ON THE RADON–NIKODYM PROPERTY IN JORDAN ALGEBRAS

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1. Introduction. Banach spaces whose duals possess the Radon–Nikodym property have been studied extensively in the past (cf. [**5**]). It has been shown recently in [**4**] that a C*-algebra is scattered if and only if its Banach dual possesses the Radon-Nikodym property. This result extends the well-known result of Pełczynski and Semandini [**8**] that a compact Hausdorff space Ω is dispersed if and only if $C(\Omega)^*$ has the Radon–Nikodym property. The purpose of this note is to give a transparent proof of a more general result for Jordan algebras which unifies the aforementioned results. We prove that the dual of a JB-algebra A possesses the Radon–Nikodym property if and only if the state space of A is the σ-convex hull of its pure states. We also consider the projective tensor products of the duals of JB-algebras in this context.

We first recall that a JB-*algebra* A is a real Jordan algebra with identity e which is also a Banach space, and where the Jordan product and the norm are related as follows:

$$\|a \circ b\| \leq \|a\| \cdot \|b\|,$$
$$\|a^2\| = \|a\|^2,$$
$$\|a^2\| \leq \|a^2 + b^2\|.$$

A JBW-*algebra* is a JB-algebra which is a Banach dual space, and the *enveloping* JBW-*algebra* of a JB-algebra A is the bidual A^{**} with the Arens product. We note that the self-adjoint part of a C*-algebra is a JB-algebra and that the self-adjoint part of a von Neumann algebra is a JBW-algebra.

Let K be the state space of a JB-algebra A and let ∂K be the set of pure states. Then, as in [**3**], for each $p \in \partial K$, one can associate a *dense representation*

$$\phi_p : A \to c(p) \circ A^{**}$$

with $\phi_p(a) = c(p) \circ a$ for each a in A, where $c(p)$ denotes the central support of p. Moreover there is a unique weakly continuous extension

$$\tilde{\phi}_p : A^{**} \to c(p) \circ A^{**}$$

with $\tilde{\phi}_p(A^{**}) = c(p) \circ A^{**}$ and $\ker \tilde{\phi}_p = (e - c(p)) \circ A^{**}$. If p and q are two pure states, then either $c(p) = c(q)$ or $c(p) \circ c(q) = 0$; in the latter case we say that p and q are *disjoint*.

A Banach space X is said to possess the *Radon–Nikodym property* if for any finite measure space (Ω, Σ, μ) and any μ-continuous vector measure $L : \Sigma \to X$ of bounded total variation, there exists a Bochner integrable function $g : \Omega \to X$ such that $L(E) = \int_E g \, d\mu$ for all E in Σ. Lindenstrauss [**7**] has proved that if X has the Radon-Nikodym property, then it has the Krein–Milman property; that is, every (norm) closed bounded convex subset of X is the (norm) closed convex hull of its extreme points. The converse is also

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true if X is a dual space. We note that subspaces of a Banach space with the Radon-Nikodym property also possess this property.

2. The main results. A JBW-algebra is called a *factor* if its centre is the scalar multiples of the identity e. A JBW-factor is of *type* I if it contains a (nonzero) minimal idempotent.

We noted in [**4**] that the Banach space $T(H)$ of trace-class operators on a Hilbert space H has the Radon-Nikodym property. Actually we have the following more general result.

PROPOSITION 1. *Let M be a type* I JBW-*factor. Then its predual M_* has the Radon–Nikodym property.*

Proof. Since reflexive spaces have the Radon–Nikodym property (cf. [**5**, p. 76]), we need only consider the case in which M is not isomorphic to a spin factor or the exceptional algebra M_3^8. Then, as in [**3**; Theorem 3.1], we may assume that M is an irreducible JW-algebra contained in the full operator algebra $B(H)$ on some (complex) Hilbert space H. Let $T(H)_{sa}$ denote the self-adjoint trace-class operators. If M is the self-adjoint part $B(H)_{sa}$ of $B(H)$, then $M_* = T(H)_{sa}$ has the Radon-Nikodym property. Otherwise, by [**3**, Theorem 3.1], there exists a *-antiautomorphism $\Phi: B(H) \to B(H)$ such that $M = \{x \in B(H)_{sa} : \Phi(x) = x\}$ and Φ^2 is the identity mapping I. Further, Φ is implemented by a conjugate linear isometry $j: H \to H$ with $\Phi(x) = j^{-1}x^*j$, for each x in $B(H)$. As usual, we identify the dual $T(H)^*$ with $B(H)$ via the duality $(t, x) \in T(H) \times B(H) \to \mathrm{Tr}(xt)$ where Tr is the canonical trace. Let $N = \{t \in T(H)_{sa} : \Phi(t) = t\}$. We show that M is isometric to the dual space N^* via the mapping

$$x \in M \to x^* \in N^* : x^*(t) = \mathrm{Tr}(xt) \qquad (t \in N).$$

First $|\mathrm{Tr}(xt)| \leq \|x\| \cdot \mathrm{Tr}(|t|)$ implies that $\|x^*\| \leq \|x\|$. The reverse inequality follows from the fact that

$$\|x\| = \sup\{|\mathrm{Tr}(xt)| : t \in T(H)_{sa}, \mathrm{Tr}(|t|) \leq 1\}$$

and if $t \in T(H)_{sa}$ with $\mathrm{Tr}(|t|) \leq 1$, then $\frac{1}{2}(t + \Phi(t)) \in N$ with

$$\mathrm{Tr}(|\tfrac{1}{2}(t + \Phi(t))|) \leq \tfrac{1}{2}\mathrm{Tr}(|t|) + \tfrac{1}{2}\mathrm{Tr}(|\Phi(t)|) = \mathrm{Tr}(|t|) \leq 1,$$

$$\mathrm{Tr}(x(\tfrac{1}{2}(t + \Phi(t)))) = \tfrac{1}{2}\mathrm{Tr}(xt) + \tfrac{1}{2}\mathrm{Tr}(x\Phi(t)) = \tfrac{1}{2}\mathrm{Tr}(xt) + \mathrm{Tr}(\Phi(tx)) = \mathrm{Tr}(xt).$$

Also, the mapping is onto N^*. Indeed, if $f \in N^*$ and $\phi \in T(H)^*_{sa}$ extends f, then there exists $x \in B(H)_{sa}$ such that $\phi(t) = \mathrm{Tr}(xt)$ for all t in $T(H)$. Let $y = \frac{1}{2}(x + \Phi(x))$. Then $y \in M$ and for $t \in N$, we have $\Phi(t) = t$ and as before $y^*(t) = \mathrm{Tr}(yt) = \mathrm{Tr}(\frac{1}{2}(x + \Phi(x))t) = \mathrm{Tr}(xt) = \phi(t) = f(t)$. So $f = y^*$.

Hence M is isometric to N^* and by the uniqueness of the predual, M_* is isometric to N which has the Radon-Nikodym property since it is a subspace of $T(H)$.

PROPOSITION 2. *Let M be a* JBW-*algebra. Then its predual M_* has the Radon–Nikodym property if and only if M is a direct sum of type I* JBW-*factors.*

Proof. The sufficiency follows from the above proposition. We prove the necessity.

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Let $K \subset M_*$ be the normal state space of M. Then the Krein–Milman property implies that K is the (norm) closed convex hull of the pure states ∂K. For each $p \in \partial K$, consider the dense representation

$$\phi_p : M \to c(p) \circ M^{**}$$

where $c(p) \in M$ and $\phi_p(M) = c(p) \circ M$ is a type I JBW-factor (cf. [**1**, §5]). We have $\bigcap_{p \in \partial K} \ker \phi_p = \{0\}$ for if $a \in \bigcap_{p \in \partial K} \ker \phi_p$, then $c(p) \circ a = 0$ for all p in ∂K and hence $a = 0$ as K is the norm-closed convex hull of ∂K. Now $\ker \phi_p = (e - c(p)) \circ M$ for each $p \in \partial K$ gives $\bigwedge_{p \in \partial K} (e - c(p)) = 0$ and so $e = \bigvee_{p \in \partial K} c(p)$. Therefore $M = \Sigma \oplus (c(p) \circ M)$ where the sum is taken over all the mutually disjoint pure normal states.

Incidentally, we have only used the Krein–Milman property in the above proof; hence the Krein–Milman property and the Radon–Nikodym property are equivalent in the preduals of JBW-algebras.

THEOREM 3. *Let A be a* JB-*algebra with state space K. Then the following conditions are equivalent.*

(i) *K is the σ-convex hull $\sigma(\partial K)$ of the pure states; that is,*

$$K = \sigma(\partial K) = \left\{ \sum_{n=1}^{\infty} \lambda_n k_n : \sum_{n=1}^{\infty} \lambda_n = 1, \lambda_n \geq 0, k_n \in \partial K \right\}$$

with norm-convergent infinite sum.

(ii) *A^* has the Radon–Nikodym property.*

(iii) *A^{**} is a direct sum of type I* JBW-*factors.*

Proof. (i) $\Rightarrow$ (iii). As in the proof of Proposition 2, we have $A^{**} = \Sigma \oplus (c(p) \circ A^{**})$ where the sum is taken over all the mutually disjoint pure states of A.

(iii) $\Rightarrow$ (ii). Proposition 2.

(ii) $\Rightarrow$ (i). By the Krein–Milman property, the state space K of A is the norm-closed convex hull of its pure states ∂K. As the σ-convex hull $\sigma(\partial K)$ is a split face of K [**2**; Corollary 5.8], it is norm-closed and it follows that $K = \sigma(\partial K)$.

REMARKS 1. The above result does not hold for arbitrary Banach spaces. For instance, every positive functional of l_1 is the sum of a sequence of pure functionals while $l_1^* = l_\infty$ does not have the Radon–Nikodym property [**5**, p. 219].

2. A compact Hausdorff space Ω is dispersed [**8**] if and only if every Radon measure on Ω is atomic. Jensen has extended this notion to C*-algebras in [**10**] where a C*-algebra is called *scattered* if every positive functional is atomic. Moreover, the atomic functionals are shown to be the sums of sequences of pure functions [**10**, Theorem 1.2]. Theorem 3 relates this notion to the Radon–Nikodym property in the wider context of Jordan algebras.

3. Tensor products. Let X and Y be Banach spaces with the Radon–Nikodym property. It is an open problem whether the projective tensor product $X \hat{\otimes} Y$ also has this property (cf. [**5**, p. 258]). We prove here a special case for the dual spaces of JB-algebras.

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Let H and K be Hilbert spaces. We first recall that $T(H)$ can be identified with the projective tensor product $H\hat{\otimes}_{\mathbb{C}}H^*$ [9, Theorem 5.12]. It follows from the associativity of the projective tensor products (cf. [6, p. 51]) that $T(H)_{\mathbb{R}}\hat{\otimes}T(K)_{sa}=(H_{\mathbb{R}}\hat{\otimes}H^*_{\mathbb{R}})\hat{\otimes}T(K)_{sa}$ has the Radon–Nikodym property (cf. [5, p. 249]). Further, it is easy to verify that $T(H)_{sa}\hat{\otimes}T(K)_{sa}\subseteq T(H)_{\mathbb{R}}\hat{\otimes}T(K)_{sa}$. Hence $T(H)_{sa}\hat{\otimes}T(K)_{sa}$ has the Radon–Nikodym property.

LEMMA 4. *Let M and N be two type I JBW-factors with preduals M_* and N_* respectively. Then the projective tensor product $M_*\hat{\otimes}N_*$ has the Radon–Nikodym property.*

Proof. As in Proposition 1, it suffices to consider the case in which $M\subseteq B(H)$ and $N\subseteq B(K)$ are irreducible JW-algebras on some Hilbert spaces H and K with *-antiautomorphisms $\Phi: B(H)\to B(H)$ and $\Psi: B(K)\to B(K)$ such that $\frac{1}{2}(I+\Phi): T(H)_{sa}\to T(H)_{sa}$ and $\frac{1}{2}(I+\Psi): T(K)_{sa}\to T(K)_{sa}$ are projections with norm ≤ 1 and $M_*=\frac{1}{2}(I+\Phi)T(H)_{sa}$, $N_*=\frac{1}{2}(I+\Psi)T(K)_{sa}$. Thus, by [9, Theorem 3.10], we have $M_*\hat{\otimes}N_*\subseteq T(H)_{sa}\hat{\otimes}T(K)_{sa}$. Therefore $M_*\hat{\otimes}N_*$ has the Radon–Nikodym property.

THEOREM 5. *Let A and B be JB-algebras such that A^* and B^* have the Radon–Nikodym property. Then the projective tensor product $A^*\hat{\otimes}B^*$ also has the Radon–Nikodym property.*

Proof. By Theorem 3, $A^{**}=\sum_\alpha M^\alpha$ and $B^{**}=\sum_\beta M^\beta$ are direct sums of type I JBW-factors M^α, M^β with $A^*=\left(\sum_\alpha M^\alpha_*\right)_{l_1}$ and $B^*=\left(\sum_\beta M^\beta_*\right)_{l_1}$ where M^α_* and M^β_* are preduals of M^α and M^β respectively. Thus we have

$$A^*\hat{\otimes}B^*=\left(\sum_\alpha M^\alpha_*\right)\hat{\otimes}\left(\sum_\beta M^\beta_*\right)$$

$$=\sum_{\alpha,\beta}(M^\alpha_*\hat{\otimes}M^\beta_*) \qquad \text{(cf [6, p. 46])}$$

which has the Radon–Nikodym property by the above lemma.

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REFERENCES

1. E. M. Alfsen, F. W. Shultz and E. Størmer, A Gelfand-Neumark theorem for Jordan algebras, *Advances in Math.* **28** (1978), 11–56.

2. E. M. Alfsen and F. M. Shultz, State spaces of Jordan algebras, *Acta Math.* **140** (1978), 115–190.

3. E. M. Alfsen, F. W. Shultz and H. Hanche-Olsen, State spaces of C*-algebras, *Acta Math.* **144** (1980), 267–305.

4. C. H. Chu, A note on scattered C*-algebras and the Radon–Nikodym property *J. London Math. Soc* **24** (1981), 533–536.

5. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Amer. Math. Soc. Survey No. 15, (1977).

6. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the Amer. Math. Soc. **16** (1966).

7. J. Lindenstrauss, On extreme points in l_1, *Israel J. Math.* **4** (1966), 59–61.

8. A. Pełczynski and Z. Semandini, Spaces of continuous functions III: spaces $C(\Omega)$ for Ω without perfect subsets, *Studia Math.* **18** (1959), 211–222.

9. R. Schatten, *A theory of cross spaces*, Ann. of Math. Studies, **26** (1950).

10. H. E. Jensen, Scattered C*-algebras, *Math. Scand.* **41** (1977), 308–314.

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