## § 2. Circumcentre.

*The perpendiculars to the sides of a triangle from the mid points of the sides are concurrent.**

The following demonstration † may be compared with the demonstration of § 5.

Figure 16.

Let A′, B′, C′ be the mid points of BC, CA, AB,

and let A′X′, B′Y′, C′Z′ be perpendicular to BC, CA, AB.

| | |
|---|---|
| Join | B′C′, C′A′, A′B′. |
| Then | B′C′ is parallel to BC; |
| therefore | A′X′ is perpendicular to B′C′. |
| Hence | B′Y′ " " " C′A′ |
| and | C′Z′ " " " A′B′. |

If therefore it be assumed as true that the perpendiculars to the sides of a triangle from the opposite vertices are concurrent,

A′X′, B′Y′, C′Z′ are concurrent.

Another demonstration is obtained at once from the theorem, ‡ *If three points be taken on the sides of a triangle such that the sums of the squares of the alternate segments taken cyclically are equal, the perpendiculars to the sides of the triangle at these points are concurrent.*

The point of concurrency, which will be denoted by O, is the centre of the circle circumscribed about ABC. This circle is often called the *circumcircle*,§ and the centre of it the *circumcentre*.§

The radius of the circumcircle is usually denoted by R.

(1) The circumcentre of a triangle is the orthocentre of its complementary triangle.

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* Euclid's *Elements*, IV. 5.

† C. Adams, *Die Lehre von den Transversalen*, p. 21 (1843).

‡ F. G. de Oppel, *Analysis Triangulorum*, p. 32 (1746).

§ These terms as well as *incircle, excircle, midcircle, incentre, excentre, midcentre* were suggested by W. H. H. H[udson]. See *Nature*, XXVIII. 7, 104 (1883). The terms *Umkreis, Inkreis, Ankreis, Mittenkreis* have been more or less in use in Germany since 1866, as may be seen from Schlömilch's *Zeitschrift*.

The perpendiculars to the sides of a triangle from the mid points of the sides are sometimes called *médiatrices* in France and Belgium.

(2) Since the complementary and the fundamental triangles are similar, and since their sides are in the ratio of 1 to 2, the distance of the circumcentre of a triangle from any side is half of the distance between the orthocentre of the triangle and the vertex opposite that side.*

(3) If O be the circumcentre of a triangle ABC, the circle on OA as diameter bisects AB and AC.

Similarly for the circles on OB and OC.

(4) *If the circle on $OA$ as diameter should cut $BC$ at $P$ and $P'$, then $AP$ is a mean proportional* † *between $BP$ and $CP$, and $AP'$ is a mean proportional between $BP'$ and $CP'$.*

FIGURE 17.

Produce AP to meet the circumcircle at Q.

The circle on OA as diameter touches the circumcircle at A; therefore A is the homothetic centre of the two circles;

therefore $AP : AQ = 1 : 2$;

therefore $BP \cdot CP = AP \cdot QP = AP^2$.

(5) *By the following construction* ‡ *a point $P$ will be found in the base $BC$ of $ABC$ such that the ratio $AP^2 : BP \cdot CP$ has a given value.*

FIGURE 18.

Join AO, and divide it at L so that $AL : LO$ has the given value; then the circle with centre L and radius LA will meet BC in two points P, P′ satisfying the condition.

Produce AP to meet the circumcircle in Q.

Then
$$\begin{aligned} AP : PQ &= AP^2 : AP \cdot PQ \\ &= AP^2 : BP \cdot PC \\ &= AL : LO \end{aligned};$$

therefore LP is parallel to OQ;
therefore $LP = LA$, since $OQ = OA$.

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* *Ladies' Diary* for 1785.

† Given without proof in the *Ladies' Diary* for 1759.

‡ Rev. J. Wolstenholme in the *Educational Times*, XXIX., 273 (1877). Four solutions are given in *Mathematical Questions from the Educational Times*, XXVII, 63-66 (1877); the one in the text is the last.

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If AP′Q′ be the other position of APQ,

then $AP : PQ = AP' : P'Q'$;

therefore QQ′ is parallel to BC;

therefore arc BQ = arc CQ′,

and AP, AP′ are isogonal with respect to ∠ A.

(6) If from a point O within or without a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB, and circles are circumscribed about the triangles OEF, OFD, ODE; the area of the triangle formed by joining the centres of these circles is one-fourth of the area * of the triangle ABC

FIGURE 19.

The centres of these circles are the mid points of OA, OB, OC.

(7) *If from a point O within triangle ABC perpendiculars OD, OE, OF be drawn to BC, CA, AB, then* †

$$2R(EF + FD + DE) = OA \cdot BC + OB \cdot CA + OC \cdot AB.$$

FIGURE 19.

For A, F, O, E lie on the circle whose diameter is OA, and the chord EF subtends the same angle A at the circumference of this circle as BC does at that of the circumcircle of ABC;

therefore $EF : OA = BC : 2R$;

therefore $2R \cdot EF = OA \cdot BC$.

(8) If O be on that arc of the circumcircle on which angle C stands, then,† by Ptolemy's theorem,

$$OA \cdot BC + OB \cdot CA - OC \cdot AB = 0;$$

therefore $EF + FD - DE = 0$;

therefore D, E, F are collinear,

which is another proof of the property of the Wallace line.

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* Todhunter's *Plane Trigonometry*, Chap. XVI., Ex. 41 (1859).

† Both (7) and (8) are due to Mr E. M. Langley, who applies the first of them to the problem of finding the triangle of minimum perimeter inscribed in a given triangle, and to the determination of the trilinear co-ordinates of the Brocard points. See *Sixteenth General Report of the Association for the Improvement of Geometrical Teaching*, pp. 34-5 (1890).

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(9) *If O be the circumcentre of ABC, and AO, BO, CO be produced to meet the circumcircle in U, V, W, the triangle UVW is congruent to ABC.*

FIGURE 20.

For $\angle AUV = \angle ABO = \angle BAO,$

$\angle AUW = \angle ACO = \angle CAO;$

therefore $\angle VUW = \angle BAC;$

therefore UVW is similar to ABC.

But these two triangles are inscribed in the same circle; therefore they are congruent.

(10) The figures BCVW, CAWU, ABUV are rectangles.

(11) If ABC be a triangle, and BW, CV be perpendicular to BC; CU, AW perpendicular to CA; AV, BU perpendicular to AB, the three straight lines AU, BV, CW are concurrent at the circumcentre of ABC, and the six points A, B, C, U, V, W are concyclic.*

FIGURE 20.

(12) *Triangles $A_1B_1C_1$, $A_2B_2C_2$ circumscribed about ABC in such a manner that their sides are perpendicular to those of ABC are congruent* † *to each other and similar to ABC.*

FIGURE 21.

Let $C_1A_1$, $A_2B_2$; $A_1B_1$, $B_2C_2$; $B_1C_1$, $C_2A_2$

meet at U, V, W.

Then $BB_1VB_2$ is a parallelogram;

therefore $BB_1 = B_2V$.

But $BW = CV$;

therefore $B_1W = CB_2$.

Again $WC_1CC_2$ is a parallelogram;

therefore $WC_1 = C_2C$;

therefore $B_1C_1 = B_1C_2$.

Similarly $C_1A_1 = C_2A_2$, $A_1B_1 = A_2B_2$.

* C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 56 (1825).

† The first part of the theorem is given by Jacobi, p. 56.

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Lastly $\angle BAC = 90° - \angle BAA$

$= \angle B_1AA_2 = \angle A$ .

Similarly $\angle ABC = \angle B_1$, $\angle ACB = C_1$.

This theorem is a particular case of a more general one.

(13) *The three lines $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent* * *at the circumcentre of ABC.*

FIGURE 21.

For AU, BV, CW are concurrent at O, the circumcentre of ABC; and O is the mid point of AU, BV, CW.
Now since $AA_1UA_2$ is a parallelogram,
therefore $A_1A_2$ passes through the mid point of AU.

Similarly for $B_1B_2$, $C_1C_2$.

(14) *If a point P be taken inside the triangle ABC, and circles be circumscribed about the triangles PBC, PCA, PAB, and their centres $O_1$, $O_2$, $O_3$ be joined, the angles of triangle $O_1O_2O_3$ are supplementary to the angles BPC, CPA, APB.*

FIGURE 22.

For $O_2O_3$, $O_3O_1$, $O_1O_2$ are respectively perpendicular to PA, PB, PC.

(15) *If through A any straight line MN be drawn meeting the circumferences PCA, PAB in M, N, then MC, NB will intersect on the circumference* † *PBC.*

Let MC, NB intersect at L.

Then $\angle M = 180° - \angle CPA$,

$\angle N = 180° - \angle APB$;

therefore $\angle M + \angle N = 360° - (\angle CPA + \angle APB)$,

$= \angle BPC$;

therefore $\angle L = 180° - \angle BPC$;

therefore L is on the circumference PBC.

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* Jacobi does not state this property, but from the way in which he letters the figures of theorems (11) and (12) it is probable that he knew it. The property is explicitly stated, along with some others, by Mr Lemoine in his paper read at the Lyons meeting (1873) of the *Association Française pour l'avancement des Sciences.*

† Rochat in Gergonne's *Annales*, II. 29 (1811). To him also are due (17) and (19).

(16) If L be any point on the circumference PBC, and if LC, LB meet the circumferences PCA, PAB again in M, N, then M, A, N are collinear.

(17) Triangle LMN is similar to $O_1O_2O_3$.

If the point P be fixed, the triangles $O_1O_2O_3$, LMN are given in species.

(18) *The angles which MN, NL, LM make with AP, BP, CP respectively are equal.*

For
$$\angle PAN = 180° - \angle PBN = \angle PBL$$
$$= 180° - \angle PCL = \angle PCM.$$

(19) *Of all the triangles such as LMN whose sides pass through A, B, C, and whose vertices are situated on the circles $O_1$, $O_2$, $O_3$, that triangle L′M′N′ is a maximum whose sides are perpendicular to AP, BP, CP.*

FIGURE 22.

For triangles L′M′N′, LMN are similar,

and PL′, PL are corresponding lines in these triangles.

Now PL′ is a diameter of the circle $O_1$;

therefore PL′ is greater than PL;

therefore L′M′N′ is greater than LMN.

(20) If O be the circumcentre of ABC, and about the triangles OBC, OCA, OAB circles be circumscribed whose centres are $O_1$, $O_2$, $O_3$, the triangle $O_1O_2O_3$ has its angles equal to 180° − 2A, 180° − 2B, 180° − 2C.

It will be found that $O_1O_2O_3$ is similar to XYZ. See § 5.

(21) *O is the incentre of the triangle $O_1O_2O_3$.*

FIGURE 22.

In the diagram suppose P to be replaced by O, and let V, W be the mid points of BO, CO.

Then the right-angled triangles $OVO_1$, $OWO_1$ have two sides of the one equal to two sides of the other;

therefore $OO_1$ bisects $\angle O_2O_1O_3$.

Similarly for $OO_2$, $OO_3$.

(22.) *If $OO_1$, $OO_2$, $OO_3$ be produced to meet the circles OBC, OCA, OAB in L′, M′, N′, the triangle L′M′N′ will be circumscribed about ABC, will be similar and similarly situated to $O_1O_2O_3$, and will have O for its incentre.*

FIGURE 22.

For $\angle OAM' + \angle OAN' = 180°$;

therefore M′, A, N′ are collinear, and M′N′ is parallel to $O_2O_3$.

Since OA, OB, OC are equal, and perpendicular to M′N′, N′L′, L′M′;

therefore O is the incentre of L′M′N′.

Many relations between the triangles $O_1O_2O_3$ and ABC may be derived from the relations between XYZ and ABC, seeing that $O_1O_2O_3$ is similar to XYZ and that the ratio of the radii of their incircles is $\frac{1}{2}R : \rho$.