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ON LOCALIZATION AT AN IDEAL

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ABSTRACT. Conditions are given under which the ring of quotients defined by an ideal is semisimple Artinian modulo its Jacobson radical.

Lambek and Michler [4] have given necessary and sufficient conditions under which $I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple Artinian, for the localization I_σ of a semiprime ideal I of a left Noetherian ring R. (Here $J(R_\sigma)$ denotes the Jacobson radical of the ring of quotients R_σ, and σ is the torsion radical of R-Mod determined by the ideal I.) When $I=(0)$ it is well-known that $R_\sigma = Q_{\max}(R)$ is semisimple Artinian if and only if R has finite left uniform dimension and zero left singular ideal. In the following note we show that this result can be extended to the localization at any ideal I. Thus we can extend and clarify Lambek and Michler's result by dropping not only the assumption that R is left Noetherian (as in [1]), but also the assumption that I is semiprime.

Throughout the paper I will denote a two-sided ideal of an associative ring R with identity, and σ will denote the torsion radical cogenerated by the R-injective envelope $E(R/I)$ of R/I. For any left R-module M,

$$\sigma M = \{m \in M \mid f(m)=0 \quad \text{for all} \quad f \in \operatorname{Hom}_R(M, E(R/I))\}.$$

The corresponding quotient functor Q_σ is defined by setting

$$Q_\sigma(M) = \{m \in E(M/\sigma M) \mid f(m)=0 \quad \text{for all} \quad f \in \operatorname{Hom}_R(E(M/\sigma M), E(R/I)) \quad \text{with} \quad f(M/\sigma M)=0\}.$$

For the natural mapping $\eta_M : M \to Q_\sigma(M)$, we have $\sigma M = \ker(\eta_M)$. The quotient category R-Mod/σ of R-Mod defined by σ is the full subcategory of modules ${}_RM$ for which η_M is an isomorphism. The quotient functor Q_σ is a left adjoint of the inclusion functor $U_\sigma : R\text{-Mod}/\sigma \to R\text{-Mod}$. Recall that the torsion radical σ is said to be perfect if R-Mod/σ coincides with R_σ-Mod (see [5, Chapter XI, Proposition 3.4]).

A module ${}_RM$ is called σ-torsion if $\sigma M = M$ and σ-torsionfree if $\sigma M = 0$. A submodule N of M is called σ-dense if M/N is σ-torsion and σ-closed if M/N is σ-torsionfree. By [5, Chapter IX, Proposition 4.4] the subobjects of $Q_\sigma(M)$ in R-Mod/σ correspond to σ-closed submodules of M. In particular, if M is

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σ-torsionfree, then $Q_\sigma(M)$ is a simple object in R-Mod/σ if and only if every non-zero submodule of M is σ-dense. If M is σ-torsionfree and $N \subseteq M$ is a σ-dense submodule, then for any $0 \neq m \in M$, $\text{Ann}(m) = \{r \in R \mid rm = 0\}$ is σ-closed, while $\{r \in R \mid rm \in N\}$ is σ-dense. Thus there exists $r \in R$ such that $0 \neq rm \in N$, which shows that N is an essential R-submodule. We can conclude that if M is σ-torsionfree and $Q_\sigma(M)$ is a simple object in R-Mod/σ, then M is a uniform R-module.

Recall that for any R-module M, the R-module $Q_\sigma(M)$ can be made into an R_σ-module as follows: for $x \in Q_\sigma(M)$, the R-homomorphism $f: R/\sigma R \to Q_\sigma(M)$ defined by $f(1) = x$ can be extended to $\rho_x : R_\sigma \to Q_\sigma(M)$, and then for all $q \in R_\sigma$ we can define $qx = \rho_x(q)$. When viewed as an R_σ-module, $Q_\sigma(M)$ will usually be denoted by M_σ. The exact sequence

$$0 \to I \to R \to R/I \to 0$$

of R-modules gives rise to the exact sequence

$$0 \to I_\sigma \to R_\sigma \to (R/I)_\sigma$$

of R_σ-modules, and so we will identify R_σ/I_σ with the corresponding submodule of $(R/I)_\sigma$. We note that although I is a two-sided ideal of R, I_σ need not be a two-sided ideal of R_0.

THEOREM. *Let I be an ideal of R, and let σ be the torsion radical defined by $E(R/I)$. Then the following conditions are equivalent.*

(1) *$Q_\sigma(R/I)$ is a finite direct sum of simple objects in R-Mod/σ.*

(2) *The ring R/I has finite left uniform dimension and zero left singular ideal.*

Proof. (1)$\Rightarrow$(2) Assume that $Q_\sigma(R/I)$ is a finite direct sum of simple objects in R-Mod/σ. Since U_σ preserves finite direct sums, the remarks preceding the theorem show that $U_\sigma Q_\sigma(R/I)$ has finite uniform dimension, and then R/I has finite uniform dimension as a left R-module since it is essential in $U_\sigma Q_\sigma(R/I)$.

Let A be any left ideal of R such that A/I is essential in R/I. Then $U_\sigma Q_\sigma(A/I)$ must be essential in $U_\sigma Q_\sigma(R/I)$, and since it is a σ-closed submodule, the intersection $U_\sigma Q_\sigma(A/I) \cap U_\sigma(X)$ must be a non-zero σ-closed submodule of $U_\sigma(X)$, for any non-zero simple subobject X of $Q_\sigma(R/I)$. Since X is simple in R-Mod/σ, this intersection must be equal to $U_\sigma(X)$, which shows that $Q_\sigma(A/I)$ contains every simple subobject of $Q_\sigma(R/I)$, and so we must have $Q_\sigma(A/I) = Q_\sigma(R/I)$. For the exact sequence

$$0 \to Q_\sigma(A/I) \to Q_\sigma(R/I) \to Q_\sigma(R/A) \to 0$$

in R-Mod/σ we must therefore have $Q_\sigma(R/A) = 0$, so that $\sigma(R/A) = R/A$. Thus A is σ-dense in R, and so $A\bar{r} \neq (0)$ for each $0 \neq \bar{r} \in R/I$, which implies that R/I has zero singular ideal.

(2)⇒(1) Assume that R/I has finite left uniform dimension and zero left singular ideal. If M is an R/I-module and $f \in \text{Hom}_R(M, E(R/I))$, then since Im($f$) is an R/I-module, it must be contained in $E_{R/I}(R/I)$, which by assumption has zero singular submodule. Thus if $N \subseteq M$ is an essential submodule, then $\text{Hom}_R(M/N, E(R/I)) = 0$. Since R/I has finite uniform dimension, it must contain an essential direct sum $A = \bigoplus_{i=1}^n U_i$ of uniform submodules U_i, $1 \le i \le n$. It follows that $\text{Hom}_R(R/A, E(R/I)) = 0$, and so A is σ-dense in R/I, which implies that $Q_\sigma(R/I) = Q_\sigma(A) = \bigoplus_{i=1}^n Q_\sigma(U_i)$. If $V_i \subseteq U_i$ is any essential R-submodule of U_i, then $\text{Hom}_R(U_i/V_i, E(R/I)) = 0$, which shows that V_i is σ-dense in U_i. By the remarks preceding the theorem, this implies that $Q_\sigma(U_i)$ is a simple object in R-Mod/σ, and so $Q_\sigma(R/I)$ is a finite direct sum of simple objects.

Corollary 1. *Let I be an ideal of R, and let σ be the torsion radical defined by $E(R/I)$. Then the following conditions are equivalent.*

(1) *R_σ/I_σ is a direct sum of simple R_σ-modules and contains an isomorphic copy of each simple left R_σ-module.*

(2) *The ring R/I has finite left uniform dimension and zero left singular ideal and the torsion radical σ is perfect.*

Proof. (1)⇒(2) It follows from [5, Chapter XI, Proposition 3.4] that σ is perfect if and only if $E(R/I)$ is a cogenerator for R_σ-Mod. Since $E(R/I)$ is injective and R_σ/I_σ has been identified with an essential submodule of $E(R/I)$, this occurs if and only if R_σ/I_σ contains an isomorphic copy of each simple R_σ-module. If σ is perfect, then epimorphisms in R-Mod/σ are onto, and so $Q_\sigma(R/I) = R_\sigma/I_\sigma$. Moreover, R-Mod/σ coincides with R_σ-Mod, and so condition (1) of the theorem is satisfied. Thus R/I has finite left uniform dimension and zero left singular ideal.

(2)⇒(1) Assume that condition (2) holds. Since σ is perfect, R_σ-Mod coincides with R-Mod/σ and R_σ/I_σ coincides with $Q_\sigma(R/I)$, so it follows from the theorem that R_σ/I_σ is a direct sum of simple R_σ-modules. Finally, R_σ/I_σ contains an isomorphic copy of each simple left R_σ-module since $E(R/I)$ must be a cogenerator for R_σ-Mod.

Corollary 2. *If the conditions of Corollary 1 are satisfied, then the localization M_σ of any R/I-module M is a direct sum of simple R_σ-modules.*

Proof. Express M as a homomorphic image of a free R/I-module, apply the quotient functor $Q_\sigma: R\text{-Mod} \to R_\sigma\text{-Mod}$ (which by assumption is exact and preserves direct sums), and use the fact that R-Mod/$\sigma = R_\sigma$-Mod.

Corollary 3. *The following conditions are equivalent.*

(1) *$I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple Artinian.*

(2) *σ is perfect, I_σ is an ideal of R_σ, and the ring R/I has finite left uniform dimension and zero left singular ideal.*

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Proof. (1)⇒(2) This follows from the theorem and Corollary 1.

(2)⇒(1) It follows immediately from Corollary 1 that $J(R_\sigma)\subseteq I_\sigma$. The reverse inclusion holds since I_σ annihilates R_σ/I_σ, which by Corollary 1 contains an isomorphic copy of each simple left R_σ-module

The final proposition investigates the relationship between R_σ/I_σ and $Q_{\max}(R/I)$. A direct proof of Corollary 3 can be given by combining the proposition with the fact that $Q_{\max}(R/I)$ is semisimple Artinian if and only if R/I has finite left uniform dimension and zero left singular ideal.

LEMMA. *Let M be a left R_σ-module which is σ-torsionfree. Then for any element $m\in M$ and any left ideal $A\subseteq R$, $A_\sigma m=(0)\Leftrightarrow Am=(0)$.*

Proof. If $A_\sigma m=(0)$, then $Am=(A+\sigma R/\sigma R)m\subseteq A_\sigma m=(0)$. On the other hand, if $A_\sigma m\neq(0)$, then $qm\neq 0$ for some $q\in A_\sigma$, and so there exists $f\in \operatorname{Hom}_R(M, E(R/I))$ with $f(qm)\neq 0$ (since $\sigma M=(0)$). Now $f(qm)=f\rho_m(q)$, so from the definition of A_σ, $f\rho_m(A_\sigma)\neq(0)$ implies that $f\rho_m(A+\sigma R/\sigma R)\neq(0)$, and hence $Am\neq(0)$.

PROPOSITION. (a) $Q_{\max}(R/I)=\{x\in(R/I)_\sigma \mid I_\sigma x=(0)\}$.

(b) *$R_\sigma/I_\sigma\subseteq Q_{\max}(R/I)$ if and only if I_σ is an ideal of R_σ; in this case R_σ/I_σ is a subring of $Q_{\max}(R/I)$.*

(c) *If σ is perfect and I_σ is an ideal, then $R_\sigma/I_\sigma=Q_{\max}(R/I)$.*

Proof. (a) The ring $Q_{\max}(R/I)$ is defined by the R/I-injective envelope of R/I, given by $E_{R/I}(R/I)=\{x\in E(R/I)\mid Ix=(0)\}$. Thus $Q_{\max}(R/I)=E_{R/I}(R/I)\cap(R/I)_\sigma$ since $E_{R/I}(R/I)$ is a fully invariant submodule of $E(R/I)$ and by definition

$$Q_{\max}(R/I)=\{x\in E_{R/I}(R/I)\mid f(x)=0 \quad \text{for all} \quad f\in \operatorname{End}(E_{R/I}(R/I)) \quad \text{such that} \quad f(R/I)=0\}.$$

The desired conclusion follows from the lemma.

(b) The left ideal I_σ is an ideal of R_σ if and only if $I_\sigma R_\sigma\subseteq I_\sigma$, that is, if and only if $I_\sigma(R_\sigma/I_\sigma)=(0)$, which occurs by part (a) if and only if $R_\sigma/I_\sigma\subseteq Q_{\max}(R/I)$. If I_σ is an ideal of R_σ, then the R-homomorphism $\pi: R_\sigma\to(R/I)_\sigma$ induced by $R\to R/I\to 0$ maps R_σ into $Q_{\max}(R/I)$, and $\pi(1)=1$. Since for $p,q\in R$, $\pi(pq)=\pi\rho_q(p)$ and $\pi(p)\pi(q)=\rho_{\pi(q)}\pi(p)$, to show that π is a ring homomorphism it suffices to show that $\pi\rho_q=\rho_{\pi(q)}\pi$, and since these map into $E(R/I)$ they will be equal if they agree on $R/\sigma R$. This completes the proof, since

$$\pi\rho_q(1)=\pi(q)=\rho_{\pi(q)}(1)=\rho_{\pi(q)}\pi(1).$$

(c) We only need that $R_\sigma/I_\sigma\subseteq Q_{\max}(R/I)$ and $R_\sigma/I_\sigma=(R/I)_\sigma$ to obtain the desired conclusion.

We note that part (a) of the above proposition generalizes Proposition 3.3 of [2], and as in Proposition 3.4 of [2] we can show that for the idealizer $\mathfrak{D}(I_\sigma)=\{q\in R_\sigma \mid I_\sigma q\subseteq I_\sigma\}$, we have $\mathfrak{D}(I_\sigma)/I_\sigma=Q_{\max}(R/I)\cap(R_\sigma/I_\sigma)$. Part (b) of the proposition extends Lemma 2.3 of [4] and part of Theorem 3.6 of [2], and in fact it can be shown that if I is a prime ideal of R, then I_σ is an ideal of R_σ if and only if R_σ/I_σ is a prime R-module. The proof of Theorem 3.7 of [2] can be extended to show that if I is a prime ideal, then the following conditions are equivalent:
(1) $(R/I)_\sigma$ is a prime R-module; (2) $(R/I)_\sigma=Q_{\max}(R/I)$; (3) $I_\sigma=\operatorname{Ann}((R/I)_\sigma)$; (4) I_σ is an ideal of R_σ and $(R/I)_\sigma$ is a prime R_σ-module.

If I_σ is an ideal of R_σ, then many properties of I_σ go up to I_σ. For example, if R/I has finite uniform dimension, then so does $E(R/I)$ and hence R_σ/I_σ; R/I has zero left singular ideal if and only if $Q_{\max}(R/I)=Q_{\max}(R_\sigma/I_\sigma)$ is von Neumann regular, and this occurs if and only if R_σ/I_σ has zero left singular ideal. As an R/I-module, R_σ/I_σ is an essential extension of R/I, and so it is easy to show that R_σ/I_σ is prime (semiprime) if R/I is prime (semiprime).

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