HILBERT MODULES REVISITED: ORTHONORMAL BASES AND HILBERT-SCHMIDT OPERATORS

by M. CABRERA, J. MARTÍNEZ and A. RODRÍGUEZ

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Introduction. The concept of a Hilbert module (over an H*-algebra) arises as a generalization of that of a complex Hilbert space when the complex field is replaced by an (associative) H*-algebra with zero annihilator. P. P. Saworotnow [**13**] introduced Hilbert modules and extended to its context some classical theorems from the theory of Hilbert spaces, J. F. Smith [**17**] gave a complete structure theory for Hilbert modules, and G. R. Giellis [**9**] obtained a nice characteristization of Hilbert modules.

Nevertheless, as far as we know, the fundamental concept of an orthonormal basis for Hilbert spaces has not been extended until now to Hilbert modules, a fact which has obstructed the development of a theory of operators of Hilbert-Schmidt type on a Hilbert module. Such a theory, which actually we develop in Section 3 of this paper, has become a crucial tool in the study of structurable H*-algebras begun by the authors [**6**]. See [**4**]. The reason is that a large class of structurable H*-algebras can be built from particular types of Hilbert modules, and structurable H*-algebras obtained in this way are of capital importance in the general theory, because every topologically simple structurable H*-algebra that is not of this form is either associative, Jordan, or finite-dimensional (see [**6**, Theorem 1.3] and [**4**, 1.11.2]). Then a large part of the proof of the main results in [**4**] (asserting that all topologically simple Lie H*-algebras can be constructed from topologically simple structurable H*-algebras by means of an infinite-dimensional extension of the finite-dimensional Allison-Kantor-Koecher-Tits construction [**1**]) relies on the theory of Hilbert-Schmidt type operators on Hilbert modules developed in this paper. The reader is referred to [**5**] for more information about the relation between Hilbert modules and structurable H*-algebras.

The key idea in our work is the appropriate concept we give of an orthonormal system and an orthonormal basis in a Hilbert module (Definition 1.2), which is in agreement with the familiar one for Hilbert spaces thanks to the simple observation that the unit of $\mathbb{C}$ is the only minimal idempotent in $\mathbb{C}$. Our approach (which is inspired by [**12**], where Hilbert modules over finite-dimensional C*-algebras were studied) follows then with minor variants the classical arguments in the case of Hilbert spaces. Thus we prove (Theorem 1.6) that an orthonormal system in a Hilbert module is an orthonormal basis if and only if either "Parseval's identity" or "Fourier expansion" are verified. We prove (Corollary 1.10) the existence of orthonormal bases for a given Hilbert module and also that all the orthonormal bases have the same cardinal (Proposition 1.11). The existence of orthonormal bases allows us to provide an easy proof of the structure theorems by J. F. Smith (Section 2). Finally, as we have already commented, Hilbert-Schmidt type operators on Hilbert modules are introduced and studied (Section 3).

1. Orthonormal bases in Hilbert modules. We recall that an *H*-algebra* is a complex algebra $\mathscr{E}$ with a conjugate-linear mapping $e \to e^*$ from $\mathscr{E}$ to $\mathscr{E}$, called the *H*-algebra involution of* $\mathscr{E}$, and a complete inner product $(.|.)$ satisfying $e^{**} = e$,

$(ef)^* = f^*e^*$ and $(ef \mid g) = (e \mid gf^*) = (f \mid e^*g)$ for all e, f, g in $\mathscr{E}$. Note that the product of any H*-algebra $\mathscr{E}$ is continuous [**7**, Proposition 2 (i)], so (by multiplying the inner product by a suitable positive number if necessary) $\mathscr{E}$ is a (complete) *normed* algebra in the usual sense of the word, a fact that will be assumed in what follows.

Given a (complex associative) H*-algebra $\mathscr{E}$ with zero annihilator, the *trace-class* $\tau(\mathscr{E})$ of $\mathscr{E}$ is defined as the set $\{ef : e, f \in \mathscr{E}\}$, and it is known ([**15**] and [**14**]) that $\tau(\mathscr{E})$ is an ideal of $\mathscr{E}$ which is a Banach *-algebra under a suitable norm $\tau(.)$ related to the given norm on $\mathscr{E}$ by $\|e\|^2 = \tau(e^*e)$ for all e in $\mathscr{E}$. There exists a canonical continuous commutative linear form on $\tau(\mathscr{E})$ (called the *trace* of $\mathscr{E}$ and denoted by tr) related with the inner product of $\mathscr{E}$ by $\mathrm{tr}(ef) = (e \mid f^*)$ for all e, f in $\mathscr{E}$. The reader is referred to [**13**], [**14**], and [**15**] for these and other interesting results about the trace class of an H*-algebra with zero annihilator.

Following [**3**, Definition 9.11], a left module over an associative complex algebra $\mathscr{E}$ is a complex vector space W together with a bilinear mapping $(e, w) \to e \circ w$ from $\mathscr{E} \times W$ to W satisfying $e \circ (f \circ w) = (ef) \circ w$ for all e, f in $\mathscr{E}$ and w in W. The original definition of Hilbert modules by P. P. Saworotnow [**13**, Definition 1], with some remarks in [**13**] and [**17**], can be formulated as follows.

Definition 1.1. A *Hilbert $\mathscr{E}$-module* is a left module W over an H*-algebra $\mathscr{E}$ with zero annihilator, provided with a mapping

$$[\ \mid\ ] : W \times W \to \tau(\mathscr{E}) \text{ (the } \tau(\mathscr{E})\text{-valued product)}$$

satisfying, for all w_1, w_1', w_2 in W, e in $\mathscr{E}$ and λ in $\mathbb{C}$, the following properties.

(i) $[\lambda w_1 \mid w_2] = \lambda [w_1 \mid w_2]$.
(ii) $[w_1 + w_1' \mid w_2] = [w_1 \mid w_2] + [w_1' \mid w_2]$.
(iii) $[e \circ w_1 \mid w_2] = e[w_1 \mid w_2]$.
(iv) $[w_1 \mid w_2]^* = [w_2 \mid w_1]$.
(v) For each nonzero w in W there is a nonzero f in $\mathscr{E}$ such that $[w \mid w] = f^*f$.
(vi) W is a Hilbert space under the inner product $(w_1 \mid w_2) := \mathrm{tr}([w_1 \mid w_2])$.

Remark 1. As an immediate consequence of the above definition and the previous comments we obtain

$$\|w\|^2 = \mathrm{tr}([w \mid w]) = \tau([w \mid w]) \text{ for all } w \text{ in } W.$$

Also from [**15**, Corollary 3] and [**13**, Theorem 2] we have

$$\|[w_1 \mid w_2]\| \le \tau([w_1 \mid w_2]) \le \|w_1\| \, \|w_2\|, \text{ for all } w_1, w_2 \text{ in } W.$$

Finally $\|e \circ w\| \le \|e\| \, \|w\|$ for all e in $\mathscr{E}$ and w in W [**13**, Lemma 1]. In what follows we shall use these facts without further comment.

Definition 1.2. Let W be a Hilbert $\mathscr{E}$-module. An element u in W is said to be a *basic element* if $e := [u \mid u]$ is a (selfadjoint) minimal idempotent in $\mathscr{E}$ (that is, e is a nonzero idempotent in $\mathscr{E}$ such that $e\mathscr{E}e = \mathbb{C}e$). We define an *orthonormal system* in W as a family of basic elements $\{u_\lambda\}_{\lambda \in \Lambda}$ satisfying $[u_\lambda \mid u_\mu] = 0$ for all λ, μ in Λ with $\lambda \ne \mu$. We say that $\{u_\lambda\}_{\lambda \in \Lambda}$ is an *orthonormal basis* in W if it is an orthonormal system generating a dense submodule of W. If N is a subset of a Hilbert module W, we define

$$N^{[\perp]} := \{w \in W : [w \mid u] = 0 \text{ for every } u \in N\}.$$

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Clearly $N^{[\perp]}$ is a closed submodule of W. It is known [**13**, Lemma 3] that, if N is a submodule of W, then $N^{[\perp]} = N^{\perp}$, the orthogonal complement of N in W regarded as a Hilbert space.

LEMMA 1.3. *Let W be a Hilbert $\mathscr{E}$-module and let u in W be such that $e := [u \mid u]$ is an idempotent in $\mathscr{E}$. Then*

$$[w \mid u] = [w \mid u]e \text{ for all } w \text{ in } W.$$

Proof. Clearly $M := \{w \in W : [w \mid u] = [w \mid u]e\}$ is a closed submodule of W and u lies in M, so that, if N denotes the closed submodule generated by u, then $N \subseteq M$. On the other hand, if w is in $N^{[\perp]}$, then w is also in $\{u\}^{[\perp]}$, so $0 = [w \mid u] = [w \mid u]e$, and so w lies in M. Now we have $W = N + N^{[\perp]} \subseteq M$. Therefore $M = W$. ■

COROLLARY 1.4. *If $\{u_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal system in a Hilbert module W and if J is a finite subset of Λ, then*

$$\left[w - \sum_{\lambda \in J} [w \mid u_\lambda] \circ u_\lambda \mid w - \sum_{\lambda \in J} [w \mid u_\lambda] \circ u_\lambda\right] = [w \mid w] - \sum_{\lambda \in J} [w \mid u_\lambda][w \mid u_\lambda]^*$$

for all w in W.

Proof. This is clear from axioms (iii) and (iv) in Definition 1.1, using the above lemma. ■

In the next proposition we study in detail the submodule generated by a basic element in a Hilbert module. We note that every H*-algebra $\mathscr{E}$ with zero annihilator, regarded as a left module over $\mathscr{E}$, has a natural structure of Hilbert $\mathscr{E}$-module by defining the $\tau(\mathscr{E})$-valued product as $[e \mid f] := ef^*$, for all e, f in $\mathscr{E}$. This structure is inherited by the closed left ideals of $\mathscr{E}$. Module isomorphisms between Hilbert $\mathscr{E}$-modules which preserve the $\tau(\mathscr{E})$-valued product are called *Hilbert module isomorphisms.*

PROPOSITION 1.5. *Let W be a Hilbert $\mathscr{E}$-module, let u be a basic element in W, and let M denote the submodule of W generated by u. Then M is closed in W, the mapping $w \to [w \mid u] \circ u$ is the orthogonal projection from W onto M, and the mapping $m \to [m \mid u]$ is a Hilbert module isomorphism from M onto $\mathscr{E}e$, where $e := [u \mid u]$. As a consequence, M is an irreducible $\mathscr{E}$-module.*

Proof. For any w in W, Lemma 1.3 gives $[w - [w \mid u] \circ u \mid u] = 0$, so $w - [w \mid u] \circ u$ lies in $\bar{M}^{[\perp]}$. This fact, together with the identity $w = [w \mid u] \circ u + (w - [w \mid u] \circ u)$, shows that $w \to [w \mid u] \circ u$ is the orthogonal projection from W onto $\bar{M}$. As a consequence, we have $\bar{M} \subseteq \mathscr{E} \circ u \subseteq M$, and so M is closed in W. On the other hand, the mapping $w \to f(w) := [w \mid u]$ from W into $\mathscr{E}e$ (see again Lemma 1.3) is a non-zero module homomorphism such that $\operatorname{Ker}(f) = \{u\}^{[\perp]} = M^{[\perp]}$. Therefore $f|_M : M \to \mathscr{E}e$ is a one to one module homomorphism. Moreover, by [**3**, Proposition 30.6], $\mathscr{E}e$ is a minimal left ideal of $\mathscr{E}$, and hence an irreducible left $\mathscr{E}$-module. Therefore actually $f|_M$ maps M onto $\mathscr{E}e$. Finally, from the equality $[m \mid u] \circ u = m$ for m in M, it follows that $f|_M$ is a Hilbert module isomorphism. ■

The following theorem provides a very useful characterization of orthonormal bases in Hilbert modules.

THEOREM 1.6. *Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal system in a Hilbert module W. Then the following statements are equivalent.*

(i) *For all w_1, w_2 in W the family $\{[w_1 \mid u_\lambda][w_2 \mid u_\lambda]^*\}_{\lambda\in\Lambda}$ is summable in the Banach space $(\tau(\mathscr{E}), \tau(.))$, with sum equal to $[w_1 \mid w_2]$.*

(ii) *For every w in W, we have $[w \mid w] = \sum_{\lambda\in\Lambda} [w \mid u_\lambda][w \mid u_\lambda]^*$ (Parseval's identity) in the Banach space $(\tau(\mathscr{E}), \tau(.))$.*

(iii) *For every w in W, we have $w = \sum_{\lambda\in\Lambda} [w \mid u_\lambda] \circ u_\lambda$ (Fourier expansion).*

(iv) *$\{u_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis in W.*

Proof. Clearly (i) implies (ii). Denote by $\mathscr{F}$ the family of all finite subsets of Λ. By Corollary 1.4,

$$\left\| w - \sum_{\lambda\in J} [w \mid u_\lambda] \circ u_\lambda \right\|^2 = \tau\left(\left[w - \sum_{\lambda\in J} [w \mid u_\lambda] \circ u_\lambda \mid w - \sum_{\lambda\in J} [w \mid u_\lambda] \circ u_\lambda \right]\right)$$
$$= \tau\left([w \mid w] - \sum_{\lambda\in J} [w \mid u_\lambda][w \mid u_\lambda]^* \right)$$

for each J in $\mathscr{F}$. Therefore (ii) is true if and only if (iii) is true. If $w = \sum_{\lambda\in\Lambda} [w \mid u_\lambda] \circ u_\lambda$, then w is a point limit of elements in the $\mathscr{E}$-submodule generated by $\{u_\lambda : \lambda \in \Lambda\}$. Therefore (iii) implies (iv). Now, let us assume that condition (iv) holds. Since by Proposition 1.5 the submodule generated by $\{u_\lambda : \lambda \in \Lambda\}$ is $\left\{ \sum_{\lambda\in J} e_\lambda \circ u_\lambda : J \in \mathscr{F}, e_\lambda \in \mathscr{E} \right\}$, we have that, for every w in W and $\varepsilon > 0$, there exist J_0 in $\mathscr{F}$ and e_λ in $\mathscr{E}(\lambda \in J_0)$ such that $\left\| w - \sum_{\lambda\in J_0} e_\lambda \circ u_\lambda \right\| < \varepsilon$. Then, for each J in $\mathscr{F}$ with $J_0 \subseteq J$, as in the proof of Proposition 1.5, we have that $\sum_{\lambda\in J} [w \mid u_\lambda] \circ u_\lambda$ is the orthogonal projection from w onto the submodule generated by $\{u_\lambda : \lambda \in J\}$ which is closed and contains $\sum_{\lambda\in J_0} e_\lambda \circ u_\lambda$, so by the orthogonal projection theorem for W regarded as a Hilbert space

$$\left\| w - \sum_{\lambda\in J} [w \mid u_\lambda] \circ u_\lambda \right\| \le \left\| w - \sum_{\lambda\in J_0} e_\lambda \circ u_\lambda \right\| < \varepsilon.$$

Thus condition (iii) is verified. Finally, (i) follows from (ii) by the polarization law. ■

REMARK 2. Parseval's identity leads to the equality $\sum_{\lambda\in\Lambda} \|[w \mid u_\lambda]\|^2 = \|w\|^2$. Indeed,

$$\sum_{\lambda\in\Lambda} \|[w \mid u_\lambda]\|^2 = \sum_{\lambda\in\Lambda} \operatorname{tr}([w \mid u_\lambda][w \mid u_\lambda]^*) = \operatorname{tr}\left(\sum_{\lambda\in\Lambda} [w \mid u_\lambda][w \mid u_\lambda]^* \right) = \operatorname{tr}([w \mid w]) = \|w\|^2.$$

Our next objective is to prove the existence of orthonormal bases.

PROPOSITION 1.7. *Let W be a nonzero submodule of a Hilbert $\mathscr{E}$-module. Then there exist basic elements in W.*

Proof. First we note that, if w is a nonzero element in W, then there exists a selfadjoint minimal idempotent e in $\mathscr{E}$ such that $e \circ w \neq 0$. Otherwise, $e \circ w = 0$ for every selfadjoint minimal idempotent e in $\mathscr{E}$ and so $\mathscr{E}e \circ w = 0$; hence $\mathscr{E} \circ w = 0$ (see [**3**,

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Theorems 34.1 and 34.16]). Now axiom (iii) in Definition 1.1 gives $[w \mid w] = 0$, which contradicts axiom (v) in the same definition. Let $w \in W \setminus \{0\}$ and let e be a selfadjoint minimal idempotent in $\mathscr{E}$ satisfying $e \circ w \neq 0$. Then $0 \neq [e \circ w \mid e \circ w] = e[w \mid w]e = \alpha e$ for some positive number α. Finally $u := \alpha^{-1/2} e \circ w$ is a basic element which lies in W. ■

From the above proposition and Proposition 1.5 we obtain the following result.

Corollary 1.8. *Every irreducible submodule of a Hilbert module is generated by a basic element; hence it is closed.*

Theorem 1.9. *Let S be a subset of a Hilbert module W. Then the following assertions are equivalent:*

(i) *S is an orthonormal basis,*

(ii) *S is a maximal orthonormal system.*

Proof. Clearly (i) implies (ii). Let us assume that condition (ii) holds, and let M be the closed submodule generated by S. If $M \neq W$, then $0 \neq M^{[\perp]}$. Therefore, by Proposition 1.7, there exists a basic element u in $M^{[\perp]}$. Obviously $S \cup \{u\}$ is an orthonormal system strictly containing S, which is a contradiction. ■

Proposition 1.7, Theorem 1.9, and Zorn's lemma lead to the following corollary.

Corollary 1.10. *Every nonzero Hilbert module has an orthonormal basis.*

We conclude this section by showing that all the orthonormal bases in a Hilbert module W have the same cardinal number, which will be called the *hilbertian dimension of W over $\mathscr{E}$.*

Proposition 1.11. *Let $\{u_\lambda\}_{\lambda \in \Lambda}$ and $\{v_\mu\}_{\mu \in M}$ be two orthonormal bases in a Hilbert $\mathscr{E}$-module W. Then* $\operatorname{card}(\Lambda) = \operatorname{card}(M)$.

Proof. Assume that $\operatorname{card}(\Lambda)$ and $\operatorname{card}(M)$ are infinite. For fixed λ in Λ, Parseval's identity gives $[u_\lambda \mid u_\lambda] = \sum_{\mu \in M} [u_\lambda \mid v_\mu][u_\lambda \mid v_\mu]^*$. Hence $M(\lambda) := \{u \in M : [u_\lambda \mid v_\mu] \neq 0\}$ is a non empty countable subset of M ([**8**, Corollary 9.9, p. 220]). Since clearly $M = \bigcup_{\lambda \in \Lambda} M(\lambda)$, we have

$$\operatorname{card}(M) \leq \sum_{\lambda \in \Lambda} \operatorname{card}(M(\lambda)) \leq \chi_0 \operatorname{card}(\Lambda) = \operatorname{card}(\Lambda).$$

By symmetry we actually have $\operatorname{card}(\Lambda) = \operatorname{card}(M)$. Now assume $\operatorname{card}(M) \geq \operatorname{card}(\Lambda) = n$, for a suitable finite number n, so that we may assume $\Lambda = \{1, \ldots, n\}$. By Fourier expansion we can write $W = \bigoplus_{i=1}^{n} \mathscr{E} \circ u_i$. If $\operatorname{card}(M) > n$ and we choose $v_1, \ldots, v_n$ different elements in $\{v_\mu\}_{\mu \in M}$, then $N = \bigoplus_{i=1}^{n} \mathscr{E} \circ v_i$ is a proper submodule in W. Since, by Proposition 1.5, each $\mathscr{E} \circ u_i$ is an irreducible submodule of W, we are in a position to apply Theorem 1 of [**10**, p. 61]. Therefore there is a non empty subset J in $\{1, \ldots, n\}$ such that

$$W = N \oplus \Big(\bigoplus_{i \in J} \mathscr{E} \circ u_i \Big) = \Big(\bigoplus_{i=1}^{n} \mathscr{E} \circ v_i \Big) \oplus \Big(\bigoplus_{i \in J} \mathscr{E} \circ u_i \Big).$$

Now Theorem 3 in [**10**, p. 62] gives $n = \text{card}(\Lambda) = n + \text{card}(J)$, a contradiction. ∎

2. The structure of Hilbert modules. In this section we will show how the structure theorems for Hilbert modules proved in [**17**] can be easily derived from our previous results on orthonormal bases.

The following two definitions provide precise methods of construction of Hilbert modules.

DEFINITION 2.1. Let $\{W_i\}_{i\in I}$ be a family of Hilbert $\mathscr{E}$-modules and let W be the vector space

$$W := \left\{\{w_i\} \in \prod_{i\in I} W_i : \sum_{i\in I} \|w_i\|^2 < +\infty\right\}.$$

For each e in $\mathscr{E}$ and $\{w_i\}$ in W the element $\{e \circ w_i\}$ lies in W; so by defining $e \circ \{w_i\} := \{e \circ w_i\}$ for all $\{w_i\}$ in W and e in $\mathscr{E}$, W becomes a left module over $\mathscr{E}$. Since for $\{w_i\}$ in W the family $\{[w_i \mid w_i]\}_{i\in I}$ is absolutely summable for the norm $\tau(.)$, the polarization formula gives us that the family $\{[w_i \mid w_i']\}_{i\in I}$ is summable, whenever $\{w_i\}$, $\{w_i'\}$ are in W. Therefore, by defining $[\{w_i\} \mid \{w_i'\}] := \sum_{i\in I} [w_i \mid w_i']$, it is straightforward to verify that W is a Hilbert $\mathscr{E}$-module called the *l^2-sum* of the given family of Hilbert modules, and denoted by $\overset{l^2}{\underset{i\in I}{\oplus}} W_i$.

Given a family $\{\mathscr{E}_i\}_{i\in I}$ of H*-algebras with zero annihilator, using that the involution of each $\mathscr{E}_i$ is isometric [**7**; Proposition 2(ix)], we can define pointwise a canonical structure of H*-algebra in the Hilbert space $\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i$.

DEFINITION 2.2. Let $\{\mathscr{E}_i\}_{i\in I}$ be a family of H*-algebras, $\{W_i\}_{i\in I}$ be a family of Hilbert $\mathscr{E}_i$-modules and let W denote the vector space

$$W := \left\{\{w_i\} \in \prod_{i\in I} W_i : \sum_{i\in I} \|w_i\|^2 < +\infty\right\}.$$

For each $\{e_i\}$ in $\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i$ and $\{w_i\}$ in W, $\{e_i \circ w_i\}$ is in W, so $\{e_i\} \circ \{w_i\} := \{e_i \circ w_i\}$ defines a $\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i$-module multiplication in W. If $\{w_i\}$ is in W and if for i in I we write $[w_i \mid w_i] = e_i e_i^*$ for a suitable e_i in $\mathscr{E}_i$, then $\sum \|e_i\|^2 = \sum \|w_i\|^2 < +\infty$, so that $\{e_i\}$ lies in $\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i$ and $\{[w_i \mid w_i]\} = \{e_i\}\{e_i\}^*$ belongs to $\tau\left(\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i\right)$. In this way W becomes a Hilbert module over $\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i$ under the $\tau\left(\overset{l^2}{\underset{i\in I}{\oplus}} \mathscr{E}_i\right)$-valued product defined by $[\{w_i\} \mid \{w_i'\}] := \{[w_i \mid w_i']\}$. This Hilbert module will be called the *mixed product* of the family $\{W_i\}_{i\in I}$ and will be denoted by $\underset{i\in I}{\times} W_i$.

To have a satisfactory structure theory for Hilbert modules, we recall [**17**] that it is

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enough to consider faithful Hilbert modules. If H is a complex Hilbert space and x, y are in H, then the operator $x \otimes y$ defined by $(x \otimes y)(z) = (z \mid y)z$ for all z in H is in $\tau(\mathcal{HS}(H))$, where $\mathcal{HS}(H)$ denotes the H*-algebra of all Hilbert-Schmidt operators on H. Every complex Hilbert space H can be regarded as a Hilbert module over $\mathcal{HS}(H)$, if the module operation is defined by the action of the operator on the vector, and the $\tau(\mathcal{HS}(H))$-valued product is defined by $[x \mid y] := x \otimes y$ for all x, y in H. We also recall that an H*-algebra $\mathscr{E}$ is called *topologically simple* if $\mathscr{E}^2 \neq 0$ and zero and $\mathscr{E}$ are the only closed ideals of $\mathscr{E}$. Clearly every topologically simple H*-algebra has zero annihilator. The following theorem collects Smith's structure theorems for Hilbert modules [**17**, Theorem 2.10 and 3.1].

THEOREM 2.3 *Let W be a faithful Hilbert $\mathscr{E}$-module.*

(i) *If $\{\mathscr{E}_i\}_{i \in I}$ is the family of all minimal closed ideals of $\mathscr{E}$, then there exists a suitable family $\{W_i\}_{i \in I}$, where, for each i in I, W_i is a faithful Hilbert module over the topologically simple H*-algebra $\mathscr{E}_i$, such that W equals the mixed product of the family $\{W_i\}_{i \in I}$.*

(ii) *If $\mathscr{E}$ is actually topologically simple, then there is a complex Hilbert space H such that $\mathscr{E} = \mathcal{HS}(H)$ and W is the l^2-sum of a suitable family of copies of H regarded as a Hilbert $\mathcal{HS}(H)$-module.*

Proof. As in the case of Hilbert spaces, using Lemma 1.3, Remark 2 and Parseval's identity, if we take an orthonormal basis $\{u_\lambda\}_{\lambda \in \Lambda}$ in W and we write $e_\lambda := [u_\lambda \mid u_\lambda]$, then the mapping $w \to \{[w \mid u_\lambda]\}$ is a Hilbert module isomorphism from W onto the l^2-sum of the family of Hilbert $\mathscr{E}$-modules $\{\mathscr{E}e_\lambda\}_{\lambda \in \Lambda}$. In this way it is enough to assume that W equals the l^2-sum of a suitable family of Hilbert $\mathscr{E}$-modules of the form $\{\mathscr{E}e_\lambda\}_{\lambda \in \Lambda}$, where the e_λ's are selfadjoint minimal idempotents in $\mathscr{E}$.

(i) Clearly, using [**3**, Theorem 34.13], for each λ in Λ, there is a unique i in I such that e_λ belongs to $\mathscr{E}_i$ and since W is a faithful module, for each i in I there exists a λ in Λ such that e_λ lies in $\mathscr{E}_i$. Therefore, defining $\Lambda_i := \{\lambda \in \Lambda : e_\lambda \in \mathscr{E}_i\}$, for i in I, the family $\{\Lambda_i : i \in I\}$ is a partition of Λ. Now, if W_i denotes the l^2-sum of the family of faithful Hilbert $\mathscr{E}_i$-modules $\{\mathscr{E}e_\lambda\}_{\lambda \in \Lambda_i}$, it is not difficult to see that, up to the natural identification $\mathscr{E} \equiv \overset{l^2}{\underset{i \in I}{\oplus}} \mathscr{E}_i$, W equals $\underset{i \in I}{\times} W_i$.

(ii) By Ambrose's theorem [**2**, Theorem 4.3] $\mathscr{E} = \mathcal{HS}(H)$, for a suitable complex Hilbert space H, so that for λ in Λ we have $e_\lambda = x_\lambda \otimes x_\lambda$ for a suitable norm-one element x_λ in H. Now the mapping $x \to x \otimes x_\lambda$ is a Hilbert module isomorphism from H (regarded as a Hilbert module over $\mathcal{HS}(H)$) onto $\mathscr{E}e_\lambda$. ■

3. The Hilbert-Schmidt class. The aim of this section is to develop a basic theory of Hilbert-Schmidt operators on Hilbert modules that extends the classical one ([**16**], [**11**]) for Hilbert spaces.

If W is a Hilbert $\mathscr{E}$-module we recall that a linear operator $F : W \to W$, with the property $F(e \circ w) = e \circ F(w)$ for all w in W and e in $\mathscr{E}$ is called an *$\mathscr{E}$-linear operator* on W. The set of all bounded $\mathscr{E}$-linear operators will be denoted by $BL_{\mathscr{E}}(W)$. It is known [**13**, Theorem 4 and Corollary] that for each F in $BL_{\mathscr{E}}(W)$, its adjoint operator F^* lies in $BL_{\mathscr{E}}(W)$ and satisfies

$$[F(w_1) \mid w_2] = [w_1 \mid F^*(w_2)].$$

LEMMA 3.1. *Let* $\{u_\lambda\}_{\lambda\in\Lambda}$ *and* $\{v_\mu\}_{\mu\in M}$ *be two orthonormal bases in a Hilbert* $\mathscr{E}$*-module* W. *Then, for* F *in* $BL_{\mathscr{E}}(W)$, *the families*

$$\{\|F(u_\lambda)\|^2\}_{\lambda\in\Lambda},\ \{\|[F(u_\lambda)\mid v_\mu]\|^2\}_{\lambda\in\Lambda,\mu\in M},\ \text{and}\ \{\|F^*(v_\mu)\|^2\}_{\mu\in M}$$

are simultaneously summable or not. Whenever they are summable, their sum is the same, independent of $\{u_\lambda\}_{\lambda\in\Lambda}$ *and* $\{v_\mu\}_{\mu\in M}$.

Proof. Using Parseval's equality and the commutativity of the trace we obtain

$$\sum_\lambda \|F(u_\lambda)\|^2 = \sum_\lambda \mathrm{tr}([F(u_\lambda)\mid F(u_\lambda)]) = \sum_\lambda \mathrm{tr}\Big(\sum_\mu [F(u_\lambda)\mid v_\mu][F(u_\lambda)\mid v_\mu]^*\Big)$$

$$= \sum_\lambda\sum_\mu \mathrm{tr}([F(u_\lambda)\mid v_\mu][F(u_\lambda)\mid v_\mu]^*) = \sum_{\lambda,\mu} \|[F(u_\lambda)\mid v_\mu]\|^2$$

$$= \sum_\mu\sum_\lambda \mathrm{tr}([u_\lambda\mid F^*(v_\mu)][u_\lambda\mid F^*(v_\mu)]^*) = \sum_\mu \|F^*(v_\mu)\|^2.$$

It follows that each of the sums written above is equal to the others (possibly equal to $+\infty$). Furthermore choosing both orthonormal bases equal to $\{v_\mu\}_{\mu\in M}$ we have $\sum_\mu \|F(v_\mu)\|^2 = \sum_\mu \|F^*(v_\mu)\|^2$. Therefore $\sum_\lambda \|F(u_\lambda)\|^2 = \sum_\mu \|F(v_\mu)\|^2$. Hence the common value of the sums is independent of the choice of $\{u_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\mu\}_{\mu\in M}$. ■

DEFINITION 3.2. Those operators F in $BL_{\mathscr{E}}(W)$ for which the common sum of the above families is finite will be called *Hilbert-Schmidt operators*, and, for such an F, $\sigma(F)$ will denote the only nonnegative real number with square equal to the above common sum. The class of all F's as above will be denoted by $\mathscr{HS}_{\mathscr{E}}(W)$.

THEOREM 3.3. *Let* W *be a Hilbert* $\mathscr{E}$*-module.*

(i) $\mathscr{HS}_{\mathscr{E}}(W)$ *is a selfadjoint ideal of* $BL_{\mathscr{E}}(W)$ *and, for* F *in* $\mathscr{HS}_{\mathscr{E}}(W)$ *and* G *in* $BL_{\mathscr{E}}(W)$, *we have*

$$\sigma(F)=\sigma(F^*),\qquad \|F\|\le\sigma(F),\qquad \sigma(FG)\le\sigma(F)\,\|G\|,\quad \text{and}\quad \sigma(GF)\le\|G\|\,\sigma(F).$$

(ii) *For every* F, G *in* $\mathscr{HS}_{\mathscr{E}}(W)$ *and for any orthonormal basis* $\{u_\lambda\}_{\lambda\in\Lambda}$ *in* W, *the family* $\{(F(u_\lambda)\mid G(u_\lambda))\}_{\lambda\in\Lambda}$ *is summable in* $\mathbb{C}$ *with sum independent of the chosen orthonormal basis, and* $(F\mid G):=\sum_{\lambda\in\Lambda}(F(u_\lambda)\mid G(u_\lambda))$ *defines an inner product in* $\mathscr{HS}_{\mathscr{E}}(W)$ *whose associated norm is* $\sigma(.)$. *With this inner product and under the involution* $*$, $\mathscr{HS}_{\mathscr{E}}(W)$ *becomes an* H**-algebra with zero annihilator.*

Proof. From Lemma 3.1, the equality $\sigma(F)=\sigma(F^*)$ is clear. If $\{u_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis in W, then, by Fourier expansion and Remark 2, we have

$$\|F(w)\| = \Big\|F\Big(\sum_{\lambda\in\Lambda}[w\mid u_\lambda]\circ u_\lambda\Big)\Big\| = \Big\|\sum_{\lambda\in\Lambda}[w\mid u_\lambda]\circ F(u_\lambda)\Big\| \le \sum_{\lambda\in\Lambda}\|[w\mid u_\lambda]\|\,\|F(u_\lambda)\|$$

$$\le \Big(\Big(\sum_{\lambda\in\Lambda}\|[w\mid u_\lambda]\|^2\Big)\Big(\sum_{\lambda\in\Lambda}\|F(u_\lambda)\|^2\Big)\Big)^{1/2} = \|w\|\,\sigma(F),$$

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so that $\|F\| \le \sigma(F)$. Now that the above inequality has been proved, the remaining assertions in the proposition follow as in the classical case of Hilbert spaces [**16**]. (Note that $\mathcal{HS}_{\mathscr{E}}(W)$, as any selfadjoint subalgebra of a C*-algebra has zero annihilator). ■

For every w_1, w_2 in W, we denote by $w_1 \square w_2$ the operator on W given by $(w_1 \square w_2)(w) := [w \mid w_2] \circ w_1$ for all w in W.

THEOREM 3.4. *Let W be a Hilbert $\mathscr{E}$-module.*
(i) *$w_1 \square w_2$ lies in $\mathcal{HS}_{\mathscr{E}}(W)$, for all w_1, w_2 in W.*
(ii) *$(w_1 \square w_2 \mid w_3 \square w_4) = \mathrm{tr}([w_4 \mid w_2][w_1 \mid w_3])$, for all w_1, w_2, w_3, w_4 in W.*
(iii) *The linear span of $\{w_1 \square w_2 : w_1, w_2 \in W\}$ is a dense ideal of $\mathcal{HS}_{\mathscr{E}}(W)$.*

Proof. Using Remark 2, (i) is readily verified. By Parseval's equality and the commutativity of the trace form, we have

$$(w_1 \square w_2 \mid w_3 \square w_4) = \sum_{\lambda \in \Lambda} (w_1 \square w_2(u_\lambda) \mid w_3 \square w_4(u_\lambda)) = \sum_{\lambda \in \Lambda} \mathrm{tr}([[u_\lambda \mid w_2] \circ w_1 \mid [u_\lambda \mid w_4] \circ w_3])$$

$$= \sum_{\lambda \in \Lambda} \mathrm{tr}([u_\lambda \mid w_2][w_1 \mid w_3][u_\lambda \mid w_4]^*) = \sum_{\lambda \in \Lambda} \mathrm{tr}([w_4 \mid u_\lambda][w_2 \mid u_\lambda]^*[w_1 \mid w_3])$$

$$= \mathrm{tr}\Big(\sum_{\lambda \in \Lambda} [w_4 \mid u_\lambda][w_2 \mid u_\lambda]^*[w_1 \mid w_3]\Big) = \mathrm{tr}([w_4 \mid w_2][w_1 \mid w_3]).$$

Thus the statement (ii) is verified. Since $F(w_1 \square w_2) = F(w_1) \square w_2$ and $(w_1 \square w_2)F = w_1 \square F^*(w_2)$ for all F in $\mathcal{HS}_{\mathscr{E}}(W)$ and w_1, w_2 in W, it follows that $\mathrm{Lin}\{w_1 \square w_2 : w_1, w_2 \in W\}$ is an ideal in $\mathcal{HS}_{\mathscr{E}}(W)$. If $\{u_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal basis in W, if F is in $\mathcal{HS}_{\mathscr{E}}(W)$, and if for each finite subset J of Λ we write

$$F_J := \sum_{\mu \in J} F(u_\mu) \square u_\mu,$$

then F_J lies in $\mathrm{Lin}\{w_1 \square w_2 : w_1, w_2 \in W\}$. Moreover, by Proposition 1.5, $F_J(u_\lambda) = F(u_\lambda)$ if λ is in J and clearly $F_J(u_\lambda) = 0$ otherwise. Therefore $(\sigma(F - F_J))^2 = \sum_{\lambda \in \Lambda} \|(F - F_J)(u_\lambda)\|^2 = \sum_{\substack{\lambda \in \Lambda \\ \lambda \notin J}} \|F(u_\lambda)\|^2$, which concludes the proof. ■

Despite what the reader might think, the closed submodule generated by an element in a Hilbert $\mathscr{E}$-module W may have infinite hilbertian dimension over $\mathscr{E}$. Even the operators of the form $w_1 \square w_2$, for w_1, w_2 in W, may have "infinite range over $\mathscr{E}$" as shown by the following example. In any case, by Fourier expansions, both dimensions must be at most countable.

EXAMPLE. Let H be the separable infinite dimensional complex Hilbert space and $\{x_m : m \in \mathbb{N}\}$ an orthonormal basis in H; let W denote the Hilbert $\mathcal{HS}(H)$-module l^2-sum of a countably infinite family of copies of H regarded as a Hilbert module over $\mathcal{HS}(H)$, and consider the element $w = \{n^{-1}x_n\}_{n \in \mathbb{N}}$ in W. Then, for every $\{y_n\}_{n \in \mathbb{N}}$ in W, we have $(w \square w)(\{y_n\}_{n \in \mathbb{N}}) = \{n^{-2}y_n\}_{n \in \mathbb{N}}$, so that quasi-null sequences are in the range of $w \square w$. As a consequence, the closure of the range of $w \square w$ equals W. Since the range of $w \square w$ is contained in the submodule of W generated by w, it follows that the closed submodule generated by w also equals W.

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Departamento De Analisis Matematico
Facultad De Ciencias
Universidad De Granada
18071-Granada
Spain

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