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ON RELATIONS BETWEEN JACOBIANS AND RESULTANTS OF POLYNOMIALS IN TWO VARIABLES

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This paper investigates some of the connections between the zeros of a polynomial vector field $F = (f, g)\colon \mathbb{C}^2 \to \mathbb{C}^2$ and the Jacobian determinant $J(f, g)$ of f and g. As a consequence, sufficient conditions are given for F to have no zeros. In addition, in the case where F has an inverse F^{-1}, it is proven that F^{-1} is also polynomial.

1. INTRODUCTION

Let $f(x,y), g(x,y)$ be nonzero polynomials with coefficients in $\mathbb{C}$, and let $F = (f,g)\colon \mathbb{C}^2 \to \mathbb{C}^2$. A zero of F is a point $(x_0, y_0) \in \mathbb{C}^2$ with the property $F(x_0, y_0) = (0,0)$.

In this paper we investigate some of the connections between zeros of F and the Jacobian determinant $J(f,g)$ of f and g. This leads to the consideration of resultants of the type $\operatorname{Res}_y(f-u, g-v) = A(x,u,v)$, $\operatorname{Res}_x(f-u, g-v) = B(y,u,v)$, where u and v are indeterminates. Let k, r be the degrees of $A(x,u,v)$ in x and $B(y,u,v)$ in y, respectively. Theorem 1 of Section 3 gives necessary and sufficient conditions for k and r to be zero in terms of $J(f,g)$. As a consequence, sufficient conditions are given for F to have no zeros.

In the case where F is $1-1$ and onto, we show (Section 4) that $k = r = 1$. Furthermore, $A(x,u,v) = ax + A_0(u,v)$, $B(y,u,v) = by + B_0(u,v)$, (Lemma 2), and this gives rise to the well-known fact that F has a polynomial inverse, F^{-1}; Proposition 1 specifically computes F^{-1}. The McKay-Wang inversion formula which generalises Cramer's rule to two polynomials in two variables, was first derived in [**3**] and rederived by Adjamagbo and van den Essen in [**1**]. Our Proposition 1 also rederives this formula by using a different approach. As a result, F is completely determined by its "border polynomials". We conclude with a conjecture regarding the nonexistence of zeros of F.

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2. Properties of Resultants

Throughout this paper $f(x,y)$ and $g(x,y)$ are polynomials in x, y with coefficients in the field $\mathbb{C}$ of complex numbers. We begin with the following definitions: The Jacobian determinant, $J(f,g)$, of f and g is defined by

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}.$$

Let

$$a(t) = a_n t^n + \cdots + a_1 t + a_0$$
$$b(t) = b_m t^m + \cdots + b_1 t + b_0$$

be nonzero polynomials of degrees n and m respectively, with coefficients in an integral domain D. The resultant of a, b with respect to t, $\mathrm{Res}_t(a,b)$, is the following $(m+n) \times (m+n)$ determinant:

$$\mathrm{Res}_t(a,b) = \begin{vmatrix} a_n & a_{n-1} & \dots & a_0 & & & \\ & a_n & \dots & a_1 & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_n & \dots & \dots & a_0 \\ b_m & \dots & \dots & \dots & b_0 & & \\ & \ddots & & & & \ddots & \\ & & b_m & \dots & \dots & \dots & b_0 \end{vmatrix}$$

There are several well-known properties of resultants, [**2**, **6**]. We state here those that will be needed later.

Property 1. There are polynomials $A(t), B(t) \in D[t]$ of degrees n', m' respectively, $n' < m$, $m' < n$ so that

$$a(t)A(t) + b(t)B(t) = \mathrm{Res}_t(a,b).$$

Property 2. $\mathrm{Res}_t(a,b) = 0 \Leftrightarrow a(t)$ and $b(t)$ have a common factor of positive degree.

Property 3. Let $a(x,y) = a_n x^n + a_{n-1}(y)x^{n-1} + \cdots + a_1(y)x + a_0(y), b(x,y) = b_m x^m + b_{m-1}(y)x^{m-1} + \cdots + b_1(y)x + b_0(y) \in \mathbb{C}[y][x]$, with a_n, b_m nonzero complex numbers, and consider $p(y) = \mathrm{Res}_x(a,b)$. If $y_0 \in \mathbb{C}$ is a root of $p(y)$, then there exists an $x_0 \in \mathbb{C}$ with the property $a(x_0,y_0) = b(x_0,y_0) = 0$.

PROPERTY 4. Let $a(t) = a_n \prod_{i=1}^{n} (t - \alpha_i)$, $b(t) = b_m \prod_{j=1}^{m} (t - \beta_j)$ be the factorisations of $a(t), b(t)$ in some splitting field E of a, b over the quotient field of D. Then

$$\operatorname{Res}_t(a,b) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j) = a_n^m \prod_{i=1}^{n} b(\alpha_i) = (-1)^{mn} b_m^n \prod_{j=1}^{m} a(\beta_j).$$

PROPERTY 5. $\operatorname{Res}_t(a, bc) = \operatorname{Res}_t(a,b) \operatorname{Res}_t(a,c)$, for any nonzero $c \in D[t]$.

DEFINITION 1. *Let $p(x,y)$ be a polynomial with coefficients in D whose degree in x is n. We say that p is quasi-regular* in x if the coefficient of x^n in $p(x,y)$ is a nonzero constant.

Let us now consider polynomials $f(x,y), g(x,y)$ so that their degress in x and in y are positive. Since we are going to consider resultants of f and g with respect to x and y, in view of Property 3, we shall henceforth assume, unless otherwise stated, that f and g are quasi-regular in both x and y.

3. A First Relation

Let u, v be indeterminates. Consider

$$A(x,u,v) = \operatorname{Res}_y(f - u, g - v),$$
$$B(y,u,v) = \operatorname{Res}_x(f - u, g - v),$$

and write

$$\begin{aligned} A(x,u,v) &= A_k(u,v)x^k + \cdots + A_1(u,v)x + A_0(u,v), \\ B(y,u,v) &= B_r(u,v)y^r + \cdots + B_1(u,v)y + B_0(u,v). \end{aligned} \tag{1}$$

Our aim is to investigate the connection between the degrees k and r of A, B and the nature of the polynomials f and g.

The following theorem provides a necessary and sufficient condition for k and r to be zero.

THEOREM 1. *Let $f(x,y)$ and $g(x,y)$ be quasi-regular in x as well as in y. Let $A(x,u,v)$, $B(y,u,v)$, k, r be as above. Then, the following conditions are equivalent:*

(i) $k = 0$.
(ii) $r = 0$.
(iii) $\exists \varphi(u,v)$, $\varphi \neq 0$, *with* $\varphi(f,g) = 0$.
(iv) $J(f,g) = 0$.

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PROOF: We first note that $A_0(u,v)B_0(u,v) \neq 0$: Since f and g are quasi-regular in y, it follows that

$$\operatorname{Res}_y(f-u, g-v)\big|_{x=0} = \operatorname{Res}_y\big(f(0,y)-u, g(0,y)-v\big) \neq 0,$$

hence $A_0(u,v) = A(0,u,v) \neq 0$. Similarly, by the regularity in x, $B_0(u,v) \neq 0$.

(i)$\Rightarrow$(ii). We argue by contradiction. Suppose then that $r \geqslant 1$. In that case pick $(u_0,v_0) \in \mathbb{C}^2$ so that $B_r(u_0,v_0)A_0(u_0,v_0) \neq 0$, and let $y_0 \in \mathbb{C}$ be such that $B(y_0,u_0,v_0) = 0$. By Property 3, we can find $x_0 \in \mathbb{C}$ with the property that $f(x_0,y_0) - u_0 = g(x_0,y_0) - v_0 = 0$. Then

$$0 = \operatorname{Res}_y\big(f(x_0,y)-u_0, g(x_0,y)-v_0\big) = \operatorname{Res}_y\big(f(x,y)-u_0, g(x,y)-v_0\big)\big|_{x=x_0} = A(x_0,u_0,v_0).$$

But the latter contradicts the hypothesis that $A(x_0,u_0,v_0) = A_0(u_0,v_0) \neq 0$.

(ii)$\Rightarrow$(iii). Using Property 1, we get that $B(y,f,g) = 0$. Since $r = 0$, $B(y,u,v) = B_0(u,v)$. Hence $B_0(f,g) = 0$. But $B_0(u,v) \neq 0$.

(iii)$\Leftrightarrow$(iv). Let $\varphi(u,v)$ be of minimal positive degree so that $\varphi(f,g) = 0$. Then

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial \varphi}{\partial u}(f,g) \\ \frac{\partial \varphi}{\partial v}(f,g) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By minimality, we note that either $\frac{\partial \varphi}{\partial u}(f,g) \neq 0$ or $\frac{\partial \varphi}{\partial v}(f,g) \neq 0$. Thus $J(f,g) = 0$.

Conversely, assume that f, g are algebraically independent. Then since $A(x,f,g) = B(y,f,g) = 0$, we see that there exist polynomials $K(x,u,v)$ and $H(y,u,v)$ of minimal positive degrees in x, y respectively, so that $K(x,f,g) = H(y,f,g) = 0$. Then

$$\begin{bmatrix} \frac{\partial K}{\partial u}(x,f,g) & \frac{\partial K}{\partial v}(x,f,g) \\ \frac{\partial H}{\partial u}(y,f,g) & \frac{\partial H}{\partial v}(y,f,g) \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial K}{\partial x}(x,f,g) & 0 \\ 0 & -\frac{\partial H}{\partial y}(y,f,g) \end{bmatrix}.$$

But $(\partial K)/(\partial x)(x,f,g) \cdot (\partial H)/(\partial y)(y,f,g) \neq 0$, and thus $J(f,g) \neq 0$.

(iv)$\Rightarrow$(i). Assume that $k \geqslant 1$. Pick $(u_0,v_0) \in \mathbb{C}^2$ so that $A_k(u_0,v_0)A_0(u_0,v_0) \neq 0$ and let $x_0 \in \mathbb{C}$ be such that $A(x_0,u_0,v_0) = 0$. By a property similar to Property 3, we can find $y_0 \in \mathbb{C}$ such that $f(x_0,y_0) - u_0 = g(x_0,y_0) - v_0 = 0$. Furthermore, we note that the polynomials $f(x,y) - u_0$ and $g(x,y) - v_0$ have no common factor of positive degree for otherwise the common factor $h(x,y)$ has positive y-degree and $A(x,u_0,v_0) = 0$ by Property 5, contradicting $A_0(u_0,v_0) \neq 0$. Let $\overline{f}(x,y) = f(x+x_0, y+y_0) - u_0$, $\overline{g}(x,y) = g(x+x_0, y+y_0) - v_0$. Then $\overline{f}(0,0) = \overline{g}(0,0) = 0$ and $J(\overline{f},\overline{g}) = 0$. Using (iii) we can find $\varphi(u,v)$ of minimal positive degree so that $\varphi(\overline{f},\overline{g}) = 0$. Furthermore, since $\varphi(0,0) = 0$, $\varphi(u,v)$ has no constant term. In that case we see that $\overline{f}(x,y)$ and $\overline{g}(x,y)$ have a common factor of positive degree, $d(x,y)$ say. The latter implies that $d(x-x_0, y-y_0)$ is a common factor of positive degree of $f(x,y) - u_0$ and $g(x,y) - v_0$. ∎

REMARK 1. Quasi-regularity cannot be dropped from the hypothesis of the theorem as the following example indicates: Let $f(x,y) = xy+1$, $g(x,y) = xy+2$. Then $J(f,g) = 0$ but $A(x,u,v) = x(u-v+1)$, $B(y,u,v) = y(u-v+1)$ and thus $k = r = 1$.

The above theorem takes a special form when $f(x,y)$ and $g(x,y)$ are homogeneous polynomials. Before we can state it we shall need the following result, due to Swan, which is an easy consequence of Property 4.

LEMMA 1. *Let $n, m \geqslant 1$, $a, b \in \mathbb{C}$, $ab \neq 0$. Then*

$$\operatorname{Res}_x(ax^n - u, bx^m - v) = (-1)^n \left(a^{m/d} v^{n/d} - b^{n/d} u^{m/d}\right)^d ,$$

where $d = \gcd(m,n)$.

In view of the above, we then have the following well-known result:

COROLLARY 1. *Let $f(x,y), g(x,y)$ be homogeneous polynomials, not necessarily quasi-regular in x, y, of positive degrees n, m respectively. Then*

$$J(f,g) = 0 \Leftrightarrow cf^{m/d} = g^{n/d} ,$$

where $c \in \mathbb{C}$, $d = \gcd(m,n)$. In particular, $cf = g$ if $m = n$.

PROOF: By a linear change of coordinates we may assume that $f(x,y)$ and $g(x,y)$ are quasi-regular in x. Suppose first that $J(f,g) = 0$. Then $\operatorname{Res}_x(f-u, g-v) = B(y,u,v) = B_0(y,u,v) = \operatorname{Res}_x(ax^n - u, bx^m - v) = (-1)^n \left(a^{m/d} v^{n/d} - b^{n/d} u^{m/d}\right)^d$, where a, b are the coefficients of x^n, x^m in $f(x,y)$, $g(x,y)$ respectively. But $B_0(f,g) = 0$. The converse is trivial. ∎

COROLLARY 2. *Let $f(x,y), g(x,y)$ be polynomials of positive degrees in x, quasi-regular in x. Then*

$$J(f,g) = 0 \Rightarrow \operatorname{Res}_x(f,g) = c, \ c \text{ is a constant.}$$

PROOF: $\operatorname{Res}_x(f,g) = B_0(0,0)$. ∎

Quasi-regularity is essential in the hypothesis of the above corollary as Remark 1 shows. Also, as a consequence of the above corollary and Theorem 1, if f, g are algebraically dependent polynomials, then they either have no zeros or they have a common factor of positive degree.

When only one parameter is allowed in (1), Theorem 1 takes a somewhat different form. We begin with the following. Let $h(x,y)$ be irreducible in $\mathbb{C}[x,y]$ of positive degree in both x and y, quasi-regular in x and y. Consider

(1a) $$p(y,u) = \operatorname{Res}_x(f-u, h) = \sum_{j=0}^{\lambda} p_j(u) y^j$$

where λ is the y-degree of $p(y,u)$. Note that $p_0(u)$ has positive degree because h has positive degree in x. The following theorem provides a necessary and sufficient condition for $\lambda = 0$.

THEOREM 2. *Let $h, p(y,u)$ be as above. Then the following conditions are equivalent:*

(i) $\lambda = 0$.
(ii) *$h(x,y)$ divides $J(f,h)$.*
(iii) *There is a unique u_0 so that h divides $f - u_0$.*
(iv) *$p_0(u) = c(u-u_0)^q$, where $q = \deg_x h(x,y), c \in \mathbb{C}$.*

PROOF: (i)$\Rightarrow$(ii). Let u_0 be such that $p_0(u_0) = 0$. Then $f - u_0 = h \cdot d$ by Property 2 and a computation shows that $J(f,h) = h \cdot J(d,h)$.
(ii)$\Rightarrow$(iii). For this we consider the following cases:
α) h divides f_y. We are going to show that $\lambda = 0$. We argue by contradiction. Suppose then that $\lambda \geqslant 1$. Pick $u_0 \in \mathbb{C}$ so that $p_\lambda(u_0) \neq 0$ and let $y_0 \in \mathbb{C}$ be such that $p(y_0,u_0) = 0$. By Property 3, we can find $x_0 \in \mathbb{C}$ with the property that $f(x_0,y_0) - u_0 = h(x_0,y_0) = 0$. We also note that h divides f_x since it divides $J(f,h)$. Therefore $f(x_0,y_0) - u_0 = f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ and $\{(x,y) \in \mathbb{C}^2 \mid f - u_0 = 0\}$ is a singular curve (over $\mathbb{C}^2$). Now let $S = \{u \in \mathbb{C} \mid p_\lambda(u) \neq 0\}$, and for each $u \in S$, let C_u be the curve $\{(x,y) \in \mathbb{C}^2 \mid f - u = 0\}$. We observe that every C_u is singular (over $\mathbb{C}^2$). But C_v is singular if and only if $v \in f[(x,y) \mid f_x = f_y = 0]$, an impossibility as Sard's theorem indicates, [5]. Thus $\lambda = 0$.
β) h does not divide f_y. Let $(x_0,y_0) \in \mathbb{C}^2$ be with the properties $h(x_0,y_0) = 0$ and $f_y(x_0,y_0) \cdot h_y(x_0,y_0) \neq 0$, and let $u_0 = f(x_0,y_0)$. In that case using the Inverse Function theorem we can find C^∞ functions $y = \varphi(x)$, $y = \psi(x)$ with $\varphi(x_0) = \psi(x_0) = y_0$ and $h(x,\varphi(x)) = f(x,\psi(x)) - u_0 = 0$ in a neighbourhood U of x_0. But since h divides $J(f,h)$ we conclude that $\varphi'(x) = \psi'(x)$ near x_0. That implies $\varphi(x) = \psi(x)$ in U, and thus h divides $f - u_0$. Finally, we note that u_0 is unique since h is irreducible.
(iii)$\Rightarrow$(iv) Let u_0, u_1 be zeros of $p_0(u)$. Then $f - u_0 = h \cdot d_0$, $f - u_1 = h \cdot d_1$, and thus $u_1 - u_0 = h(d_0 - d_1)$ or $u_1 = u_0$. Therefore $p_0(u) = c(u-u_0)^k$, for some $c \in \mathbb{C}$, $k \geqslant 1$. But then since h is quasi-regular in x, $h(x,0)$ has q zeros—counted with multiplicities. That, along with Property 4, shows that $k = q$. ∎

Now let $g = h_1^{n_1} h_2^{n_2} \cdots h_s^{n_s}$, $n_j \geqslant 1$, be the prime factorisation of g in $\mathbb{C}[x,y]$, and let $q_i = \deg h_i(x,y)$, $i = 1,\dots,s$. Note that every $h_i(x,y)$ is quasi-regular in x and y.

Using the previous theorem and Property 5 we can obtain the following:

COROLLARY 3. *Let f, g, h_i be as above. Then*

(i) $\operatorname{Res}_x(f-u,g) = p(u) \Leftrightarrow h_i$ *divides $J(f,h_i)$ for all $i = 1,\dots,s$.*

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(ii) *Suppose that* $\operatorname{Res}_x(f-u,g)=p(u)$. *Then*

$$p(u)=c\prod_{i=1}^{s}(u-u_i)^{m_i},\ u_i,c\in\mathbb{C}\ \text{and}\ m_i=n_i\cdot q_i, i=1,\ldots,s\,.$$

4. The Invertibility of F

Let $f(x,y),g(x,y)$ be as before and let $n_1=\deg f(x,0)$, $n_2=\deg f(0,y)$, $m_1=\deg g(x,0)$, $m_2=\deg g(0,y)$ and $F=(f,g):\mathbb{C}^2\to\mathbb{C}^2$. For convenience we assume that $F(0,0)=(0,0)$. In this section we shall state a necessary and sufficient condition, in terms of the polynomials $A(x,u,v)$ and $B(y,u,v)$, for the map F to be $1-1$ and onto. Our results are similar to the ones in [**1**], and they come as a natural by-product of our earlier considerations.

Lemma 2. *Let* $f,g,F,A(x,u,v)$ *and* $B(y,u,v)$ *be as above. Then* F *is* $1-1$ *and onto* $\Rightarrow A(x,u,v)=ax+A_0(u,v)$, $B(y,u,v)=by+B_0(u,v)$, *where* $a,b\in\mathbb{C},ab\neq 0$.

Proof: We shall first prove that $k=r=1$. We first note by Property 1 that $k,r\geqslant 1$ since F is onto. Pick $(u_0,v_0)\in\mathbb{C}^2$ so that $A_k(u_0,v_0)\neq 0$. Since F is an automorphism, f and g are quasi-regular in x [**3**], and thus by Property 3 the polynomial $p(x)=A(x,u_0,v_0)$ has only one root, say x_0. We are going to compute $p'(x_0)$. Let the unique y_0 be such that $f(x_0,y_0)-u_0=g(x_0,y_0)-v_0=0$. Now let $\overline{x}=x,\overline{y}=y-y_0$ and write $f(x,y)=\overline{f}(\overline{x},\overline{y})$, $g(x,y)=\overline{g}(\overline{x},\overline{y})$. Observe that $\deg f(x_0,y)=\deg\overline{f}(x_0,\overline{y})$ and $\deg g(x_0,y)=\deg\overline{g}(x_0,\overline{y})$. Now using the Chain rule for resultants [**4**, Theorem 6, p.349] we have :

$$\operatorname{Res}_y\left(f(x,y)-u_0,g(x,y)-v_0\right)=\operatorname{Res}_{\overline{y}}\left(\overline{f}(\overline{x},\overline{y})-u_0,\overline{g}(\overline{x},\overline{y})-v_0\right). \tag{2}$$

We get $p'(x_0)$ by differentiating the above resultant with respect to $\overline{x}$ and evaluating the result at $\overline{x}=x_0$. This amounts to differentiating the last column only of the determinant defining the resultant, since $\overline{f}(x_0,0)-u_0=\overline{g}(x_0,0)-u_0=0$. By expanding the resulting determinant by its last column and then expanding the two cofactors by their last columns, we get:

$$p'(x_0)=(-1)^{n_2}\left[\operatorname{Res}_{\overline{y}}\left(\frac{\overline{f}(x_0,\overline{y})-u_0}{\overline{y}},\frac{\overline{g}(x_0,\overline{y})-v_0}{\overline{y}}\right)\right]\cdot J\left(\overline{f},\overline{g}\right)(x_0,0)\,. \tag{3}$$

The above shows that $p'(x_0)\neq 0$, since 0 is the only common root of $\overline{f}(x_0,\overline{y})-u_0$ and $\overline{g}(x_0,\overline{y})-v_0$, and thus $k=1$. Similarly, we can prove that $r=1$. Now suppose that $A_1(u,v)$ is not a nonzero constant. In that case we pick $(u_1,v_1)\in\mathbb{C}^2$ so that $A_1(u_1,v_1)=0$. Then, depending upon whether $A_0(u_1,v_1)$ is nonzero or zero, the

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polynomials $f-u_1$ and $g-v_1$ will either have no common zero or will have a common factor of positive degree. But this is a contradiction to F being $1-1$ and onto. ∎

We say that F has a polynomial inverse if there is a polynomial map $G(x,y) = (p(x,y),q(x,y))$ so that $G\circ F(x,y) = (x,y)$. For a polynomial map $F=(f,g)$ we define its degree, $\deg F(x,y)$, to be the highest degree of the monomials in $f(x,y)$ and $g(x,y)$.

The following Proposition describes precisely what the inverse G of F is, in the case where F is $1-1$ and onto.

Proposition 1. *Let F, a, b, $A_0(u,v)$, $B_0(u,v)$ be as in Lemma 2 and $G(x,y) = (-(A_0(x,y))/a\,, -(B_0(x,y))/b\,)$. Then G is the inverse of $F(x,y)$. Furthermore, $\deg F(x,y) = \deg G(x,y)$.*

Proof: In view of Lemma 2 and Property 1 we have:

$$G\circ F(x,y) = G(f,g) = \left(-\frac{A_0(f,g)}{a}\,, -\frac{B_0(f,g)}{b}\right) = (x,y)\,.$$

For the second assertion, we note that $A_0(u,v) = \mathrm{Res}_y\,(f(0,y)-u, g(0,y)-v)$, and thus $\deg A_0(u,v) = \max(n_2,m_2)$ and, similarly $\deg B_0(u,v) = \max(n_1,m_1)$. Thus, $\deg G = \max(n_1,n_2,m_1,m_2)$. But since the Newton polygons of an automorphism are triangles we see that $\deg F(x,y) = \deg G(x,y)$, [**3**]. ∎

Finally, we may use the so-called "border polynomials" of $F(x,y)$ to describe explicitly its inverse, $G(x,y)$. These are $f(x,0)$, $g(x,0)$, $f(0,y)$ and $g(0,y)$. Using Lemma 2 we get:

(4)
$$A_0(u,v) = \mathrm{Res}_y\,(f(0,y)-u, g(0,y)-v)\,,\ B_0(u,v) = \mathrm{Res}_x\,(f(x,0)-u, g(x,0)-v)\,.$$

Furthermore, (3) together with the Chain rule for resultants shows that

(5) $$a = (-1)^{n_2}\left[\mathrm{Res}_y\left(\frac{f(0,y)}{y}, \frac{g(0,y)}{y}\right)\right]\cdot J(f,g)(0,0).$$

Likewise

(6) $$b = (-1)^{n_1}\left[\mathrm{Res}_x\left(\frac{f(x,0)}{x}, \frac{g(x,0)}{x}\right)\right]\cdot J(f,g)(0,0).$$

Thus,

Proposition 2. *If $F=(f,g)$ is $1-1$ and onto, F has a polynomial inverse G which is completely determined by the border polynomials of F.*

5. A Conjecture

Let $F = (f, g)$ be as before, but not necessarily $F(0,0) = (0,0)$. It is clear that F has no zeros if there exist polynomials $\varphi(x,y)$, $\psi(x,y)$ so that $f\varphi + g\psi = 1$. The latter is equivalent–by Property 3–to the fact that $\mathrm{Res}_x(f,g) = c_1$ and $\mathrm{Res}_y(f,g) = c_2$, $c_1 c_2 \neq 0$, $c_i \in \mathbb{C}$.

In Section 3 we saw that if $J(f,g) = 0$, then $\mathrm{Res}_x(f,g) = c, c \in \mathbb{C}$. Furthermore, (Corollary 3), whenever every irreducible factor h of g divides $J(f,h)$, then $\mathrm{Res}_x(f,g) = c$. We believe that a partial converse of Corollary 2 is true; we state it in the form of the following:

Conjecture. *Let $f(x,y), g(x,y)$ be quasi-regular in x. Then $\mathrm{Res}_x(f,g) = c$, $c \in \mathbb{C} \Rightarrow$ there exists a point $(x_0, y_0) \in \mathbb{C}^2$ for which $J(f,g)(x_0,y_0) = 0$.*

Remark 2. We note that the above conjecture is trivially true in the case where $c = 0$.

We conclude with a consequence of the above conjecture in relation to the Jacobian conjecture.

Remark 3. Let $F = (f,g) : \mathbb{C}^2 \to \mathbb{C}^2$ be such that $J(f,g) = 1$. Then, if the above conjecture is true, F is onto.

Proof: By a linear change of coordinates we may suppose that f, g are quasi-regular in x. Now, if there exists a point (s,t) for which $F^{-1}(s,t) = \emptyset$, then $\mathrm{Res}_x(f - s, g - t) = c \neq 0$. But $J(f-s, g-t) = J(f,g) = 1$, contradicting the conjecture. ∎

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