\alpha}\|D_t^r\nabla D_\beta^j v\|_2)\\
&\quad+\sum_{j=0}^{s-2}\|D_t^r\tilde\Delta D_\alpha^j v\|_2^2+\sum_{j=0}^{s-1}\|D_t^r\nabla D_\alpha^j v\|_2^2+\sum_{j=0}^{s}\|D_t^rD_\alpha^s v\|_2^2\Big\}\\
&\quad+\sum_{j=0}^{s-1}\|D_t^rD_xD_\alpha^j w\|_2\{\|D_t^rD_xD_\alpha^j w\|_2+\|D_t^rD_\alpha^j w\|_2\}\\
&\quad+\sum_{j=0}^{s-1}\|D_t^rD_x^2D_\alpha^j w\|_2^2+\sum_{j=0}^{s}\|D_t^r\nabla D_\alpha^j w\|_2^2+\sum_{j=0}^{s}\|D_t^rD_\alpha^s w\|_2^2\}\\
&\quad+\sum_{\substack{h,j,\ell\ge0\\ 2h+j\ge r+\frac{s}{2}}}^{r,s,s-\ell}\|D_t^hD_\alpha^j w\|_2\|D_t^hD_xD_\alpha^j w\|_2(\|D_t^{r-h}\nabla D_\alpha^{s-j}v\|_2^2+\|D_t^{r-h}D_xD_\alpha^{s-j}w\|_2^2)\\
&\quad+\sum_{\substack{h,\ell,j\ge0\\ 2h+\frac{\ell}{2}\ge r+\frac{j}{2}}}^{r,s-1,s-\ell-1}\|D_t^hD_xD_\alpha^\ell w\|_2\|D_t^hD_x^2D_\alpha^\ell w\|_2(\|D_t^{r-h}\nabla D_\alpha^{s-j}v\|_2^2+\|D_t^{r-h}D_xD_\alpha^{s-j}w\|_2^2)\\
&\quad+\sum_{h,j=0}^{r,s-1}\sum_{\ell=0,\ell\le s}^{[j/2+r-2h]}(\|D_t^h\nabla D_\alpha^\ell v\|_2^2+\|D_t^hD_xD_\alpha^\ell w\|_2^2)\\
&\qquad\{\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2\|D_t^{r-h}D_x^2D_\alpha^{j-\ell}w\|_2+\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2^2\|D_t^{r-h}D_\alpha^{j-\ell}w\|_2^2\}\\
&\quad+\sum_{h,j=0}^{r,s-1}\sum_{\ell=0,\ell\le s}^{[j/2+r-2h]}\{\|D_t^hD_xD_\alpha^\ell w\|_2^2
\end{aligned}
$$

$$\times(\|D_t^{r-h}\nabla D_\alpha^{j-\ell}v\|_2\|D_t^{r-h}\tilde{\Delta}D_\alpha^{j-\ell}v\|_2+\sum_{\beta\le\alpha}\|D_t^{r-h}\nabla D_\beta^{j-\ell}v\|_2^2)$$
$$+\|D_t^h\nabla D_\alpha^\ell v\|_2^2(\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2\|D_t^{r-h}D_x^2D_\alpha^{j-\ell}w\|_2$$
$$+\|D_t^{r-h}D_xD_\alpha^{j-\ell}w\|_2^2\|D_t^{r-h}D_\alpha^{j-\ell}w\|_2^2)\}$$
$$+\sum_{h,j=0}^{r,s-1}\sum_{\ell=0,\ell\le s}^{[j/2+r-2h]}\{\|D_t^hD_\alpha^\ell B\|_2^2+\sum_{m=0}^{\ell}\|D_t^h\nabla D_\alpha^m B\|_{6/5}^2\}$$
$$+\sum_{h,j=0}^{r,s-1}\sum_{\ell=0,\ell\le s}^{[j/2+r-2h]}\{\|D_t^h\nabla D_\beta^j v\|_2(\|D_t^h\tilde{\Delta}D_\beta^j v\|_2+\sum_{\beta\le\alpha}\|D_t^h\nabla D_\beta^j v\|_2)$$
$$+\|D_t^hD_xD_\beta^j w\|_2(\|D_t^hD_x^2D_\beta^j w\|_2+\|D_t^hD_xD_\beta^j w\|_2)\}$$
$$+\sum_{j=0}^{s}\|D_t^{r+1}D_\alpha^j w\|_2^2+\|D_t^rD_x^2D_\alpha^{s-1}w\|_2^2+\|D_t^rD_\alpha^s B\|_{6/5}^2$$
$$+\sum_{j=0}^{s-1}\|D_t^{r+1}D_xD_\beta^j w\|_{6/5}$$
$$+\sum_{j=0}^{s}\|D_t^{r+1}D_\alpha^j w\|_{6/5}^2+\|D_t^{r+1}D_\alpha^{s-1}w\|_2^2+\|D_t^rD_\alpha^{s-1}B\|_2^2+1\}$$

11. **Tangential Derivative Estimates.** As in the initial value problem [4, § 3] we now apply Young's inequality to the various products of derivative norms on the right hand side of the main inequality (10.13). Up to this stage, the functions v and w have appeared essentially in a symmetric way in many of the sums on the right hand side. But while the above process of bounding by powers of all the norms should be carried out completely for all terms in w, this will not be done for all products containing v. As in [4, § 13], the terms containing Stokes operators acting on v are accompanied by powers of the corresponding gradient norms, which should be retained in such a way that no power of a Stokes operator norm in v or its time and tangential derivatives higher than the square appears on the right side. After the cancellation against the left side of any remaining terms in $\|D_t^r\nabla D_\alpha^s v\|_2^2$ and $\|D_t^r\tilde{\Delta}D_\alpha^{s-1}v\|_2^2$, this yields an inequality of the following form, where a prime on the summation sign again indicates omission of the *upper right corner* term $h=r, j=s$ in the first two occurrences, and $h=r, j=s-1$ in the third. The body force terms are carried over without significant change apart from being expressed as powers of derivative norms. For $h=j=0$ replace $4h+2j-1$ by $+1$ henceforth.

$$D_t\{\|D_t^rD_\alpha^s v\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}v\|_2^2\}+\frac{5\nu}{4}\{\|D_t^r\nabla D_\alpha^s v\|_2^2+\|D_t^r\tilde{\Delta}D_\alpha^{s-1}v\|_2^2\}$$
$$\le C_{r,s}\{\|D_t^rD_\alpha^s v\|_2^{2\frac{4r+2s+1}{4r+2s-1}}+\|D_t^r\nabla D_\alpha^{s-1}v\|_2^{2\frac{4r+2s+1}{4r+2s-1}}$$
$$+\|\nabla v\|_2^{8r+4s+2}+\sum_{h,j=0}^{r,s}{}'\|D_t^hD_\alpha^j v\|_2^{2\frac{4r+2s+1}{4h+2j-1}}$$
$$+\sum_{h,j=0}^{r,s}{}'\|D_t^h\nabla D_\alpha^j v\|_2^{2\frac{4r+2s+1}{4h+2j+1}}+\nu\sum_{j=0}^{s-2}\|D_t^r\tilde{\Delta}D_\alpha^j v\|_2^2$$

$$
\begin{aligned}
&+ \sum_{h,j=0}^{r,s-1}{}' \|D_t^h \nabla D_\alpha^j v\|_2^{2\frac{4(r-h)+2(s-j-1)}{4h+2j-1}} \|D_t^h \tilde{\Delta} D_\alpha^j v\|_2^2\} \\
&+ C'_{r,s}\{ \sum_{h,j=0}^{r,s} \|D_t^h D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j-1}} + \sum_{h,j=0}^{r,s-1} \|D_t^h D_x D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j+1}} \\
&+ \sum_{h,j=0}^{r,s-1} \|D_t^h D_x^2 D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j+3}} + \sum_{j=0}^{s} \|D_t^{r+1} D_\alpha^j w\|_{6/5}^2 \\
&+ \sum_{j=0}^{s-1} \|D_t^{r+1} D_x D_\alpha^j w\|_{6/5}^2 + \|D_t^{r+1} D_\alpha^{s-1} w\|_2^2 \\
&+ \|D_t^r D_\alpha^s B\|_{6/5}^2 + \|D_t^r D_\alpha^{s-1} B\|_2^2 \\
&+ \sum_{h,j=0}^{r,s-1} (\|D_t^h D_\alpha^j B\|_2^2 + \|D_t^h \nabla D_\alpha^j B\|_{6/5}^2) + 1\}
\end{aligned}
\tag{11.1}
$$

The presence of the unit term on the right side enables us to reduce the total number of terms by bounding lower powers of any norm by means of a constant plus a higher power term. Thus only the highest powers, having formally the singularity index $4r+2s+1$ need to be retained, except in the case of the Stokes operator terms. As in [4, § 13], the constant $C_{r,s}$ on the right side of (11.1) will be henceforth fixed in value.

We now define, for $r = 0, 1, 2, \cdots$ and $s = 0, 1, 2, \cdots$, as in [4, § 13].

$$
\begin{aligned}
F_{r,s}(t) = {} & \|D_t^r D_\alpha^s v\|_2^2 + \|D_t^r \nabla D_\alpha^{s-1} v\|_2^2 + \frac{5}{4} \sum_{h=0}^{r} \sum_{j=0}^{s+1}{}'' F_{h,j}(t)^{\frac{4r+2s-1}{4h+2j-1}} \\
& + \frac{5}{2} C_{r,s} \sum_{h,j=0}^{r,s-1} F_{h,j}(t) + 1
\end{aligned}
\tag{11.2}
$$

$$
\begin{aligned}
G_{r,s}(t) = {} & \nu(\|D_t^r \nabla D_\alpha^s v\|_2^2 + \|D_t^r \tilde{\Delta} D_\alpha^{s-1} v\|_2^2) \\
& + \sum_{h=0}^{r} \sum_{j=0}^{s+1}{}'' \frac{4r+2s-1}{4h+2j-1} F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}} G_{h,j}(t) \\
& + C_{r,s} \sum_{h=0}^{r} \sum_{j=0}^{s-1} G_{h,j}(t)
\end{aligned}
\tag{11.3}
$$

Here the double prime on the summation sign denotes omission of the terms $h = r, j = s$ and $j = s+1$. Terms in which $4r+2s-1 \leq 0$ or $4h+2j-1 \leq 0$ are also omitted. We also define

$$
\begin{aligned}
N_{r,s}(t) = {} & M_{r,s}(t) + \frac{5}{4} \sum_{h=0}^{r} \sum_{j=0}^{s+1}{}'' \frac{4r+2s-1}{4h+2j-1} F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}} N_{h,j}(t) \\
& + C_{r,s} \sum_{h=0}^{r} \sum_{j=0}^{s-1} N_{h,j}(t)
\end{aligned}
\tag{11.4}
$$

where $M_{r,s}(t)$ denotes the nonhomogeneous term

$$\begin{aligned}
M_{r,s}(t) = C'_{r,s} \sum_{h,j=0}^{r,s} & \|D_t^h D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j-1}} \\
&+ \sum_{h,j=0}^{r,s-1} \|D_t^h \nabla D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j+1}} \\
&+ \sum_{h,j=0}^{r,s-1} \|D_t^h D_x D_\alpha^j w\|_2^{2\frac{4r+2s+1}{4h+2j+3}} \\
&+ \sum_{h,j=0}^{r,s-1} \left(\|D_t^h D_\alpha^j B\|_2^2 + \|D_t^h \nabla D_\alpha^j B\|_{6/5}^2\right) \\
&+ \sum_{j=0}^{s} \|D_t^{r+1} D_\alpha^j w\|_{6/5}^2 + \sum_{j=0}^{s-1} \|D_t^{r+1} D_x D_\beta^j w\|_{6/5} \\
&+ \|D_t^{r+1} D_\alpha^{s-1} w\|_2^2 + \|D_t^r D_\alpha^s B\|_{6/5}^2 + \|D_t^r D_\alpha^{s-1} B\|_2^2 + 1.
\end{aligned} \tag{11.5}$$

As in (12,4) of [4] we note the relations

$$\begin{aligned}
\|D_t^h D_\alpha^j v\|_2^{2\frac{4r+2s+1}{4h+2j-1}} &\le F_{h,j}(t)^{\frac{4r+2s+1}{4h+2j-1}} \\
\|D_t^h \nabla D_\alpha^j v\|_2^{2\frac{4r+2s+1}{4h+2j+1}} &\le F_{h,j+1}(t)^{\frac{4r+2s+1}{4h+2j+1}} \\
\nu \|D_t^h \tilde{\Delta} D_\alpha^j v\|_2^2 &\le G_{h,j+1}(t) \\
\nu \|D_t^h \nabla D_\alpha^j v\|_2^{2\frac{4(r-h)+2(s-j-1)}{4h+2j+1}} \|D_t^h \tilde{\Delta} D_\alpha^j v\|_2^2 &\le F_{h,j+1}(t)^{\frac{4(r-h)+2(s-j-1)}{4h+2j+1}} G_{h,j+1}(t).
\end{aligned} \tag{11.6}$$

As in [4, § 13] we can show that (11.1) can be expressed in the standard form

$$\frac{d}{dt} F_{r,s}(t) + G_{r,s}(t) \le K_{r,s} F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}} + N_{r,s}(t) \tag{11.7}$$

with the aid of previous inequalities of the same type. To do this we proceed by induction on s, assuming $F_{r,s}(t) \in L^{\frac{1}{4r+2s-1}}(0,T)$ and showing by Lemma 1 that $G_{r,s}(t) \in L^{\frac{1}{4r+2s-1}}(0,T)$. Thus we assume (11.7) for $r < r_1$ and $s \le s_1 + 1$, and for $r = r_1$ and $s < s_1$ and deduce it for $r = r_1$, $s = s_1$. To do this change r, s in (11.7) to h, j, then multiply by $\frac{5}{4}\frac{4r+2s-1}{4h+2j-1} F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}}$ and add for $h = 0, 1, \cdots r; j = 0, 1, \cdots s+1$ but omitting $h = r, j = s$ and $s+1$, to (11.1). Also add on (11.7) with s replaced by j and multiplied by $\frac{5}{2} C_{r,s}$, for $h = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$ but omitting $h = r$, $\ell = s$.

This yields, with the aid of (11.6),

$$
\begin{aligned}
D_tF_{r,s}(t) &+ \frac{5}{4}G_{r,s}(t) + C_{r,s}\sum_{h,j=0}^{r,s}{}' G_{h,j}(t) \\
&\leq C_{r,s}\{\,\|D_t^rD_\alpha^s v\|_2^{2\frac{4r+2s+1}{4r+2s-1}} + \|D_t^r\nabla D_\alpha^{s-1}v\|_2^{2\frac{4r+2s+1}{4r+2s-1}} \\
&+ 2\sum_{h,j=0}^{r,s+1} F_{h,j}(t)^{\frac{4r+2s+1}{4h+2j-1}} \\
&+ \sum_{h,j=0}^{r,s}{}' F_{h,j+1}(t)^{\frac{4r+2s+1}{4h+2j+1}} + \nu\sum_{h,j=0}^{r,s-1}\|D_t^h\tilde{\Delta}D_\alpha^j v\|_2^2\} \\
&+ \frac{1}{8}\sum_{h,j=0}^{r,s-1}{}' F_{h,j+1}(t)^{\frac{4(r-h)+2(s-j-1)}{4h+2j+1}} G_{h,j+1}(t) \qquad (11.8)\\
&+ \frac{5}{4}\sum_{h,\ell=0}^{r,s+1}{}'' \frac{4r+2s-1}{4h+2\ell-1}F_{h,\ell}(t)^{\frac{4r+2s+1}{4h+2\ell-1}} + \frac{5}{2}C_{r,s}\sum_{h,\ell=0}^{r,s}{}' C'_{h,\ell}F_{h,\ell}(t)^{\frac{4h+2\ell+1}{4h+2\ell-1}} \\
&+ M_{r,s}(t) + \frac{5}{4}\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}'' \frac{4r+2s-1}{4h+2j-1}F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}}N_{h,j}(t) \\
&+ C_{r,s}\sum_{h=0}^{r}\sum_{j=0}^{s-1}N_{h,j}(t).
\end{aligned}
$$

Now the Stokes operator terms on the right are majorized by the sum of $G_{h,j}(t)$ function terms on the left and so can be canceled off against them in the inequality. Also the sum on the right side of (11.8) containing the products of F and G terms is less, term by term, than the corresponding sum in (11.3). Hence this term will cancel against $\frac{1}{4}G_{r,s}(t)$ on the left hand side of (11.8) without changing the sign of that inequality. Thus we now obtain, as in (13.8) of [4] and with the aid of (13.5)

$$
\begin{aligned}
D_tF_{r,s}(t) + G_{r,s}(t) &\leq C\{\,F_{r,s}(t)^{\frac{4r+2s+1}{4h+2s-1}} + 2\sum_{h,j=0}^{r,s-1} F_{h,j}(t)^{\frac{4r+2s+1}{4h+2s-1}} \\
&+ \frac{9}{4}\sum_{h,j=0}^{r,s}{}' F_{h,j+1}(t)^{\frac{4r+2s+1}{4h+2(j+1)-1}} \qquad (11.9)\\
&+ \sum_{h=0}^{r}\sum_{j=0}^{s-1} F_{h,j}(t)^{\frac{4r+2s+1}{4r+2j-1}}\} + N_{r,s}(t) \\
&\leq CF_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}} + N_{r,s}(t)
\end{aligned}
$$

and this establishes (11.7) for the given values of r, s.

Next we show that the condition $N_{r,s}(t) = F_{r,s}(t)^a\,Q_1(t)$ holds, where $Q_1(t) \in L^1(0,T)$ and $a = \frac{4r+2s}{4r+2s-1}$. We assume the hypotheses of the Theorem for a given ρ, and let r, s be non-negative integers such that $\rho = 4r + 2s + 1$. Then it is a straightforward matter to verify that $M_{r,s}(t) \in L^1(0,T)$.

By the induction hypothesis for earlier stages,

$$
N_{h,j}(t) = F_{h,j}(t)^{1+\frac{1}{4h+2j-1}}Q_{h,j}(t), \qquad (11.10)
$$

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where $Q_{h,j}(t) \in L^1(0,T)$ and $h = 0,1,\dots,r; j = 0,1,2,\dots,s+1$ with $h = r,\ j = s,\ s+1$ omitted. Now, by (11.4)

$$
\begin{aligned}
N_{r,s}(t) &= Q_{r,s}(t) + \frac{5}{4}\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}'' F_{h,j}(t)^{\frac{4(r-h)+2(s-j)+4h+2j}{4h+2j-1}} Q_{h,j}(t) \\
&\quad + C_{r,s}\sum_{h=0}^{r}\sum_{j=0}^{s-1} F_{h,j}(t)^{\frac{4r+2s}{4h+2j-1}} Q_{h,j}(t) \\
&\le Q_{r,s}(t) + \frac{5}{4}\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}'' F_{r,s}(t)^{\frac{4h+2j-1}{4r+2s-1}\cdot\frac{4r+2s}{4h+2j-1}} Q_{h,j}(t) \\
&\quad + C_{r,s}\sum_{h=0}^{r}\sum_{j=0}^{s-1} F_{r,s}(t)^{\frac{4h+2j-1}{4r+2s-1}\cdot\frac{4r+2s}{4h+2j-1}} Q_{h,j}(t) \\
&\le Q_{r,s}(t) + \frac{5}{4}\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}'' F_{r,s}(t)^{\frac{4r+2s}{4r+2s-1}} Q_{h,j}(t) \\
&\quad + C_{r,s}\sum_{h,j=0}^{r,s-1} F_{r,s}(t)^{\frac{4r+2s}{4r+2s-1}} Q_{h,j}(t) \qquad (11.11)\\
&\le CQ_1(t)F_{r,s}(t)^{\frac{4r+2s}{4r+2s-1}}
\end{aligned}
$$

This shows the condition of the lemma is satisfied, and the lemma therefore applies to (11.9). Hence $G_{r,s}(t) \in L^{\frac{1}{4r+2s+1}}(0,T)$ and this now yields $\|D_t^r\nabla D_\alpha^s v\|_2$ and $\|D_t^r\tilde{\Delta}D_\alpha^{s-1}v\|_2 \in L^{\frac{2}{4r+2s+1}}(0,T)$ as required.

Since this induction has already been shown for $s = 0, 1$ and all r in § 8, the induction over s will run separately for each value of r, for all positive integers s, and subsequently for $r = 0, 1, 2, \cdots$ in succession for all applicable values of r and s.

Because $\|D_t^rD_\alpha^s v\|_2 < C\|D_t^r\nabla D_\alpha^{s-1}v\|_2$, it follows that this norm will have been estimated at the preceding stage in s so that $\|D_t^rD_\alpha^s v\|_2 \in L^{\frac{2}{4r+2s-1}}(0,T)$. We have therefore estimated $\|D_t^rD_\alpha^s v\|_2$, $\|D_t^r\nabla D_\alpha^s v\|_2$ and $\|D_t^r\tilde{\Delta}D_\alpha^s v\|_2$ which latter has been shown, as in § 14 of [4], to be equivalent to estimating $\|D_t^rD_iD_jD_\alpha^s v\|_2$. Thus all tangential derivative norms, together with their first and second space derivative norms, have been estimated as required, for v. As these results hold for w by hypothesis, they now hold for $u = v + w$.

Likewise, by (3.12) we have

$$
\begin{aligned}
\max|D_t^rD_\alpha^s u| &\equiv \|D_t^rD_\alpha^s u\|_\infty \\
&\le C(\|D_t^r\nabla D_\alpha^s u\|_2^{\frac{1}{2}}\|D_t^r\tilde{\Delta}D_\alpha^s u\|_2^{\frac{1}{2}} \\
&\quad + \|D_t^rD_\alpha^s u\|_2^{\frac{1}{2}}\|D_t^r\tilde{\Delta}D_\alpha^s u\|_2^{\frac{1}{2}} \qquad (11.12)\\
&\quad + \|D_t^rD_\alpha^s u\|_2) \in L^{\frac{1}{2r+s+1}}(0,T)
\end{aligned}
$$

with corresponding results for v and w separately. As in (14.2) of [4] only derivatives with respect to time and tangential variables are present on the left side in (11.12), but no normal derivatives as yet.

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12. **Normal Derivatives of Higher Order.** Remaining to be estimated are the $L^2(\Omega)$ norms of the partial derivatives containing three or more normal derivations.

By (4.7) the desired conditions hold for w at every stage of calculation since $\rho \geq 7$ for every step involving a third order space derivative. Repeating the calculations of [4, § 15] we find the third normal derivative of u is bounded as desired, and the same result then automatically follows for $v = u - w$. As in [4, § 15] the main calculation can be done by induction on the normal order, the only new circumstance being the presence of body force terms in the vorticity equation. Since the Laplacian and the body force term appear as separate linear terms in the momentum equations, the normal derivative of u under study after any number of differentiations will always have exactly two orders of space derivatives more than the body force terms.

Thus if h, k, are integers such that $2h + j = \frac{1}{2}(\rho - 3)$, then $4h + 2j + 3 = \rho$ so that by (4.9) $\|D_t^h D_x^j B\|_2 \in L^2(0, T)$. By the estimates of [4, § 15] and the remark above, it follows that $\|D_t^h D_x^{j+2} u\|_2 \in L^{\frac{2}{4h+2j-1}}(0, T)$ and this is the highest derivative norm guaranteed by the hypotheses of the Theorem, since $4h + 2(j + 2) = \rho + 1$ so that $2h + (j + 2) = \frac{1}{2}(\rho + 1)$, as stated in (4.7). The result for the maximum norm then follows in the usual way. This completes the proof of part (b) of the Theorem.

Finally, the proof of part (c) of the Theorem is immediate when (b) is proved for all odd $\rho > 0$. Continuity of any given derivative of w or B is a consequence, for example, of inclusion in $L^{6/5}\big(\Omega \times (0, T)\big)$ of all partial derivatives of order four higher.

13. **Unbounded domains.** The analytical operations involved in our application of the basic Integrability Lemma are the differentiation with respect to t for the space integral defining $F(t)$, and the integration by parts of a term containing the Laplacian. For a bounded domain Ω, our smoothness hypothesis on the solution $u(x, t)$ makes the justification of these operations straightforward and elementary. When the domain Ω is unbounded, however, further considerations of uniform convergence and integrability appear. Here we describe a justification of these operations based on the unified integration approach of McShane [11]. The necessary conditions for infinite domains require that the value of ρ in Theorem 1 be increased by 2 relative to bounded domains.

For brevity we treat only one stage of the induction, namely the derivation of (6.14), leaving to the reader the adaptation of the method to later stages. We accordingly choose $\rho = 5$, and note that the hypotheses of Theorems 1 and 2, and Corollaries 1 and 2, of [9, Chap. 4, § 2] will be satisfied if we consider a time interval $E_k : T_{k-1} < t < T_k$ wherein $u(x, t)$ is regular and a suitable initial instant $t_I \in E_k$ is chosen (such t_I are dense in E_k).

We now consider the nonlinear terms such as $v_k v_{i,k}$ and any other quadratic terms as non-homogeneous terms placed on the right hand side of the Navier-Stokes equations. In applying the above results for compact subsets of E_k we may assume that the prior stage (6.8) of the induction has been established and $\|v\|_2$ and $\|\nabla v\|_2$ are therefore bounded and continuous functions of t therein.

By the uniqueness Theorem 2 of [9, p. 89] the solution described there must coincide with $u = v+w$ of the present paper. It now follows from Theorem 1 and from Corollaries 1 and 2 of [9, Chap. 4, § 2] in succession that Δv, $\tilde{\Delta} v$, $D_x v$, v_t, ∇v_t and ∇p are all elements of $L^2(\Omega)$ with norms depending continuously on t in E_k, hence locally uniformly convergent there [9, p. 529, Theorem 2].

To justify the operations in (6.13) we may now note that differentiation with respect to t as in

$$\frac{d}{dt}F_3(t) = \frac{d}{dt}\|\nabla v\|_2^2 = 2\int_\Omega \nabla v \cdot \nabla v_t \, dx$$

is justified by the absolute convergence, locally uniform with respect to t by [8, p. 529, Theorem 2], of the integral on the right side [11, p. 137 and p. 259]. Likewise the integration by parts in

$$\int_\Omega \nabla v \cdot \nabla v_t \, dx = -\int_\Omega \Delta v \cdot v_t \, dx = -\int_\Omega \tilde{\Delta} v \cdot v_t \, dx$$

can be justified term by term using absolute convergence in the one-dimensional integration by parts Theorem 9.4 of [11, p. 126] and the results on iterated integration of [11, Chap. 4, § 4, pp. 261–273 or § 7, pp. 300–304]. These results now justify the first term on the left of (6.13). As the remaining terms are obtained directly, our demonstration is complete.

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